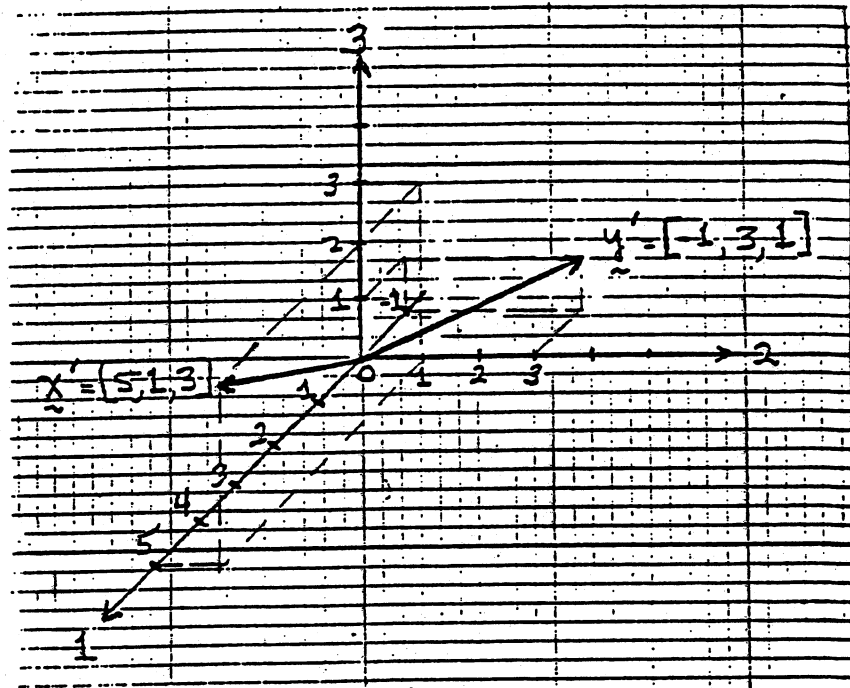


Chapter 2

2.1 a)



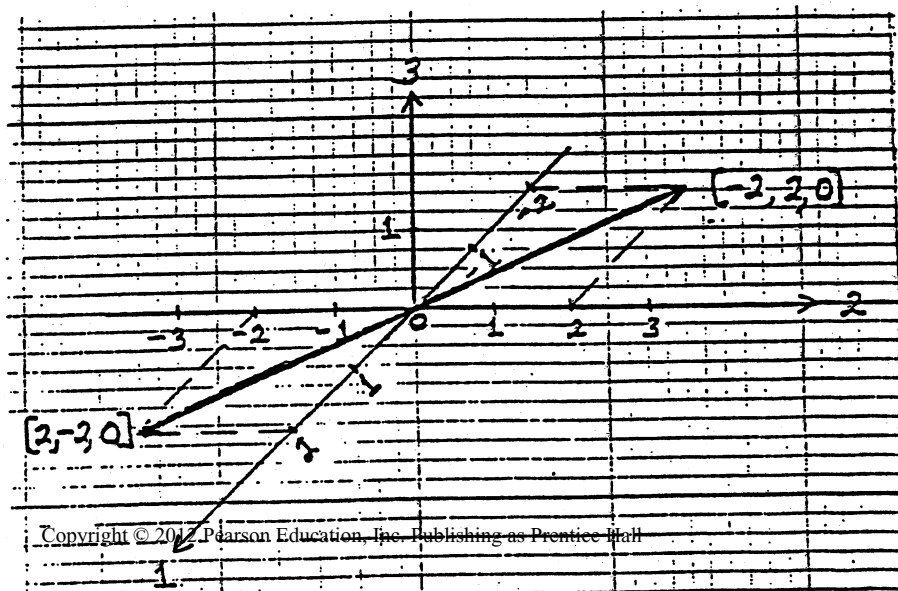
$$b) \quad i) \quad L_{\underline{x}} = \sqrt{\underline{x}'\underline{x}} = \sqrt{35} = 5.916$$

$$ii) \quad \cos(\theta) = \frac{\underline{x}'\underline{y}}{L_{\underline{x}}L_{\underline{y}}} = \frac{1}{19.621} = .051$$

$$\theta = \arccos(.051) \doteq 87^\circ$$

$$iii) \quad \text{projection of } \underline{y} \text{ on } \underline{x} \text{ is } \left[\frac{\underline{y}'\underline{x}}{\underline{x}'\underline{x}} \right] \underline{x} = \frac{1}{35} \underline{x} = \left[\frac{1}{7}, \frac{1}{35}, \frac{3}{35} \right]'$$

c)



$$2.2 \quad a) \quad 5A = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix} \quad b) \quad BA = \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$$

$$c) \quad A'B' = \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix} \quad d) \quad C'B = [12, -7]$$

e) No.

$$2.3 \quad a) \quad A' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A \quad \text{so} \quad (A')' = A' = A$$

$$b) \quad C' = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}; \quad (C')^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} -\frac{2}{10} & \frac{4}{10} \\ \frac{3}{10} & -\frac{1}{10} \end{bmatrix}; \quad (C^{-1})' = \begin{bmatrix} -\frac{2}{10} & \frac{3}{10} \\ \frac{4}{10} & -\frac{1}{10} \end{bmatrix} = (C')^{-1}$$

$$c) \quad (AB)' = \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix} = (AB)'$$

d) AB has (i,j) th entry

$$a_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj} = \sum_{\ell=1}^k a_{i\ell}b_{\ell j}$$

Consequently, $(AB)'$ has (i,j) th entry

$$c_{ji} = \sum_{\ell=1}^k a_{j\ell}b_{\ell i}$$

Next B' has i th row $[b_{i1}, b_{i2}, \dots, b_{ik}]$ and A' has j th

column $[a_{j1}, a_{j2}, \dots, a_{jk}]'$ so $B'A'$ has $(i, j)^{\text{th}}$ entry

$$b_{1i}a_{j1} + b_{2i}a_{j2} + \dots + b_{ki}a_{jk} = \sum_{\ell=1}^k a_{j\ell}b_{\ell i} = c_{ji}$$

Since i and j were arbitrary choices, $(AB)' = B'A'$.

- 2.4 a) $I = I'$ and $AA^{-1} = I = A^{-1}A$. Thus $I' = I = (AA^{-1})' = (A^{-1})'A'$ and $I = (A^{-1}A)' = A'(A^{-1})'$. Consequently, $(A^{-1})'$ is the inverse of A' or $(A')^{-1} = (A^{-1})'$.

- b) $(B^{-1}A^{-1})AB = B^{-1}(\underbrace{A^{-1}A}_I)B = B^{-1}B = I$ so AB has inverse $(AB)^{-1} = B^{-1}A^{-1}$. It was sufficient to check for a left inverse but we may also verify $AB(B^{-1}A^{-1}) = A(\underbrace{BB^{-1}}_I)A^{-1} = AA^{-1} = I$.

2.5

$$QQ' = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{bmatrix} = \begin{bmatrix} \frac{169}{169} & 0 \\ 0 & \frac{169}{169} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q'Q.$$

2.6

- a) Since $A = A'$, A is symmetric.
- b) Since the quadratic form

$$\underline{x}'A\underline{x} = [x_1, x_2] \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 9x_1^2 - 4x_1x_2 + 6x_2^2$$

$$= (2x_1 - x_2)^2 + 5(x_1^2 + x_2^2) > 0 \text{ for } [x_1, x_2] \neq [0, 0]$$

we conclude that A is positive definite.

2.7

- a) Eigenvalues: $\lambda_1 = 10$, $\lambda_2 = 5$.

Normalized eigenvectors: $\underline{e}_1 = [2/\sqrt{5}, -1/\sqrt{5}] = [.894, -.447]$

$\underline{e}_2 = [1/\sqrt{5}, 2/\sqrt{5}] = [.447, .894]$

$$b) A = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix} = 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, -1/\sqrt{5}] + 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, 2/\sqrt{5}]$$

$$c) A^{-1} = \frac{1}{9(6) - (-2)(-2)} \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} .12 & .04 \\ .04 & .18 \end{bmatrix}$$

$$d) \text{Eigenvalues: } \lambda_1 = .2, \lambda_2 = .1$$

$$\text{Normalized eigenvectors: } \underline{e}_1' = [1/\sqrt{5}, 2/\sqrt{5}]$$

$$\underline{e}_2' = [2/\sqrt{5}, -1/\sqrt{5}]$$

2.8

$$\text{Eigenvalues: } \lambda_1 = 2, \lambda_2 = -3$$

$$\text{Normalized eigenvectors: } \underline{e}_1' = [2/\sqrt{5}, 1/\sqrt{5}]$$

$$\underline{e}_2' = [1/\sqrt{5}, -2/\sqrt{5}]$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = 2 \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, 1/\sqrt{5}] - 3 \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, -2/\sqrt{5}]$$

2.9

$$a) A^{-1} = \frac{1}{1(-2) - 2(2)} \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$$

$$b) \text{Eigenvalues: } \lambda_1 = 1/2, \lambda_2 = -1/3$$

$$\text{Normalized eigenvectors: } \underline{e}_1' = [2/\sqrt{5}, 1/\sqrt{5}]$$

$$\underline{e}_2' = [1/\sqrt{5}, -2/\sqrt{5}]$$

$$c) A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} [2/\sqrt{5}, 1/\sqrt{5}] - \frac{1}{3} \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} [1/\sqrt{5}, -2/\sqrt{5}]$$

2.10

$$B^{-1} = \frac{1}{4(4.002001) - (4.001)^2} \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= 333,333 \begin{bmatrix} 4.002001 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4(4.002) - (4.001)^2} \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$= -1,000,000 \begin{bmatrix} 4.002 & -4.001 \\ -4.001 & 4 \end{bmatrix}$$

$$\text{Thus } A^{-1} = (-3)B^{-1}$$

2.11

With $p = 1$, $|a_{11}| = a_{11}$ and with $p = 2$

$$\begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22} - 0(0) = a_{11}a_{22}$$

Proceeding by induction, we assume the result holds for any $(p-1) \times (p-1)$ diagonal matrix A_{11} . Then writing

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A_{11} & \\ 0 & & & \end{bmatrix}$$

we expand $|A|$ according to Definition 2A.24 to find

$$|A| = a_{11} |A_{11}| + 0 + \cdots + 0. \text{ Since } |A_{11}| = a_{22}a_{33} \cdots a_{pp}$$

$$\text{by the induction hypothesis, } |A| = a_{11}(a_{22}a_{33} \cdots a_{pp}) =$$

$$a_{11}a_{22}a_{33} \cdots a_{pp}.$$

2.12

By (2-20), $A = PAP'$ with $PP' = P'P = I$. From Result 2A.11(e) $|A| = |P| |\Lambda| |P'| = |\Lambda|$. Since Λ is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_p$, we can apply Exercise 2.11 to get $|A| = |\Lambda| = \prod_{i=1}^p \lambda_i$.

2.14

Let λ be an eigenvalue of A . Thus $0 = |A - \lambda I|$. If Q is orthogonal, $QQ' = I$ and $|Q||Q'| = 1$ by Exercise 2.13. Using Result 2A.11(e) we can then write

$$0 = |Q| |A - \lambda I| |Q'| = |QAQ' - \lambda I|$$

and it follows that λ is also an eigenvalue of QAQ' if Q is orthogonal.

2.16

$(A'A)' = A'(A')' = A'A$ showing $A'A$ is symmetric.

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = A\underline{x}. \text{ Then } 0 \leq y_1^2 + y_2^2 + \dots + y_p^2 = \underline{y}'\underline{y} = \underline{x}'A'A\underline{x}$$

and $A'A$ is non-negative definite by definition.

2.18

Write $c^2 = \underline{x}'A\underline{x}$ with $A = \begin{bmatrix} 4 & -\sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$. The eigenvalue-normalized eigenvector pairs for A are:

$$\lambda_1 = 2, \quad \underline{e}_1' = [.577, .816]$$

$$\lambda_2 = 5, \quad \underline{e}_2' = [.816, -.577]$$

For $c^2 = 1$, the half lengths of the major and minor axes of the ellipse of constant distance are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} = .707 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} = .447$$

respectively. These axes lie in the directions of the vectors \underline{e}_1 and \underline{e}_2 respectively.

For $c^2 = 4$, the half lengths of the major and minor axes are

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{2}} = 1.414 \quad \text{and} \quad \frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{5}} = .894 .$$

As c^2 increases the lengths of the major and minor axes increase.

2.20

Using matrix A in Exercise 2.3, we determine

$$\lambda_1 = 1.382, \quad \underline{e}_1 = [.8507, \quad -.5257]'$$

$$\lambda_2 = 3.618, \quad \underline{e}_2 = [.5257, \quad .8507]'$$

We know

$$A^{1/2} = \sqrt{\lambda_1} \underline{e}_1 \underline{e}_1' + \sqrt{\lambda_2} \underline{e}_2 \underline{e}_2' = \begin{bmatrix} 1.376 & .325 \\ .325 & 1.701 \end{bmatrix}$$

$$A^{-1/2} = \frac{1}{\sqrt{\lambda_1}} \underline{e}_1 \underline{e}_1' + \frac{1}{\sqrt{\lambda_2}} \underline{e}_2 \underline{e}_2' = \begin{bmatrix} .7608 & -.1453 \\ -.1453 & .6155 \end{bmatrix}$$

We check

$$A^{1/2} A^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1/2} A^{1/2}$$

2.21 (a)

$$A'A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

$$0 = |A'A - \lambda I| = (9 - \lambda)^2 - 1 = (10 - \lambda)(8 - \lambda) \quad , \text{ so } \lambda_1 = 10 \text{ and } \lambda_2 = 8.$$

Next,

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives } e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives } e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(b)

$$AA' = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$0 = |AA' - \lambda I| = \begin{vmatrix} 2 - \lambda & 0 & 4 \\ 0 & 8 - \lambda & 0 \\ 4 & 0 & 8 - \lambda \end{vmatrix}$$

$$= (2 - \lambda)(8 - \lambda)^2 - 4^2(8 - \lambda) = (8 - \lambda)(\lambda - 10)\lambda \quad \text{so } \lambda_1 = 10, \lambda_2 = 8, \text{ and } \lambda_3 = 0.$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 10 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\text{gives } \begin{matrix} 4e_3 = 8e_1 \\ 8e_2 = 10e_2 \end{matrix} \quad \text{so } e_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 8 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\text{gives } \begin{matrix} 4e_3 = 6e_1 \\ 4e_1 = 0 \end{matrix} \quad \text{so } e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Also, } e_3 = [-2/\sqrt{5}, 0, 1/\sqrt{5}]'$$

(c)

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix} = \sqrt{10} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right] + \sqrt{8} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right]$$

2.22 (a)

$$AA' = \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} = \begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix}$$

$0 = |AA' - \lambda I| = (144 - \lambda)(126 - \lambda) - (12)^2 = (150 - \lambda)(120 - \lambda)$, so $\lambda_1 = 150$ and $\lambda_2 = 120$. Next,

$$\begin{bmatrix} 144 & -12 \\ -12 & 126 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad \text{gives } e_1 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

and $\lambda_2 = 120$ gives $e_2 = [1/\sqrt{5}, 2/\sqrt{5}]'$.

(b)

$$A'A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \\ 8 & -9 \end{bmatrix} \begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix}$$

$$0 = |A'A - \lambda I| = \begin{vmatrix} 25 - \lambda & 50 & 5 \\ 50 & 100 - \lambda & 10 \\ 5 & 10 & 145 - \lambda \end{vmatrix} = (150 - \lambda)(\lambda - 120)\lambda$$

so $\lambda_1 = 150$, $\lambda_2 = 120$, and $\lambda_3 = 0$. Next,

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 150 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\text{gives } \begin{cases} -120e_1 + 60e_2 = 0 \\ -25e_1 + 5e_3 = 0 \end{cases} \quad \text{or } e_1 = \frac{1}{\sqrt{30}} \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 25 & 50 & 5 \\ 50 & 100 & 10 \\ 5 & 10 & 145 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 120 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\text{gives } \begin{array}{l} 60e_1 + 60e_3 = 0 \\ -120e_2 + -240e_3 = 0 \end{array} \text{ or } e_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\text{Also, } e_3 = [2/\sqrt{5}, -1/\sqrt{5}, 0]'$$

(c)

$$\begin{bmatrix} 4 & 8 & 8 \\ 3 & 6 & -9 \end{bmatrix}$$

$$= \sqrt{150} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{30}} \quad \frac{2}{\sqrt{30}} \quad \frac{5}{\sqrt{30}} \right] + \sqrt{120} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \left[\frac{1}{\sqrt{6}} \quad \frac{2}{\sqrt{6}} \quad -\frac{1}{\sqrt{6}} \right]$$

2.24

$$\text{a) } \ddagger^{-1} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{b) } \begin{array}{l} \lambda_1 = 4, \quad \underline{e}_1 = [1, 0, 0]' \\ \lambda_2 = 9, \quad \underline{e}_2 = [0, 1, 0]' \\ \lambda_3 = 1, \quad \underline{e}_3 = [0, 0, 1]' \end{array}$$

$$\text{c) For } \ddagger^{-1}: \begin{array}{l} \lambda_1 = 1/4, \quad \underline{e}'_1 = [1, 0, 0]' \\ \lambda_2 = 1/9, \quad \underline{e}'_2 = [0, 1, 0]' \\ \lambda_3 = 1, \quad \underline{e}'_3 = [0, 0, 1]' \end{array}$$

2.25

$$a) \quad v^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \quad \rho = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -.2 & .267 \\ -.2 & 1 & .167 \\ .267 & .167 & 1 \end{bmatrix}$$

$$b) \quad v^{1/2} \rho v^{1/2} =$$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 4/3 \\ -2/5 & 2 & 1/3 \\ 4/5 & 1/2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \mathbf{\Sigma}$$

2.26

$$a) \quad \rho_{13} = \sigma_{13} / \sigma_{11}^{1/2} \sigma_{22}^{1/2} = 4 / \sqrt{25} \sqrt{9} = 4/15 = .267$$

$$b) \quad \text{Write } X_1 = 1 \cdot X_1 + 0 \cdot X_2 + 0 \cdot X_3 = \underline{c}_1' X \quad \text{with } \underline{c}_1' = [1, 0, 0]$$

$$\frac{1}{2} X_2 + \frac{1}{2} X_3 = \underline{c}_2' X \quad \text{with } \underline{c}_2' = [0, \frac{1}{2}, \frac{1}{2}]$$

Then $\text{Var}(X_1) = \sigma_{11} = 25$. By (2-43),

$$\text{Var}\left(\frac{1}{2} X_2 + \frac{1}{2} X_3\right) = \underline{c}_2' \mathbf{\Sigma} \underline{c}_2 = \frac{1}{4} \sigma_{22} + \frac{2}{4} \sigma_{23} + \frac{1}{4} \sigma_{33} = 1 + \frac{1}{2} + \frac{9}{4}$$

$$= \frac{15}{4} = 3.75$$

By (2-45), (see also hint to Exercise 2.28),

$$\text{Cov}\left(X_1, \frac{1}{2} X_2 + \frac{1}{2} X_3\right) = \underline{c}_1' \mathbf{\Sigma} \underline{c}_2 = \frac{1}{2} \sigma_{12} + \frac{1}{2} \sigma_{13} = -1 + 2 = 1$$

so

$$\text{Corr}(X_1, \frac{1}{2}X_1 + \frac{1}{2}X_2) = \frac{\text{Cov}(X_1, \frac{1}{2}X_1 + \frac{1}{2}X_2)}{\sqrt{\text{Var}(X_1)} \sqrt{\text{Var}(\frac{1}{2}X_1 + \frac{1}{2}X_2)}} = \frac{1}{5\sqrt{3.75}} = .103$$

2.27

a) $\mu_1 - 2\mu_2, \sigma_{11} + 4\sigma_{22} - 4\sigma_{12}$

b) $-\mu_1 + 3\mu_2, \sigma_{11} + 9\sigma_{22} - 6\sigma_{12}$

c) $\mu_1 + \mu_2 + \mu_3, \sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{23}$

d) $\mu_1 + 2\mu_2 - \mu_3, \sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}$

e) $3\mu_1 - 4\mu_2, 9\sigma_{11} + 16\sigma_{22}$ since $\sigma_{12} = 0$.

2.29

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} \end{bmatrix} = \begin{bmatrix} \dagger_{11} & \dagger_{12} \\ \dagger_{21} & \dagger_{22} \end{bmatrix}$$

2.31 (a)

$$E[\mathbf{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad (\text{b}) \quad \mathbf{A}\boldsymbol{\mu}^{(1)} = [1 \quad -1] \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 1$$

(c)

$$\text{Cov}(\mathbf{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = [1 \quad -1] \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$

(e)

$$E[\mathbf{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (\text{f}) \quad \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(g)

$$\text{Cov}(\mathbf{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\mathbf{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 48 & -8 \\ -8 & 4 \end{bmatrix}$$

(i)

$$\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}' = [1 \quad -1] \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = [0 \quad 2]$$

2.32 (a)

$$E[\mathbf{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad (\text{b}) \quad \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

(c)

$$\text{Cov}(\mathbf{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

(e)

$$E[\mathbf{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \quad (\text{f}) \quad \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

(g)

$$\text{Cov}(\mathbf{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(h)

$$\begin{aligned} \text{Cov}(\mathbf{B}\mathbf{X}^{(2)}) &= \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 9 & 24 \end{bmatrix} \end{aligned}$$

(i)

$$\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

(j)

$$\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}) = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}'$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2.33 (a)

$$E[\mathbf{X}^{(1)}] = \boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \quad (\text{b}) \quad \mathbf{A}\boldsymbol{\mu}^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

(c)

$$\text{Cov}(\mathbf{X}^{(1)}) = \boldsymbol{\Sigma}_{11} = \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix}$$

(d)

$$\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}) = \mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & -1 & \frac{1}{2} \\ -1 & 3 & 1 \\ \frac{1}{2} & 1 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 4 \\ 4 & 63 \end{bmatrix}$$

(e)

$$E[\mathbf{X}^{(2)}] = \boldsymbol{\mu}^{(2)} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad (\text{f}) \quad \mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(g)

$$\text{Cov}(\mathbf{X}^{(2)}) = \boldsymbol{\Sigma}_{22} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(h)

$$\text{Cov}(\mathbf{B}\mathbf{X}^{(2)}) = \mathbf{B}\boldsymbol{\Sigma}_{22}\mathbf{B}' = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 6 \end{bmatrix}$$

(i)

$$\text{Cov}(X^{(1)}, X^{(2)}) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$$

(j)

$$\text{Cov}(AX^{(1)}, BX^{(2)}) = A\Sigma_{12}B'$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -4.5 & 4.5 \end{bmatrix}$$

$$2.34 \quad \underline{\underline{b}}' \underline{\underline{b}} = 4 + 1 + 16 + 0 = 21, \quad \underline{\underline{d}}' \underline{\underline{d}} = 15 \quad \text{and} \quad \underline{\underline{b}}' \underline{\underline{d}} = -2 - 3 - 8 + 0 = -13$$

$$(\underline{\underline{b}}' \underline{\underline{d}})^2 = 169 \leq 21(15) = 315$$

$$2.35 \quad \underline{\underline{b}}' \underline{\underline{d}} = -4 + 3 = -1$$

$$\underline{\underline{b}}' B \underline{\underline{b}} = [-4, 3] \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} -4 \\ 3 \end{bmatrix} = [-14 \quad 23] \begin{bmatrix} -4 \\ 3 \end{bmatrix} = 125$$

$$\underline{\underline{d}}' B^{-1} \underline{\underline{d}} = [1, 1] \begin{bmatrix} 5/6 & 2/6 \\ 2/6 & 2/6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 11/6$$

$$\text{so} \quad 1 = (\underline{\underline{b}}' \underline{\underline{d}})^2 \leq 125 (11/6) = 229.17$$

$$2.36 \quad 4x_1^2 + 4x_2^2 + 6x_1x_2 = \underline{\underline{x}}' A \underline{\underline{x}} \quad \text{where} \quad A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

$(4 - \lambda)^2 - 3^2 = 0$ gives $\lambda_1 = 7, \lambda_2 = 1$. Hence the maximum is 7 and the minimum is 1.

$$2.37 \quad \text{From (2-51),} \quad \max_{\underline{\underline{x}}' \underline{\underline{x}}=1} \underline{\underline{x}}' A \underline{\underline{x}} = \max_{\underline{\underline{x}} \neq \underline{\underline{0}}} \frac{\underline{\underline{x}}' A \underline{\underline{x}}}{\underline{\underline{x}}' \underline{\underline{x}}} = \lambda_1$$

where λ_1 is the largest eigenvalue of A . For A given in Exercise 2.6, we have from Exercise 2.7, $\lambda_1 = 10$ and

$\underline{\underline{e}}_1' = [.894, -.447]$. Therefore $\max_{\underline{\underline{x}}' \underline{\underline{x}}=1} \underline{\underline{x}}' A \underline{\underline{x}} = 10$ and this

maximum is attained for $\underline{\underline{x}} = \underline{\underline{e}}_1$.

2.38

Using computer, $\lambda_1 = 18, \lambda_2 = 9, \lambda_3 = 9$. Hence the maximum is 18 and the minimum is 9.

$$2.41 \text{ (a)} \quad E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$(b) \quad \text{Cov}(\mathbf{AX}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_x\mathbf{A}' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

(c) All pairs of linear combinations have zero covariances.

$$2.42 \text{ (a)} \quad E(\mathbf{AX}) = \mathbf{A}E(\mathbf{X}) = \mathbf{A}\mu_x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$(b) \quad \text{Cov}(\mathbf{AX}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}' = \mathbf{A}\Sigma_x\mathbf{A}' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 24 \end{bmatrix}$$

(c) All pairs of linear combinations have zero covariances.