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Solutions — Chapter 2

2.1.1. *Commutativity of Addition*:

$$(x+\mathrm{i}\,y)+(u+\mathrm{i}\,v)=(x+u)+\mathrm{i}\,(y+v)=(u+\mathrm{i}\,v)+(x+\mathrm{i}\,y).$$

Associativity of Addition:

$$\begin{aligned}(x+\mathrm{i}\,y)+\big[\,(u+\mathrm{i}\,v)+(p+\mathrm{i}\,q)\,\big]&=(x+\mathrm{i}\,y)+\big[\,(u+p)+\mathrm{i}\,(v+q)\,\big]\\&=(x+u+p)+\mathrm{i}\,(y+v+q)\\&=\big[\,(x+u)+\mathrm{i}\,(y+v)\,\big]+(p+\mathrm{i}\,q)=\big[\,(x+\mathrm{i}\,y)+(u+\mathrm{i}\,v)\,\big]+(p+\mathrm{i}\,q).\end{aligned}$$

Additive Identity: $\mathbf{0}=0=0+\mathrm{i}\,0$ and

$$(x+\mathrm{i}\,y)+0=x+\mathrm{i}\,y=0+(x+\mathrm{i}\,y).$$

Additive Inverse: $-(x+\mathrm{i}\,y)=(-x)+\mathrm{i}\,(-y)$ and

$$(x+\mathrm{i}\,y)+\big[\,(-x)+\mathrm{i}\,(-y)\,\big]=0=\big[\,(-x)+\mathrm{i}\,(-y)\,\big]+(x+\mathrm{i}\,y).$$

Distributivity:

$$(c+d)\,(x+\mathrm{i}\,y)=(c+d)\,x+\mathrm{i}\,(c+d)\,y=(c\,x+d\,x)+\mathrm{i}\,(c\,y+d\,y)=c\,(x+\mathrm{i}\,y)+d\,(x+\mathrm{i}\,y),$$
$$c[\,(x+\mathrm{i}\,y)+(u+\mathrm{i}\,v)\,]=c\,(x+u)+(y+v)=(c\,x+c\,u)+\mathrm{i}\,(c\,y+c\,v)=c\,(x+\mathrm{i}\,y)+c\,(u+\mathrm{i}\,v).$$

Associativity of Scalar Multiplication:

$$c\,[\,d\,(x+\mathrm{i}\,y)\,]=c\,[\,(d\,x)+\mathrm{i}\,(d\,y)\,]=(c\,d\,x)+\mathrm{i}\,(c\,d\,y)=(c\,d)\,(x+\mathrm{i}\,y).$$

Unit for Scalar Multiplication: $1\,(x+\mathrm{i}\,y)=(1\,x)+\mathrm{i}\,(1\,y)=x+\mathrm{i}\,y$.

Note: Identifying the complex number $x+\mathrm{i}\,y$ with the vector $(\,x,y\,)^T\in\mathbb{R}^2$ respects the operations of vector addition and scalar multiplication, and so we are in effect reproving that $\mathbb{R}^2$ is a vector space.

2.1.2. *Commutativity of Addition*:

$$(x_1,y_1)+(x_2,y_2)=(x_1\,x_2,y_1\,y_2)=(x_2,y_2)+(x_1,y_1).$$

Associativity of Addition:

$$(x_1,y_1)+\big[\,(x_2,y_2)+(x_3,y_3)\,\big]=(x_1\,x_2\,x_3,y_1\,y_2\,y_3)=\big[\,(x_1,y_1)+(x_2,y_2)\,\big]+(x_3,y_3).$$

Additive Identity: $\mathbf{0}=(1,1)$, and

$$(x,y)+(1,1)=(x,y)=(1,1)+(x,y).$$

Additive Inverse:

$$-(x,y)=\left(\frac{1}{x},\frac{1}{y}\right)\qquad\text{and}\qquad(x,y)+\big[-(x,y)\big]=(1,1)=\big[-(x,y)\big]+(x,y).$$

Distributivity:

$$\begin{aligned}(c+d)\,(x,y)&=(x^{c+d},y^{c+d})=(x^c\,x^d,y^c\,y^d)=(x^c,y^c)+(x^d,y^d)=c\,(x,y)+d\,(x,y)\\c\big[\,(x_1,y_1)+(x_2,y_2)\,\big]&=((x_1\,x_2)^c,(y_1\,y_2)^c)=(x_1^c\,x_2^c,y_1^c\,y_2^c)\\&=(x_1^c,y_1^c)+(x_2^c,y_2^c)=c\,(x_1,y_1)+c\,(x_2,y_2).\end{aligned}$$

Associativity of Scalar Multiplication:

$$c\,(d\,(x,y))=c\,(x^d,y^d)=(x^{c\,d},y^{c\,d})=(c\,d)\,(x,y).$$

Unit for Scalar Multiplication: $1\,(x,y)=(x,y)$.

Note: We can uniquely identify a point $(x,y) \in Q$ with the vector $(\log x, \log y)^T \in \mathbb{R}^2$. Then the indicated operations agree with standard vector addition and scalar multiplication in $\mathbb{R}^2$, and so Q is just a disguised version of $\mathbb{R}^2$.

♢ 2.1.3. We denote a typical function in $\mathcal{F}(S)$ by $f(x)$ for $x \in S$.

Commutativity of Addition:

$$(f+g)(x) = f(x) + g(x) = (f+g)(x).$$

Associativity of Addition:

$$[f+(g+h)](x) = f(x)+(g+h)(x) = f(x)+g(x)+h(x) = (f+g)(x)+h(x) = [(f+g)+h](x).$$

Additive Identity: $0(x) = 0$ for all x, and $(f+0)(x) = f(x) = (0+f)(x)$.

Additive Inverse: $(-f)(x) = -f(x)$ and

$$[f+(-f)](x) = f(x) + (-f)(x) = 0 = (-f)(x) + f(x) = [(-f)+f](x).$$

Distributivity:

$$[(c+d)f](x) = (c+d)f(x) = c f(x) + d f(x) = (c f)(x) + (d f)(x),$$
$$[c(f+g)](x) = c f(x) + c g(x) = (c f)(x) + (c g)(x).$$

Associativity of Scalar Multiplication:

$$[c(d f)](x) = c d f(x) = [(c d) f](x).$$

Unit for Scalar Multiplication: $(1 f)(x) = f(x)$.

2.1.4. (a) $(1,1,1,1)^T$, $(1,-1,1,-1)^T$, $(1,1,1,1)^T$, $(1,-1,1,-1)^T$. (b) Obviously not.

2.1.5. One example is $f(x) \equiv 0$ and $g(x) = x^3 - x$.

2.1.6. (a) $f(x) = -4x + 3$; (b) $f(x) = -2x^2 - x + 1$.

2.1.7.

(a) $\begin{pmatrix} x-y \\ xy \end{pmatrix}$, $\begin{pmatrix} e^x \\ \cos y \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, which is a constant function.

(b) Their sum is $\begin{pmatrix} x-y+e^x+1 \\ xy+\cos y+3 \end{pmatrix}$. Multiplied by -5 is $\begin{pmatrix} -5x+5y-5e^x-5 \\ -5xy-5\cos y-15 \end{pmatrix}$.

(c) The zero element is the constant function $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

♢ 2.1.8. This is the same as the space of functions $\mathcal{F}(\mathbb{R}^2, \mathbb{R}^2)$. Explicitly:

Commutativity of Addition:

$$\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix} = \begin{pmatrix} v_1(x,y)+w_1(x,y) \\ v_2(x,y)+w_2(x,y) \end{pmatrix} = \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix} + \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

Associativity of Addition:

$$\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} + \left[\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix} \right] = \begin{pmatrix} u_1(x,y)+v_1(x,y)+w_1(x,y) \\ u_2(x,y)+v_2(x,y)+w_2(x,y) \end{pmatrix}$$
$$= \left[\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} + \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} \right] + \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix}.$$

Additive Identity: $\mathbf{0} = (0,0)$ for all x, y, and

$$\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \mathbf{0} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \mathbf{0} + \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

Additive Inverse: $-\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} -v_1(x,y) \\ -v_2(x,y) \end{pmatrix}$, and

$$\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \begin{pmatrix} -v_1(x,y) \\ -v_2(x,y) \end{pmatrix} = \mathbf{0} = \begin{pmatrix} -v_1(x,y) \\ -v_2(x,y) \end{pmatrix} + \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

Distributivity:

$$(c+d)\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} (c+d)\,v_1(x,y) \\ (c+d)\,v_2(x,y) \end{pmatrix} = c\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + d\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix},$$

$$c\left[\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + \begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix}\right] = \begin{pmatrix} c\,v_1(x,y) + c\,w_1(x,y) \\ c\,v_2(x,y) + c\,w_2(x,y) \end{pmatrix} = c\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} + c\begin{pmatrix} w_1(x,y) \\ w_2(x,y) \end{pmatrix}.$$

Associativity of Scalar Multiplication:

$$c\left[d\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}\right] = \begin{pmatrix} c\,d\,v_1(x,y) \\ c\,d\,v_2(x,y) \end{pmatrix} = (c\,d)\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

Unit for Scalar Multiplication:

$$1\begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix}.$$

♡ 2.1.9. We identify each sample value with the matrix entry $m_{ij} = f(i\,h, j\,k)$. In this way, every sampled function corresponds to a uniquely determined $m \times n$ matrix and conversely. Addition of sample functions, $(f+g)(i\,h, j\,k) = f(i\,h, j\,k) + g(i\,h, j\,k)$ corresponds to matrix addition, $m_{ij} + n_{ij}$, while scalar multiplication of sample functions, $c\,f(i\,h, j\,k)$, corresponds to scalar multiplication of matrices, $c\,m_{ij}$.

2.1.10. $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots)$, $c\,\mathbf{a} = (c\,a_1, c\,a_2, c\,a_3, \ldots)$. Explicity verification of the vector space properties is straightforward. An alternative, smarter strategy is to identify $\mathbb{R}^\infty$ as the space of functions $f: \mathbb{N} \to \mathbb{R}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers and we identify the function f with its sample vector $\mathbf{f} = (f(1), f(2), \ldots)$.

2.1.11. (*i*) $\mathbf{v} + (-1)\mathbf{v} = 1\,\mathbf{v} + (-1)\mathbf{v} = \big(1 + (-1)\big)\mathbf{v} = 0\,\mathbf{v} = \mathbf{0}$.

(*j*) Let $\mathbf{z} = c\,\mathbf{0}$. Then $\mathbf{z} + \mathbf{z} = c(\mathbf{0} + \mathbf{0}) = c\,\mathbf{0} = \mathbf{z}$, and so, as in the proof of (*h*), $\mathbf{z} = \mathbf{0}$.

(*k*) Suppose $c \neq \mathbf{0}$. Then $\mathbf{v} = 1\,\mathbf{v} = \left(\dfrac{1}{c}\cdot c\right)\mathbf{v} = \dfrac{1}{c}\,(c\,\mathbf{v}) = \dfrac{1}{c}\,\mathbf{0} = \mathbf{0}$.

◇ 2.1.12. If $\mathbf{0}$ and $\widetilde{\mathbf{0}}$ both satisfy axiom (*c*), then $\mathbf{0} = \widetilde{\mathbf{0}} + \mathbf{0} = \mathbf{0} + \widetilde{\mathbf{0}} = \widetilde{\mathbf{0}}$.

◇ 2.1.13. *Commutativity of Addition*:

$$(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) = (\mathbf{v} + \widehat{\mathbf{v}}, \mathbf{w} + \widehat{\mathbf{w}}) = (\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) + (\mathbf{v}, \mathbf{w}).$$

Associativity of Addition:

$$(\mathbf{v}, \mathbf{w}) + \big[(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}) + (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}})\big] = (\mathbf{v} + \widehat{\mathbf{v}} + \widetilde{\mathbf{v}}, \mathbf{w} + \widehat{\mathbf{w}} + \widetilde{\mathbf{w}}) = \big[(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\big] + (\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}).$$

Additive Identity: the zero element is $(\mathbf{0}, \mathbf{0})$, and

$$(\mathbf{v}, \mathbf{w}) + (\mathbf{0}, \mathbf{0}) = (\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0}) + (\mathbf{v}, \mathbf{w}).$$

Additive Inverse: $-(\mathbf{v}, \mathbf{w}) = (-\mathbf{v}, -\mathbf{w})$ and

$$(\mathbf{v}, \mathbf{w}) + (-\mathbf{v}, -\mathbf{w}) = (\mathbf{0}, \mathbf{0}) = (-\mathbf{v}, -\mathbf{w}) + (\mathbf{v}, \mathbf{w}).$$

Distributivity:

$$(c+d)(\mathbf{v}, \mathbf{w}) = ((c+d)\mathbf{v}, (c+d)\mathbf{w}) = c(\mathbf{v}, \mathbf{w}) + d(\mathbf{v}, \mathbf{w}),$$

$$c\big[(\mathbf{v}, \mathbf{w}) + (\widehat{\mathbf{v}}, \widehat{\mathbf{w}})\big] = (c\mathbf{v} + c\widehat{\mathbf{v}}, c\mathbf{v} + c\widehat{\mathbf{w}}) = c(\mathbf{v}, \mathbf{w}) + c(\widehat{\mathbf{v}}, \widehat{\mathbf{w}}).$$

Associativity of Scalar Multiplication:

$$c(d(\mathbf{v}, \mathbf{w})) = (c\,d\,\mathbf{v}, c\,d\,\mathbf{w}) = (c\,d)(\mathbf{v}, \mathbf{w}).$$

Unit for Scalar Multiplication: $1(\mathbf{v}, \mathbf{w}) = (1\,\mathbf{v}, 1\,\mathbf{w}) = (\mathbf{v}, \mathbf{w})$.

2.1.14. Here $V = \mathrm{C}^0$ while $W = \mathbb{R}$, and so the indicated pairs belong to the Cartesian product vector space $\mathrm{C}^0 \times \mathbb{R}$. The zero element is the pair $\mathbf{0} = (0, 0)$ where the first 0 denotes the identically zero function, while the second 0 denotes the real number zero. The laws of vector addition and scalar multiplication are

$$(f(x), a) + (g(x), b) = (f(x) + g(x), a + b), \qquad c(f(x), a) = (c\,f(x), c\,a).$$

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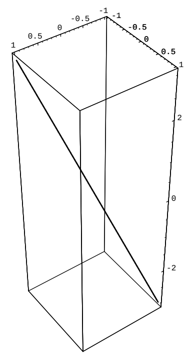
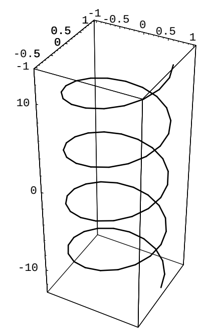
2.2.1.
(*a*) If $\mathbf{v} = (\,x, y, z\,)^T$ satisfies $x - y + 4z = 0$ and $\widetilde{\mathbf{v}} = (\,\widetilde{x}, \widetilde{y}, \widetilde{z}\,)^T$ also satisfies $\widetilde{x} - \widetilde{y} + 4\widetilde{z} = 0$, so does $\mathbf{v} + \widetilde{\mathbf{v}} = (\,x + \widetilde{x}, y + \widetilde{y}, z + \widetilde{z}\,)^T$ since $(x + \widetilde{x}) - (y + \widetilde{y}) + 4(z + \widetilde{z}) = (x - y + 4z) + (\widetilde{x} - \widetilde{y} + 4\widetilde{z}) = 0$, as does $c\mathbf{v} = (\,cx, cy, cz\,)^T$ since $(cx) - (cy) + 4(cz) = c(x - y + 4z) = 0$.
(*b*) For instance, the zero vector $\mathbf{0} = (\,0, 0, 0\,)^T$ does not satisfy the equation.

2.2.2. (*b*,*c*,*d*,*g*,*i*) are subspaces; the rest are not. Case (*j*) consists of the 3 coordinate axes and the line $x = y = z$.

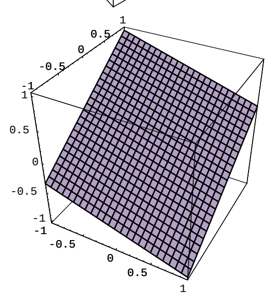
2.2.3. (*a*) Subspace:



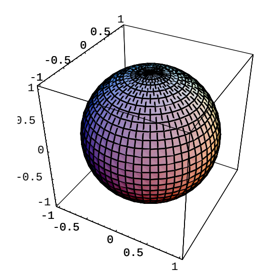
(*b*) Not a subspace:



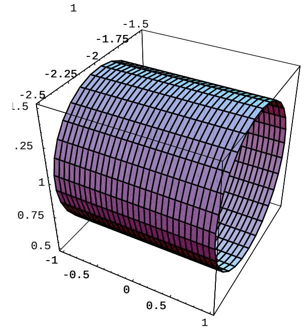
(*c*) Subspace:



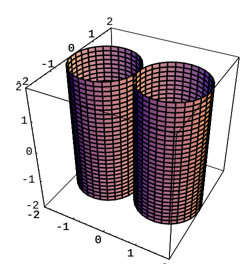
(*d*) Not a subspace:



(*e*) Not a subspace:



(*f*) Even though the cylinders are not subspaces, their intersection is the z axis, which is a subspace:



2.2.4. Any vector of the form $a\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} a + 2b \\ 2a - c \\ -a + b + 3c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will belong to W. The coefficient matrix $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ -1 & 1 & 3 \end{pmatrix}$ is nonsingular, and so for any

$\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ we can arrange suitable values of a, b, c by solving the linear system. Thus, every vector in $\mathbb{R}^3$ belongs to W and so $W = \mathbb{R}^3$.

2.2.5. False, with two exceptions: $[0,0] = \{0\}$ and $(-\infty, \infty) = \mathbb{R}$.

2.2.6.
(a) Yes. For instance, the set $S = \{(x,0\} \cup \{(0,y)\}$ consisting of the coordinate axes has the required property, but is not a subspace. More generally, any (finite) collection of 2 or more lines going through the origin satisfies the property, but is not a subspace.
(b) For example, $S = \{(x,y) \mid x, y \geq 0\}$ — the positive quadrant.

2.2.7. (a,c,d) are subspaces; (b,e) are not.

2.2.8. Since $\mathbf{x} = \mathbf{0}$ must belong to the subspace, this implies $\mathbf{b} = A\mathbf{0} = \mathbf{0}$. For a homogeneous system, if $\mathbf{x}, \mathbf{y}$ are solutions, so $A\mathbf{x} = \mathbf{0} = A\mathbf{y}$, so are $\mathbf{x}+\mathbf{y}$ since $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$, as is $c\mathbf{x}$ since $A(c\mathbf{x}) = cA\mathbf{x} = \mathbf{0}$.

2.2.9. L and M are strictly lower triangular if $l_{ij} = 0 = m_{ij}$ whenever $i \leq j$. Then $N = L + M$ is strictly lower triangular since $n_{ij} = l_{ij} + m_{ij} = 0$ whenever $i \leq j$, as is $K = cL$ since $k_{ij} = c\,l_{ij} = 0$ whenever $i \leq j$.

♢ 2.2.10. Note $\operatorname{tr}(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \operatorname{tr} A + \operatorname{tr} B$ and $\operatorname{tr}(cA) = \sum_{i=1}^{n} c\,a_{ii} = c \sum_{i=1}^{n} a_{ii} = c \operatorname{tr} A$. Thus, if $\operatorname{tr} A = \operatorname{tr} B = 0$, then $\operatorname{tr}(A+B) = 0 = \operatorname{tr}(cA)$, proving closure.

2.2.11.
(a) No. The zero matrix is not an element.
(b) No if $n \geq 2$. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy $\det A = 0 = \det B$, but $\det(A+B) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$, so $A + B$ does not belong to the set.

2.2.12. (d,f,g,h) are subspaces; the rest are not.

2.2.13. (a) Vector space; (b) not a vector space: $(0,0)$ does not belong; (c) vector space; (d) vector space; (e) not a vector space: If f is non-negative, then $-1\,f = -f$ is not (unless $f \equiv 0$); (f) vector space; (g) vector space; (h) vector space.

2.2.14. If $f(1) = 0 = g(1)$, then $(f+g)(1) = 0$ and $(cf)(1) = 0$, so both $f+g$ and cf belong to the subspace. The zero function does not satisfy $f0) = 1$. For a subspace, a can be anything, while $b = 0$.

2.2.15. All cases except (e,g) are subspaces. In (g), $|\,x\,|$ is not in C^1.

2.2.16. (a) Subspace; (b) subspace; (c) Not a subspace: the zero function does not satisfy the condition; (d) Not a subspace: if $f(0) = 0$, $f(1) = 1$, and $g(0) = 1$, $g(1) = 0$, then f and g are in the set, but $f + g$ is not; (e) subspace; (f) Not a subspace: the zero function does not satisfy the condition; (g) subspace; (h) subspace; (i) Not a subspace: the zero function does not satisfy the condition.

2.2.17. If $u'' = xu$, $v'' = xv$, are solutions, and c, d constants, then $(cu + dv)'' = cu'' + dv'' = cxu + dxv = x(cu + dv)$, and hence $cu + dv$ is also a solution.

2.2.18. For instance, the zero function $u(x) \equiv 0$ is not a solution.

2.2.19.
(a) It is a subspace of the space of all functions $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$, which is a particular instance of Example 2.7. Note that $\mathbf{f}(t) = (f_1(t), f_2(t))^T$ is continuously differentiable if and

only if its component functions $f_1(t)$ and $f_2(t)$ are. Thus, if $\mathbf{f}(t) = (\, f_1(t), f_2(t)\,)^T$ and $\mathbf{g}(t) = (\, g_1(t), g_2(t)\,)^T$ are continuously differentiable, so are

$$(\mathbf{f}+\mathbf{g})(t) = (\, f_1(t)+g_1(t), f_2(t)+g_2(t)\,)^T \text{ and } (c\,\mathbf{f})(t) = (\, c f_1(t), c f_2(t)\,)^T.$$

(*b*) Yes: if $\mathbf{f}(0) = \mathbf{0} = \mathbf{g}(0)$, then $(c\,\mathbf{f} + d\,\mathbf{g})(0) = \mathbf{0}$ for any $c, d \in \mathbb{R}$.

2.2.20. $\nabla \cdot (c\,\mathbf{v} + d\,\mathbf{w}) = c\,\nabla\cdot\mathbf{v} + d\,\nabla\cdot\mathbf{w} = 0$ whenever $\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{w} = 0$ and $c, d, \in \mathbb{R}$.

2.2.21. Yes. The sum of two convergent sequences is convergent, as is any constant multiple of a convergent sequence.

2.2.22.

(*a*) If $\mathbf{v}, \mathbf{w} \in W \cap Z$, then $\mathbf{v}, \mathbf{w} \in W$, so $c\,\mathbf{v} + d\,\mathbf{w} \in W$ because W is a subspace, and $\mathbf{v}, \mathbf{w} \in Z$, so $c\,\mathbf{v} + d\,\mathbf{w} \in Z$ because Z is a subspace, hence $c\,\mathbf{v} + d\,\mathbf{w} \in W \cap Z$.

(*b*) If $\mathbf{w} + \mathbf{z}, \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}} \in W + Z$ then $c\,(\mathbf{w} + \mathbf{z}) + d\,(\widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}) = (c\,\mathbf{w} + d\,\widetilde{\mathbf{w}}) + (c\,\mathbf{z} + d\,\widetilde{\mathbf{z}}) \in W + Z$, since it is the sum of an element of W and an element of Z.

(*c*) Given any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, then $\mathbf{w}, \mathbf{z} \in W \cup Z$. Thus, if $W \cup Z$ is a subspace, the sum $\mathbf{w} + \mathbf{z} \in W \cup Z$. Thus, either $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} \in W$ or $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{z}} \in Z$. In the first case $\mathbf{z} = \widetilde{\mathbf{w}} - \mathbf{w} \in W$, while in the second $\mathbf{w} = \widetilde{\mathbf{z}} - \mathbf{z} \in Z$. We conclude that for any $\mathbf{w} \in W$ and $\mathbf{z} \in Z$, either $\mathbf{w} \in Z$ or $\mathbf{z} \in W$. Suppose $W \not\subset Z$. Then we can find $\mathbf{w} \in W \setminus Z$, and so for any $\mathbf{z} \in Z$, we must have $\mathbf{z} \in W$, which proves $Z \subset W$.

♢ 2.2.23. If $\mathbf{v}, \mathbf{w} \in \bigcap W_i$, then $\mathbf{v}, \mathbf{w} \in W_i$ for each i and so $c\,\mathbf{v} + d\,\mathbf{w} \in W_i$ for any $c, d \in \mathbb{R}$ because W_i is a subspace. Since this holds for all i, we conclude that $c\,\mathbf{v} + d\,\mathbf{w} \in \bigcap W_i$.

♡ 2.2.24.

(*a*) They clearly only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix}$ can be written as a sum of vectors on the two axes.

(*b*) Since the only common solution to $x = y$ and $x = 3y$ is $x = y = 0$, the lines only intersect at the origin. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 3b \\ b \end{pmatrix}$, where $a = -\frac{1}{2}x + \frac{3}{2}y$, $b = \frac{1}{2}x - \frac{1}{2}y$, can be written as a sum of vectors on each line.

(*c*) A vector $\mathbf{v} = (\, a, 2a, 3a\,)^T$ in the line belongs to the plane if and only if $a + 2(2a) + 3(3a) = 14a = 0$, so $a = 0$ and the only common element is $\mathbf{v} = \mathbf{0}$. Moreover, every $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{14}\begin{pmatrix} x + 2y + 3z \\ 2(x + 2y + 3z) \\ 3(x + 2y + 3z) \end{pmatrix} + \frac{1}{14}\begin{pmatrix} 13x - 2y - 3z \\ -2x + 10y - 6z \\ -3x - 6y + 5z \end{pmatrix}$ can be written as a sum of a vector in the line and a vector in the plane.

(*d*) If $\mathbf{w} + \mathbf{z} = \widetilde{\mathbf{w}} + \widetilde{\mathbf{z}}$, then $\mathbf{w} - \widetilde{\mathbf{w}} = \widetilde{\mathbf{z}} - \mathbf{z}$. The left hand side belongs to W, while the right hand side belongs to Z, and so, by the first assumption, they must both be equal to $\mathbf{0}$. Therefore, $\mathbf{w} = \widetilde{\mathbf{w}}$, $\mathbf{z} = \widetilde{\mathbf{z}}$.

2.2.25.

(*a*) $(\mathbf{v}, \mathbf{w}) \in V_0 \cap W_0$ if and only if $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{0})$ and $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{w})$, which means $\mathbf{v} = \mathbf{0}, \mathbf{w} = \mathbf{0}$, and hence $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0})$ is the only element of the intersection. Moreover, we can write any element $(\mathbf{v}, \mathbf{w}) = (\mathbf{v}, \mathbf{0}) + (\mathbf{0}, \mathbf{w})$.

(*b*) $(\mathbf{v}, \mathbf{w}) \in D \cap A$ if and only if $\mathbf{v} = \mathbf{w}$ and $\mathbf{v} = -\mathbf{w}$, hence $(\mathbf{v}, \mathbf{w}) = (\mathbf{0}, \mathbf{0})$. Moreover, we can write $(\mathbf{v}, \mathbf{w}) = (\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}, \frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}) + (\frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{w}, -\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w})$ as the sum of an element of D and an element of A.

2.2.26.

(*a*) If $f(-x) = f(x)$, $\widetilde{f}(-x) = \widetilde{f}(x)$, then $(c f + d\widetilde{f})(-x) = c f(-x) + d\widetilde{f}(-x) = c f(x) + d\widetilde{f}(x) = (c f + d\widetilde{f})(x)$ for any $c, d, \in \mathbb{R}$, and hence it is a subspace.

(*b*) If $g(-x) = -g(x)$, $\widetilde{g}(-x) = -\widetilde{g}(x)$, then $(cg + d\widetilde{g})(-x) = cg(-x) + d\widetilde{g}(-x) = -cg(x) - d\widetilde{g}(x) = -(cg + d\widetilde{g})(x)$, proving it is a subspace. If $f(x)$ is both even and odd, then $f(x) = f(-x) = -f(x)$ and so $f(x) \equiv 0$ for all x. Moreover, we can write any function $h(x) = f(x) + g(x)$ as a sum of an even function $f(x) = \frac{1}{2}\left[\,h(x) + h(-x)\,\right]$ and an odd function $g(x) = \frac{1}{2}\left[\,h(x) - h(-x)\,\right]$.

(*c*) This follows from part (*b*), and the uniqueness follows from Exercise 2.2.24(*d*).

2.2.27. If $A = A^T$ and $A = -A^T$ is both symmetric and skew-symmetric, then $A = \mathrm{O}$. Given any square matrix, write $A = S + J$ where $S = \frac{1}{2}\left(A + A^T\right)$ is symmetric and $J = \frac{1}{2}\left(A - A^T\right)$ is skew-symmetric. This verifies the two conditions for complementary subspaces. Uniqueness of the decomposition $A = S + J$ follows from Exercise 2.2.24(*d*).

♦ 2.2.28.

(*a*) By induction, we can show that

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-1/x} = Q_n(x)\,\frac{e^{-1/x}}{x^n},$$

where $P_n(y)$ and $Q_n(x) = x^n P_n(1/x)$ are certain polynomials of degree n. Thus,

$$\lim_{x\to 0} f^{(n)}(x) = \lim_{x\to 0} Q_n(x)\,\frac{e^{-1/x}}{x^n} = Q_n(0) \lim_{y\to\infty} y^n\, e^{-y} = 0,$$

because the exponential e^{-y} goes to zero faster than any power of y goes to ∞.

(*b*) The Taylor series at $a = 0$ is $0 + 0x + 0x^2 + \cdots \equiv 0$, which converges to the zero function, not to $e^{-1/x}$.

2.2.29.

(*a*) The Taylor series is the geometric series $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$.

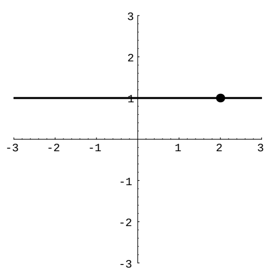
(*b*) The ratio test can be used to prove that the series converges precisely when $|\,x\,| < 1$.

(*c*) Convergence of the Taylor series to $f(x)$ for x near 0 suffices to prove analyticity of the function at $x = 0$.

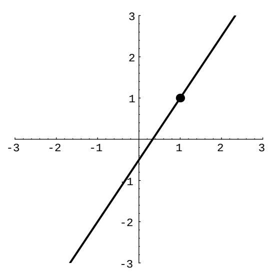
♡ 2.2.30.

(*a*) If $\mathbf{v}+\mathbf{a}, \mathbf{w}+\mathbf{a} \in A$, then $(\mathbf{v}+\mathbf{a})+(\mathbf{w}+\mathbf{a}) = (\mathbf{v}+\mathbf{w}+\mathbf{a})+\mathbf{a} \in A$ requires $\mathbf{v}+\mathbf{w}+\mathbf{a} = \mathbf{u} \in V$, and hence $\mathbf{a} = \mathbf{u} - \mathbf{v} - \mathbf{w} \in A$.

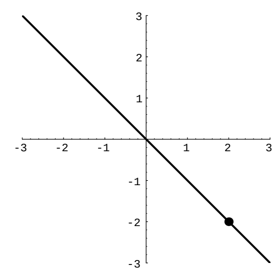
(*b*) (*i*)



(*ii*)



(*iii*)



(*c*) Every subspace $V \subset \mathbb{R}^2$ is either a point (the origin), or a line through the origin, or all of $\mathbb{R}^2$. Thus, the corresponding affine subspaces are the point $\{\mathbf{a}\}$; a line through $\mathbf{a}$, or all of $\mathbb{R}^2$ since in this case $\mathbf{a} \in V = \mathbb{R}^2$.

(*d*) Every vector in the plane can be written as $(x, y, z)^T = (\widetilde{x}, \widetilde{y}, \widetilde{z})^T + (1, 0, 0)^T$ where $(\widetilde{x}, \widetilde{y}, \widetilde{z})^T$ is an arbitrary vector in the subspace defined by $\widetilde{x} - 2\widetilde{y} + 3\widetilde{x} = 0$.

(*e*) Every such polynomial can be written as $p(x) = q(x) + 1$ where $q(x)$ is any element of the subspace of polynomials that satisfy $q(1) = 0$.

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2.3.1. $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$

2.3.2. $\begin{pmatrix} -3 \\ 7 \\ 6 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -3 \\ -2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} -2 \\ 6 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \\ 6 \\ -7 \end{pmatrix}.$

2.3.3.

(a) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$;

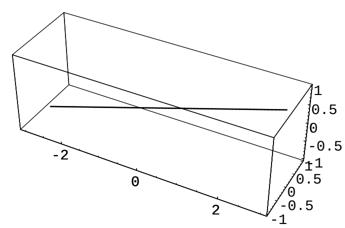
(b) Yes, since $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{3}{10} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \frac{7}{10} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \frac{4}{10} \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$;

(c) No, since the vector equation $\begin{pmatrix} 3 \\ 0 \\ -1 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ does not have a solution.

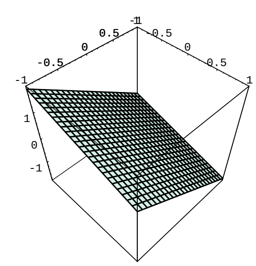
2.3.4. Cases (b), (c), (e) span $\mathbb{R}^2$.

2.3.5.

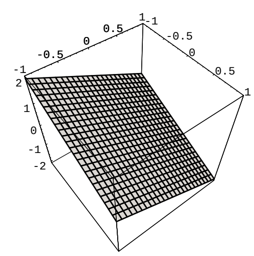
(a) The line $(3t, 0, t)^T$:



(b) The plane $z = -\frac{3}{5}x - \frac{6}{5}y$:



(c) The plane $z = -x - y$:



2.3.6. They are the same. Indeed, since $\mathbf{v}_1 = \mathbf{u}_1 + 2\mathbf{u}_2$, $\mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$, every vector $\mathbf{v} \in V$ can be written as a linear combination $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = (c_1 + c_2)\mathbf{u}_1 + (2c_1 + c_2)\mathbf{u}_2$ and hence belongs to U. Conversely, since $\mathbf{u}_1 = -\mathbf{v}_1 + 2\mathbf{v}_2$, $\mathbf{u}_2 = \mathbf{v}_1 - \mathbf{v}_2$, every vector $\mathbf{u} \in U$ can be written as a linear combination $\mathbf{u} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 = (-c_1 + c_2)\mathbf{v}_1 + (2c_1 - c_2)\mathbf{v}_2$, and hence belongs to U.

2.3.7. (a) Every symmetric matrix has the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

(*b*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$

2.3.8.
(*a*) They span $\mathcal{P}^{(2)}$ since $ax^2+bx+c = \frac{1}{2}(a-2b+c)(x^2+1)+\frac{1}{2}(a-c)(x^2-1)+b(x^2+x+1)$.
(*b*) They span $\mathcal{P}^{(3)}$ since $ax^3+bx^2+cx+d = a(x^3-1)+b(x^2+1)+c(x-1)+(a-b+c+d)1$.
(*c*) They do not span $\mathcal{P}^{(3)}$ since $ax^3+bx^2+cx+d = c_1x^3+c_2(x^2+1)+c_3(x^2-x)+c_4(x+1)$ cannot be solved when $b+c-d \neq 0$.

2.3.9. (*a*) Yes. (*b*) No. (*c*) No. (*d*) Yes: $\cos^2 x = 1-\sin^2 x$. (*e*) No. (*f*) No.

2.3.10. (*a*) $\sin 3x = \cos\left(3x - \frac{1}{2}\pi\right)$; (*b*) $\cos x - \sin x = \sqrt{2}\,\cos\left(x + \frac{1}{4}\pi\right)$,
(*c*) $3\cos 2x + 4\sin 2x = 5\cos\left(2x - \tan^{-1}\frac{4}{3}\right)$, (*d*) $\cos x \sin x = \frac{1}{2}\sin 2x = \frac{1}{2}\cos\left(2x - \frac{1}{2}\pi\right)$.

2.3.11. (*a*) If u_1 and u_2 are solutions, so is $u = c_1 u_1 + c_2 u_2$ since $u'' - 4u' + 3u = c_1(u_1'' - 4u_1' + 3u_1) + c_2(u_2'' - 4u_2' + 3u_2) = 0$. (*b*) span $\{e^x, e^{3x}\}$; (*c*) 2.

2.3.12. Each is a solution, and the general solution $u(x) = c_1 + c_2\cos x + c_3 \sin x$ is a linear combination of the three independent solutions.

2.3.13. (*a*) e^{2x}; (*b*) $\cos 2x, \sin 2x$; (*c*) $e^{3x}, 1$; (*d*) e^{-x}, e^{-3x}; (*e*) $e^{-x/2}\cos\frac{\sqrt{3}}{2}x$, $e^{-x/2}\sin\frac{\sqrt{3}}{2}x$; (*f*) $e^{5x}, 1, x$; (*g*) $e^{x/\sqrt{2}}\cos\frac{x}{\sqrt{2}}, e^{x/\sqrt{2}}\sin\frac{x}{\sqrt{2}}, e^{-x/\sqrt{2}}\cos\frac{x}{\sqrt{2}}, e^{-x/\sqrt{2}}\sin\frac{x}{\sqrt{2}}$.

2.3.14. (*a*) If u_1 and u_2 are solutions, so is $u = c_1 u_1 + c_2 u_2$ since $u'' + 4u = c_1(u_1'' + 4u_1) + c_2(u_2'' + 4u_2) = 0$, $u(0) = c_1 u_1(0) + c_2 u_2(0) = 0$, $u(\pi) = c_1 u_1(\pi) + c_2 u_2(\pi) = 0$.
(*b*) span $\{\sin 2x\}$

2.3.15. (*a*) $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\,\mathbf{f}_1(x) + \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (*b*) not in the span; (*c*) $\begin{pmatrix} 1-2x \\ -1-x \end{pmatrix} = \mathbf{f}_1(x) - \mathbf{f}_2(x) - \mathbf{f}_3(x)$; (*d*) not in the span; (*e*) $\begin{pmatrix} 2-x \\ 0 \end{pmatrix} = 2\,\mathbf{f}_1(x) - \mathbf{f}_3(x)$.

2.3.16. True, since $\mathbf{0} = 0\,\mathbf{v}_1 + \cdots + 0\,\mathbf{v}_n$.

2.3.17. False. For example, if $\mathbf{z} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then $\mathbf{z} = \mathbf{u} + \mathbf{v}$, but the equation $\mathbf{w} = c_1\,\mathbf{u} + c_2\,\mathbf{v} + c_3\,\mathbf{z} = \begin{pmatrix} c_1 + c_3 \\ c_2 + c_3 \\ 0 \end{pmatrix}$ has no solution.

◇ 2.3.18. By the assumption, any $\mathbf{v} \in V$ can be written as a linear combination

$$\mathbf{v} = c_1\,\mathbf{v}_1 + \cdots + c_m\,\mathbf{v}_m = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_m + 0\,\mathbf{v}_{m+1} + \cdots + 0\,\mathbf{v}_n$$

of the combined collection.

◇ 2.3.19.
(*a*) If $\mathbf{v} = \sum_{j=1}^{m} c_j\,\mathbf{v}_j$ and $\mathbf{v}_j = \sum_{i=1}^{n} a_{ij}\,\mathbf{w}_i$, then $\mathbf{v} = \sum_{i=1}^{n} b_i\,\mathbf{v}_i$ where $b_i = \sum_{j=1}^{m} a_{ij}\,c_j$, or, in vector language, $\mathbf{b} = A\,\mathbf{c}$.
(*b*) Every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and hence, by part (a), a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_m$, which shows that $\mathbf{w}_1, \ldots, \mathbf{w}_m$ also span V.

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♢ 2.3.20.

(a) If $\mathbf{v} = \sum_{i=1}^{m} a_i \mathbf{v}_i$, $\mathbf{w} = \sum_{i=1}^{n} b_i \mathbf{v}_i$, are two finite linear combinations, so is

$$c\mathbf{v} + d\mathbf{w} = \sum_{i=1}^{\max\{m,n\}} (c a_i + d b_i)\mathbf{v}_i \quad \text{where we set } a_i = 0 \text{ if } i > m \text{ and } b_i = 0 \text{ if } i > n.$$

(b) The space $\mathcal{P}^{(\infty)}$ of all polynomials, since every polynomial is a finite linear combination of monomials and vice versa.

2.3.21. (a) Linearly independent; (b) linearly dependent; (c) linearly dependent; (d) linearly independent; (e) linearly dependent; (f) linearly dependent; (g) linearly dependent; (h) linearly independent; (i) linearly independent.

2.3.22. (a) The only solution to the homogeneous linear system

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ -2 \\ 1 \\ -1 \end{pmatrix} = \mathbf{0} \qquad \text{is} \qquad c_1 = c_2 = c_3 = 0.$$

(b) All but the second lie in the span. (c) $a - c + d = 0$.

2.3.23.

(a) The only solution to the homogeneous linear system

$$A\,\mathbf{c} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} = \mathbf{0}$$

with nonsingular coefficient matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ is $\mathbf{c} = \mathbf{0}$.

(b) Since A is nonsingular, the inhomogeneous linear system

$$\mathbf{v} = A\,\mathbf{c} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

has a solution $\mathbf{c} = A^{-1}\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^4$.

(c)
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \tfrac{3}{8} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \tfrac{1}{8} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \tfrac{3}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} - \tfrac{1}{4} \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

2.3.24. (a) Linearly dependent; (b) linearly dependent; (c) linearly independent; (d) linearly dependent; (e) linearly dependent; (f) linearly independent.

2.3.25. False:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \mathrm{O}.$$

2.3.26. False — the zero vector always belongs to the span.

2.3.27. Yes, when it is the zero vector.

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2.3.28. Because $\mathbf{x}, \mathbf{y}$ are linearly independent, $\mathbf{0} = c_1 \mathbf{u} + c_2 \mathbf{v} = (a c_1 + c c_2)\mathbf{x} + (b c_1 + d c_2)\mathbf{y}$ if and only if $a c_1 + c c_2 = 0, b c_1 + d c_2 = 0$. The latter linear system has a nonzero solution $(c_1, c_2) \neq \mathbf{0}$, and so $\mathbf{u}, \mathbf{v}$ are linearly dependent, if and only if the determinant of the coefficient matrix is zero: $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = a d - b c = 0$, proving the result. The full collection $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ is linearly dependent since, for example, $a\mathbf{x} + b\mathbf{y} - \mathbf{u} + 0\mathbf{v} = \mathbf{0}$ is a nontrivial linear combination.

2.3.29. The statement is false. For example, any set containing the zero element that does not span V is linearly dependent.

♢ 2.3.30. (b) If the only solution to $A\mathbf{c} = \mathbf{0}$ is the trivial one $\mathbf{c} = \mathbf{0}$, then the only linear combination which adds up to zero is the trivial one with $c_1 = \cdots = c_k = 0$, proving linear independence. (c) The vector $\mathbf{b}$ lies in the span if and only if $\mathbf{b} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = A\mathbf{c}$ for some $\mathbf{c}$, which implies that the linear system $A\mathbf{c} = \mathbf{b}$ has a solution.

♢ 2.3.31.

(a) Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent,
$$\mathbf{0} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k + 0\mathbf{v}_{k+1} + \cdots + 0\mathbf{v}_n$$
if and only if $c_1 = \cdots = c_k = 0$.

(b) This is false. For example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, are linearly dependent, but the subset consisting of just $\mathbf{v}_1$ is linearly independent.

2.3.32.

(a) They are linearly dependent since $(x^2 - 3) + 2(2 - x) - (x - 1)^2 \equiv 0$.

(b) They do not span $\mathcal{P}^{(2)}$.

2.3.33. (a) Linearly dependent; (b) linearly independent; (c) linearly dependent; (d) linearly independent; (e) linearly dependent; (f) linearly dependent; (g) linearly independent; (h) linearly independent; (i) linearly independent.

2.3.34. When $x > 0$, we have $f(x) - g(x) \equiv 0$, proving linear dependence. On the other hand, if $c_1 f(x) + c_2 g(x) \equiv 0$ for all x, then at, say $x = 1$, we have $c_1 + c_2 = 0$ while at $x = -1$, we must have $-c_1 + c_2 = 0$, and so $c_1 = c_2 = 0$, proving linear independence.

♡ 2.3.35.

(a) $0 = \sum_{i=1}^{k} c_i p_i(x) = \sum_{j=0}^{n} \sum_{i=1}^{k} c_i a_{ij} x^j$ if and only if $\sum_{j=0}^{n} \sum_{i=1}^{k} c_i a_{ij} = 0$, $j = 0, \ldots, n$, or, in matrix notation, $A^T \mathbf{c} = \mathbf{0}$. Thus, the polynomials are linearly independent if and only if the linear system $A^T \mathbf{c} = \mathbf{0}$ has only the trivial solution $\mathbf{c} = \mathbf{0}$ if and only if its $(n+1) \times k$ coefficient matrix has rank $A^T = \operatorname{rank} A = k$.

(b) $q(x) = \sum_{j=0}^{n} b_j x^j = \sum_{i=1}^{k} c_i p_i(x)$ if and only if $A^T \mathbf{c} = \mathbf{b}$.

(c) $A = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 4 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 4 & -1 \end{pmatrix}$ has rank 4 and so they are linearly dependent.

(d) $q(x)$ is not in the span.

♢ 2.3.36. Suppose the linear combination $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \equiv 0$ for all x. Thus, every real x is a root of $p(x)$, but the Fundamental Theorem of Algebra says this is only possible if $p(x)$ is the zero polynomial with coefficients $c_0 = c_1 = \cdots = c_n = 0$.

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♡ 2.3.37.
(*a*) If $c_1 f_1(x) + \cdots + c_n f_n(x) \equiv 0$, then $c_1 f_1(x_i) + \cdots + c_n f_n(x_i) = 0$ at all sample points, and so $c_1 \mathbf{f}_1 + \cdots + c_n \mathbf{f}_n = \mathbf{0}$. Thus, linear dependence of the functions implies linear dependence of their sample vectors.
(*b*) Sampling $f_1(x) = 1$ and $f_2(x) = x^2$ at $-1, 1$ produces the linearly dependent sample vectors $\mathbf{f}_1 = \mathbf{f}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
(*c*) Sampling at $0,\ \frac14\pi,\ \frac12\pi,\ \frac34\pi,\ \pi$, leads to the linearly independent sample vectors
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ \frac{\sqrt2}{2} \\ 0 \\ -\frac{\sqrt2}{2} \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \frac{\sqrt2}{2} \\ 1 \\ \frac{\sqrt2}{2} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

2.3.38.
(*a*) Suppose $c_1 \mathbf{f}_1(t) + \cdots + c_n \mathbf{f}_n(t) \equiv \mathbf{0}$ for all t. Then $c_1 \mathbf{f}_1(t_0) + \cdots + c_n \mathbf{f}_n(t_0) = \mathbf{0}$, and hence, by linear independence of the sample vectors, $c_1 = \cdots = c_n = 0$, which proves linear independence of the functions.
(*b*) $c_1 \mathbf{f}_1(t) + c_2 \mathbf{f}_1(t) = \begin{pmatrix} 2c_2 t + (c_1 - c_2) \\ 2c_2 t^2 + (c_1 - c_2)t \end{pmatrix} \equiv \mathbf{0}$ if and only if $c_2 = 0,\ c_1 - c_2 = 0$, and so $c_1 = c_2 = 0$, proving linear independence. However, at any t_0, the vectors $\mathbf{f}_2(t_0) = (2t_0 - 1)\mathbf{f}_1(t_0)$ are scalar multiples of each other, and hence linearly dependent.

♡ 2.3.39.
(*a*) Suppose $c_1 f(x) + c_2 g(x) \equiv 0$ for all x for some $\mathbf{c} = (c_1, c_2)^T \neq \mathbf{0}$. Differentiating, we find $c_1 f'(x) + c_2 g'(x) \equiv 0$ also, and hence $\begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}$ for all x. The homogeneous system has a nonzero solution if and only if the coefficient matrix is singular, which requires its determinant $W[f(x), g(x)] = 0$.
(*b*) This is the contrapositive of part (*a*), since if f, g were not linearly independent, then their Wronskian would vanish everywhere.
(*c*) Suppose $c_1 f(x) + c_2 g(x) = c_1 x^3 + c_2 |x|^3 \equiv 0$. then, at $x = 1$, $c_1 + c_2 = 0$, whereas at $x = -1$, $-c_1 + c_2 = 0$. Therefore, $c_1 = c_2 = 0$, proving linear independence. On the other hand, $W[x^3, |x|^3] = x^3(3x^2 \operatorname{sign} x) - (3x^2)|x|^3 \equiv 0$.

2.4.1. Only (*a*) and (*c*) are bases.

2.4.2. Only (*b*) is a basis.

2.4.3. (*a*) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$; (*b*) $\begin{pmatrix} \frac34 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac14 \\ 0 \\ 1 \end{pmatrix}$; (*c*) $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

2.4.4.
(*a*) They do not span $\mathbb{R}^3$ because the linear system $A\mathbf{c} = \mathbf{b}$ with coefficient matrix $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 0 & -1 & -1 & -1 \\ 2 & 1 & -1 & 3 \end{pmatrix}$ does not have a solution for all $\mathbf{b}$ since $\operatorname{rank} A = 2$.
(*b*) 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent.

(c) No, because if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ don't span $\mathbb{R}^3$, no subset of them will span it either.
(d) 2, because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent and span the subspace, and hence form a basis.

2.4.5.
(a) They span $\mathbb{R}^3$ because the linear system $A\mathbf{c} = \mathbf{b}$ with coefficient matrix $A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & -2 & 3 \\ 2 & 5 & 1 & -1 \end{pmatrix}$ has a solution for all $\mathbf{b}$ since $\operatorname{rank} A = 3$.
(b) 4 vectors in $\mathbb{R}^3$ are automatically linearly dependent.
(c) Yes, because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ also span $\mathbb{R}^3$ and so form a basis.
(d) 3 because they span all of $\mathbb{R}^3$.

2.4.6.
(a) Solving the defining equation, the general vector in the plane is $\mathbf{x} = \begin{pmatrix} 2y+4z \\ y \\ z \end{pmatrix}$ where y, z are arbitrary. We can write $\mathbf{x} = y \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = (y+2z) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + (y+z) \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ and hence both pairs of vectors span the plane. Both pairs are linearly independent since they are not parallel, and hence both form a basis.
(b) $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix};$
(c) Any two linearly independent solutions, e.g., $\begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix}$, will form a basis.

♡ 2.4.7. (a) (i) Left handed basis; (ii) right handed basis; (iii) not a basis; (iv) right handed basis. (b) Switching two columns or multiplying a column by -1 changes the sign of the determinant. (c) If $\det A = 0$, its columns are linearly dependent and hence can't form a basis.

2.4.8.
(a) $\left(-\frac{2}{3}, \frac{5}{6}, 1, 0\right)^T, \left(\frac{1}{3}, -\frac{2}{3}, 0, 1\right)^T$; $\dim = 2$.
(b) The condition $p(1) = 0$ says $a + b + c = 0$, so $p(x) = (-b-c)x^2 + bx + c = b(-x^2+x) + c(-x^2+1)$. Therefore $-x^2+x,\ -x^2+1$ is a basis, and so $\dim = 2$.
(c) $e^x,\ \cos 2x,\ \sin 2x$, is a basis, so $\dim = 3$.

2.4.9. (a) $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\dim = 1$; (b) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$, $\dim = 2$; (c) $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$, $\dim = 3$.

2.4.10. (a) We have $a + bt + ct^2 = c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2)$ provided $a = c_1 + c_3$, $b = c_2 + 2c_3$, $c = c_1 + c_2 + c_3$. The coefficient matrix of this linear system, $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, is nonsingular, and hence there is a solution for any a, b, c, proving that they span the space of quadratic polynomials. Also, they are linearly independent since the linear combination is zero if and only if c_1, c_2, c_3 satisfy the corresponding homogeneous linear system $c_1 + c_3 = 0$, $c_2 + 2c_3 = 0$, $c_1 + c_2 + c_3 = 0$, and hence $c_1 = c_2 = c_3 = 0$. (Or, you can use the fact that $\dim \mathcal{P}^{(2)} = 3$ and the spanning property to conclude that they form a basis.)

(b) $1+4t+7t^2 = 2(1+t^2)+6(t+t^2)-(1+2t+t^2)$

2.4.11. (a) $a+bt+ct^2+dt^3 = c_1+c_2(1-t)+c_3(1-t)^2+c_4(1-t)^3$ provided $a = c_1+c_2+c_3+c_4$, $b = -c_2-2c_3-3c_4$, $c = c_3+3c_4$, $d = -c_4$. The coefficient matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is nonsingular, and hence they span $\mathcal{P}^{(3)}$. Also, they are linearly independent since the linear combination is zero if and only if $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the corresponding homogeneous linear system. (Or, you can use the fact that $\dim \mathcal{P}^{(3)} = 4$ and the spanning property to conclude that they form a basis.) (b) $1+t^3 = 2-3(1-t)+3(1-t)^2-(1-t)^3$.

2.4.12. (a) They are linearly dependent because $2p_1 - p_2 + p_3 \equiv 0$. (b) The dimension is 2, since p_1, p_2 are linearly independent and span the subspace, and hence form a basis.

2.4.13.

(a) The sample vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ are linearly independent and hence form a basis for $\mathbb{R}^4$ — the space of sample functions.

(b) Sampling x produces $\begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2+\sqrt{2}}{8}\begin{pmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} - \frac{2-\sqrt{2}}{8}\begin{pmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}$.

2.4.14.

(a) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis since we can uniquely write any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a E_{11} + b E_{12} + c E_{21} + d E_{22}$.

(b) Similarly, the matrices E_{ij} with a 1 in position (i,j) and all other entries 0, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, form a basis for $\mathcal{M}_{m\times n}$, which therefore has dimension mn.

2.4.15. $k \neq -1, 2$.

2.4.16. A basis is given by the matrices E_{ii}, $i = 1, \ldots, n$ which have a 1 in the i^{th} diagonal position and all other entries 0.

2.4.17.

(a) $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; dimension $= 3$.

(b) A basis is given by the matrices E_{ij} with a 1 in position (i,j) and all other entries 0 for $1 \le i \le j \le n$, so the dimension is $\frac{1}{2}n(n+1)$.

2.4.18. (a) Symmetric: dim $= 3$; skew-symmetric: dim $= 1$; (b) symmetric: dim $= 6$; skew-symmetric: dim $= 3$; (c) symmetric: dim $= \frac{1}{2}n(n+1)$; skew-symmetric: dim $= \frac{1}{2}n(n-1)$.

♡ 2.4.19.

(a) If a row (column) of A adds up to a and the corresponding row (column) of B adds up to b, then the corresponding row (column) of $C = A + B$ adds up to $c = a + b$. Thus, if all row and column sums of A and B are the same, the same is true for C. Similarly, the row (column) sums of $c\,A$ are c times the row (column) sums of A, and hence all the same if A is a semi-magic square.

(*b*) A matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$ is a semi-magic square if and only if

$$a+b+c=d+e+f=g+h+j=a+d+e=b+e+h=c+f+j.$$

The general solution to this system is

$$A = e\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + j\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (e-g)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (g+j-e)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} +$$

$$+ f\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + (h-f)\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

which is a linear combination of permutation matrices.

(*c*) The dimension is 5, with any 5 of the 6 permutation matrices forming a basis.

(*d*) Yes, by the same reasoning as in part (*a*). Its dimension is 3, with basis

$$\begin{pmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

(*e*) $A = c_1\begin{pmatrix} 2 & 2 & -1 \\ -2 & 1 & 4 \\ 3 & 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix} + c_3\begin{pmatrix} -1 & 2 & 2 \\ 4 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$ for any c_1, c_2, c_3.

♢ 2.4.20. For instance, take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_1 + \mathbf{v}_3$. In fact, there are infinitely many different ways of writing this vector as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

♢ 2.4.21.

(a) By Theorem 2.31, we only need prove linear independence. If $\mathbf{0} = c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n = A(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)$, then, since A is nonsingular, $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$, and hence $c_1 = \cdots = c_n = 0$.

(*b*) $A\mathbf{e}_i$ is the i^{th} column of A, and so a basis consists of the column vectors of the matrix.

♢ 2.4.22. Since $V \neq \{\mathbf{0}\}$, at least one $\mathbf{v}_i \neq \mathbf{0}$. Let $\mathbf{v}_{i_1} \neq \mathbf{0}$ be the first nonzero vector in the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then, for each $k = i_1 + 1, \ldots, n-1$, suppose we have selected linearly independent vectors $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$ from among $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}, \mathbf{v}_{k+1}$ form a linearly independent set, we set $\mathbf{v}_{i_{j+1}} = \mathbf{v}_{k+1}$; otherwise, $\mathbf{v}_{k+1}$ is a linear combination of $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_j}$, and is not needed in the basis. The resulting collection $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_m}$ forms a basis for V since they are linearly independent by design, and span V since each $\mathbf{v}_i$ either appears in the basis, or is a linear combination of the basis elements that were selected before it. We have $\dim V = n$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and so form a basis for V.

♢ 2.4.23. This is a special case of Exercise 2.3.31(*a*).

♢ 2.4.24.

(*a*) $m \leq n$ as otherwise $\mathbf{v}_1, \ldots, \mathbf{v}_m$ would be linearly dependent. If $m = n$ then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and hence, by Theorem 2.31 span all of $\mathbb{R}^n$. Since every vector in their span also belongs to V, we must have $V = \mathbb{R}^n$.

(*b*) Starting with the basis $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V with $m < n$, we choose any $\mathbf{v}_{m+1} \in \mathbb{R}^n \setminus V$. Since $\mathbf{v}_{m+1}$ does not lie in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$ are linearly independent and span an $m+1$ dimensional subspace of $\mathbb{R}^n$. Unless $m+1 = n$ we can then choose another vector $\mathbf{v}_{m+2}$ not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{m+1}$, and so $\mathbf{v}_1, \ldots, \mathbf{v}_{m+2}$ are also linearly independent. We continue on in this fashion until we arrive at n linearly independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which necessarily form a basis of $\mathbb{R}^n$.

(c) (i) $\left(1, 1, \frac{1}{2}\right)^T, (1,0,0)^T, (0,1,0)^T$; (ii) $(1,0,-1)^T, (0,1,-2)^T, (1,0,0)^T$.

♢ 2.4.25.

(a) If $\dim V = \infty$, then the inequality is trivial. Also, if $\dim W = \infty$, then one can find infinitely many linearly independent elements in W, but these are also linearly independent as elements of V and so $\dim V = \infty$ also. Otherwise, let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ form a basis for W. Since they are linearly independent, Theorem 2.31 implies $n \le \dim V$.

(b) Since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent, if $n = \dim V$, then by Theorem 2.31, they form a basis for V. Thus every $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_n$, and hence, since W is a subspace, $\mathbf{v} \in W$ too. Therefore, $W = V$.

(c) Example: $V = \mathrm{C}^0[a,b]$ and $W = \mathcal{P}^{(\infty)}$.

♢ 2.4.26. (a) Every $\mathbf{v} \in V$ can be uniquely decomposed as $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{w} \in W, \mathbf{z} \in Z$. Write $\mathbf{w} = c_1 \mathbf{w}_1 + \ldots + c_j \mathbf{w}_j$ and $\mathbf{z} = d_1 \mathbf{z}_1 + \cdots + d_k \mathbf{z}_k$. Then $\mathbf{v} = c_1 \mathbf{w}_1 + \ldots + c_j \mathbf{w}_j + d_1 \mathbf{z}_1 + \cdots + d_k \mathbf{z}_k$, proving that $\mathbf{w}_1, \ldots, \mathbf{w}_j, \mathbf{z}_1, \ldots, \mathbf{z}_k$ span V. Moreover, by uniqueness, $\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{w} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$, and so the only linear combination that sums up to $\mathbf{0} \in V$ is the trivial one $c_1 = \cdots = c_j = d_1 = \cdots = d_k = 0$, which proves linear independence of the full collection. (b) This follows immediately from part (a): $\dim V = j+k = \dim W + \dim Z$.

♢ 2.4.27. Suppose the functions are linearly independent. This means that for every $\mathbf{0} \neq \mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$, there is a point $x_{\mathbf{c}} \in \mathbb{R}$ such that $\sum_{i=1}^{n} c_i f_i(x_{\mathbf{c}}) \neq 0$. The assumption says that $\{\mathbf{0}\} \neq V_{x_1,\ldots,x_m}$ for all choices of sample points. Recursively define the following sample points. Choose x_1 so that $f_1(x_1) \neq 0$. (This is possible since if $f_1(x) \equiv 0$, then the functions are linearly dependent.) Thus $V_{x_1} \subsetneq \mathbb{R}^m$ since $\mathbf{e}_1 \notin V_{x_1}$. Then, for each $m = 1, 2, \ldots$, given $x_1, \ldots, x_m$, choose $\mathbf{0} \neq \mathbf{c}_0 \in V_{x_1,\ldots,x_m}$, and set $x_{m+1} = x_{\mathbf{c}_0}$. Then $\mathbf{c}_0 \notin V_{x_1,\ldots,x_{m+1}} \subsetneq V_{x_1,\ldots,x_m}$ and hence, by induction, $\dim V_m \le n - m$. In particular, $\dim V_{x_1,\ldots,x_n} = 0$, so $V_{x_1,\ldots,x_n} = \{\mathbf{0}\}$, which contradicts our assumption and proves the result. Note that the proof implies we only need check linear dependence at all possible collections of n sample points to conclude that the functions are linearly dependent.

2.5.1.

(a) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $\frac{3}{4} b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$.

(b) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ such that $2 b_1 + b_2 = 0$; kernel spanned by $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

(c) Range: all $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ such that $-2 b_1 + b_2 + b_3 = 0$; kernel spanned by $\begin{pmatrix} -\frac{5}{4} \\ -\frac{7}{8} \\ 1 \end{pmatrix}$.

(d) Range: all $\mathbf{b} = (b_1, b_2, b_3, b_4)^T$ such that $-2 b_1 - b_2 + b_3 = 2 b_1 + 3 b_2 + b_4 = 0$; kernel spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

2.5.2. (a) $\begin{pmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$: plane; (b) $\begin{pmatrix} \frac{1}{4} \\ \frac{3}{8} \\ 1 \end{pmatrix}$: line; (c) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$: plane;
(d) $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$: line; (e) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: point; (f) $\begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ 1 \end{pmatrix}$: line.

2.5.3.
(a) Kernel spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; range spanned by $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$;
(b) compatibility: $-\frac{1}{2}a + \frac{1}{4}b + c = 0$.

2.5.4. (a) $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$; (b) $\mathbf{x} = \begin{pmatrix} 1+t \\ 2+t \\ 3+t \end{pmatrix}$ where t is arbitrary.

2.5.5. In each case, the solution is $\mathbf{x} = \mathbf{x}^\star + \mathbf{z}$, where $\mathbf{x}^\star$ is the particular solution and $\mathbf{z}$ belongs to the kernel:

(a) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$; (b) $\mathbf{x}^\star = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} -\frac{2}{7} \\ \frac{1}{7} \\ 1 \end{pmatrix}$;

(c) $\mathbf{x}^\star = \begin{pmatrix} -\frac{7}{9} \\ \frac{2}{9} \\ \frac{10}{9} \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$; (d) $\mathbf{x}^\star = \begin{pmatrix} \frac{5}{6} \\ 1 \\ -\frac{2}{3} \end{pmatrix}$, $\mathbf{z} = \mathbf{0}$; (e) $\mathbf{x}^\star = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $\mathbf{z} = v\begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

(f) $\mathbf{x}^\star = \begin{pmatrix} \frac{11}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = r\begin{pmatrix} -\frac{13}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$; (g) $\mathbf{x}^\star = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{z} = z\begin{pmatrix} 6 \\ 2 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$.

2.5.6. The i^{th} entry of $A\,(1,1,\ldots,1)^T$ is $a_{i1} + \ldots + a_{in}$ which is n times the average of the entries in the i^{th} row. Thus, $A\,(1,1,\ldots,1)^T = \mathbf{0}$ if and only if each row of A has average 0.

2.5.7. The kernel has dimension $n-1$, with basis $-r^{k-1}\mathbf{e}_1 + \mathbf{e}_k = \left(-r^{k-1}, 0, \ldots, 0, 1, 0, \ldots, 0\right)^T$ for $k = 2, \ldots n$. The range has dimension 1, with basis $(1, r^n, r^{2n} \ldots, r^{(n-1)n})^T$.

♢ 2.5.8. (a) If $\mathbf{w} = P\mathbf{w}$, then $\mathbf{w} \in \operatorname{rng} P$. On the other hand, if $\mathbf{w} \in \operatorname{rng} P$, then $\mathbf{w} = P\mathbf{v}$ for some $\mathbf{v}$. But then $P\mathbf{w} = P^2\mathbf{v} = P\mathbf{v} = \mathbf{w}$. (b) Given $\mathbf{v}$, set $\mathbf{w} = P\mathbf{v}$. Then $\mathbf{v} = \mathbf{w} + \mathbf{z}$ where $\mathbf{z} = \mathbf{v} - \mathbf{w} \in \ker P$ since $P\mathbf{z} = P\mathbf{v} - P\mathbf{w} = P\mathbf{v} - P^2\mathbf{v} = P\mathbf{v} - P\mathbf{v} = \mathbf{0}$. Moreover, if $\mathbf{w} \in \ker P \cap \operatorname{rng} P$, then $\mathbf{0} = P\mathbf{w} = \mathbf{w}$, and so $\ker P \cap \operatorname{rng} P = \{\mathbf{0}\}$, proving complementarity.

2.5.9. False. For example, if $A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in both $\ker A$ and $\operatorname{rng} A$.

♢ 2.5.10. Let $\mathbf{r}_1, \ldots, \mathbf{r}_{m+k}$ be the rows of C, so $\mathbf{r}_1, \ldots, \mathbf{r}_m$ are the rows of A. For $\mathbf{v} \in \ker C$, the i^{th} entry of $C\mathbf{v} = \mathbf{0}$ is $\mathbf{r}_i\mathbf{v} = 0$, but then this implies $A\mathbf{v} = \mathbf{0}$ and so $\mathbf{v} \in \ker A$. As an example, $A = (\,1 \quad 0\,)$ has kernel spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, while $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has $\ker C = \{\mathbf{0}\}$.

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◇ 2.5.11. If $\mathbf{b} = A\mathbf{x} \in \operatorname{rng} A$, then $\mathbf{b} = C\mathbf{z}$ where $\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}$, and so $\mathbf{b} \in \operatorname{rng} C$. As an example, $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has $\operatorname{rng} A = \{\mathbf{0}\}$, while the range of $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the x axis.

2.5.12. $\mathbf{x}_1^\star = \begin{pmatrix} -2 \\ \frac{3}{2} \end{pmatrix}, \quad \mathbf{x}_2^\star = \begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix}; \quad \mathbf{x} = \mathbf{x}_1^\star + 4\mathbf{x}_2^\star = \begin{pmatrix} -6 \\ \frac{7}{2} \end{pmatrix}.$

2.5.13. $\mathbf{x}^\star = 2\mathbf{x}_1^\star + \mathbf{x}_2^\star = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}.$

2.5.14.
(a) By direct matrix multiplication: $A\mathbf{x}_1^\star = A\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$
(b) The general solution is $\mathbf{x} = \mathbf{x}_1^\star + t(\mathbf{x}_2^\star - \mathbf{x}_1^\star) = (1-t)\mathbf{x}_1^\star + t\mathbf{x}_2^\star = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}.$

2.5.15. 5 meters.

2.5.16. The mass will move 6 units in the horizontal direction and -6 units in the vertical direction.

2.5.17. $\mathbf{x} = c_1\mathbf{x}_1^\star + c_2\mathbf{x}_2^\star$ where $c_1 = 1 - c_2$.

2.5.18. False: in general, $(A+B)\mathbf{x}^\star = (A+B)\mathbf{x}_1^\star + (A+B)\mathbf{x}_2^\star = \mathbf{c} + \mathbf{d} + B\mathbf{x}_1^\star + A\mathbf{x}_2^\star$, and the third and fourth terms don't necessarily add up to $\mathbf{0}$.

◇ 2.5.19. $\operatorname{rng} A = \mathbb{R}^n$, and so A must be a nonsingular matrix.

◇ 2.5.20.
(a) If $A\mathbf{x}_i = \mathbf{e}_i$, then $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ which, by (2.13), is the i^{th} column of the matrix A^{-1}.
(b) The solutions to $A\mathbf{x}_i = \mathbf{e}_i$ in this case are $\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 2 \\ -\frac{1}{2} \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ -1 \end{pmatrix}, \mathbf{x}_3 = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix},$
which are the columns of $A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$

2.5.21.
(a) range: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$; kernel: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
(b) range: $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -8 \\ -1 \\ 6 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$; kernel: $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; cokernel: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.
(c) range: $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$; corange: $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 2 \end{pmatrix}$; kernel: $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

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(*d*) range: $\begin{pmatrix}1\\0\\2\\3\\1\end{pmatrix}, \begin{pmatrix}-3\\3\\-3\\-3\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\0\\3\\3\end{pmatrix}$; corange: $\begin{pmatrix}1\\-3\\2\\2\\1\end{pmatrix}, \begin{pmatrix}0\\3\\-6\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\4\end{pmatrix}$;

kernel: $\begin{pmatrix}4\\2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\0\\1\\0\end{pmatrix}$; cokernel: $\begin{pmatrix}-2\\-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\-1\\1\end{pmatrix}$.

2.5.22. $\begin{pmatrix}-1\\2\\-3\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}-3\\1\\0\end{pmatrix}$, which are its first, third and fourth columns;

Second column: $\begin{pmatrix}2\\-4\\6\end{pmatrix} = 2\begin{pmatrix}-1\\2\\-3\end{pmatrix}$; fifth column: $\begin{pmatrix}5\\-4\\8\end{pmatrix} = -2\begin{pmatrix}-1\\2\\-3\end{pmatrix} + \begin{pmatrix}0\\1\\2\end{pmatrix} - \begin{pmatrix}-3\\1\\0\end{pmatrix}$.

2.5.23. range: $\begin{pmatrix}1\\2\\-3\end{pmatrix}, \begin{pmatrix}0\\4\\1\end{pmatrix}$; corange: $\begin{pmatrix}1\\-3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\4\end{pmatrix}$; second column: $\begin{pmatrix}-3\\-6\\9\end{pmatrix} = -3\begin{pmatrix}1\\2\\-3\end{pmatrix}$;

second and third rows: $\begin{pmatrix}2\\-6\\4\end{pmatrix} = 2\begin{pmatrix}1\\-3\\0\end{pmatrix} + \begin{pmatrix}0\\0\\4\end{pmatrix}$, $\begin{pmatrix}-3\\9\\1\end{pmatrix} = -3\begin{pmatrix}1\\-3\\0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0\\0\\4\end{pmatrix}$.

2.5.24.

(*i*) rank $= 1$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 1$, $\dim \ker A = \dim \operatorname{coker} A = 1$;
kernel basis: $\begin{pmatrix}-2\\1\end{pmatrix}$; cokernel basis: $\begin{pmatrix}2\\1\end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;
example: $\mathbf{b} = \begin{pmatrix}1\\-2\end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix} + z\begin{pmatrix}-2\\1\end{pmatrix}$.

(*ii*) rank $= 1$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 1$, $\dim \ker A = 2$, $\dim \operatorname{coker} A = 1$; kernel basis: $\begin{pmatrix}\frac{1}{3}\\1\\0\end{pmatrix}, \begin{pmatrix}\frac{2}{3}\\0\\1\end{pmatrix}$; cokernel basis: $\begin{pmatrix}2\\1\end{pmatrix}$; compatibility conditions: $2b_1 + b_2 = 0$;
example: $\mathbf{b} = \begin{pmatrix}3\\-6\end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}\frac{1}{3}\\1\\0\end{pmatrix} + z\begin{pmatrix}\frac{2}{3}\\0\\1\end{pmatrix}$.

(*iii*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = 0$, $\dim \operatorname{coker} A = 1$;
kernel: $\{\mathbf{0}\}$; cokernel basis: $\begin{pmatrix}-\frac{20}{13}\\\frac{3}{13}\\1\end{pmatrix}$; compatibility conditions: $-\frac{20}{13}b_1 + \frac{3}{13}b_2 + b_3 = 0$;
example: $\mathbf{b} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$.

(*iv*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = \dim \operatorname{coker} A = 1$;
kernel basis: $\begin{pmatrix}-2\\-1\\1\end{pmatrix}$; cokernel basis: $\begin{pmatrix}-2\\1\\1\end{pmatrix}$; compatibility conditions:
$-2b_1 + b_2 + b_3 = 0$; example: $\mathbf{b} = \begin{pmatrix}2\\1\\3\end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix}1\\0\\0\end{pmatrix} + z\begin{pmatrix}-2\\-1\\1\end{pmatrix}$.

(*v*) rank $= 2$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 2$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; kernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -\frac{9}{4} \\ \frac{1}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$; compatibility: $-\frac{9}{4}\, b_1 + \frac{1}{4}\, b_2 + b_3 = 0$, $\frac{1}{4}\, b_1 - \frac{1}{4}\, b_2 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 2 \\ 6 \\ 3 \\ 1 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(*vi*) rank $= 3$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 3$, $\dim \ker A = \dim \operatorname{coker} A = 1$; kernel basis: $\begin{pmatrix} \frac{13}{4} \\ \frac{13}{8} \\ -\frac{7}{2} \\ 1 \end{pmatrix}$; cokernel basis: $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$; compatibility conditions: $-b_1 - b_2 + b_3 + b_4 = 0$; example: $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}$, with solution $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} \frac{13}{4} \\ \frac{13}{8} \\ -\frac{7}{2} \\ 1 \end{pmatrix}$.

(*vii*) rank $= 4$; $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 4$, $\dim \ker A = 1$, $\dim \operatorname{coker} A = 0$; kernel basis: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; cokernel is $\{\mathbf{0}\}$; no conditions; example: $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -3 \end{pmatrix}$, with $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

2.5.25. (*a*) $\dim = 2$; basis: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$; (*b*) $\dim = 1$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$;

(*c*) $\dim = 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}$; (*d*) $\dim = 3$; basis: $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -8 \\ 7 \end{pmatrix}$;

(*e*) $\dim = 3$; basis: $\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 1 \end{pmatrix}$.

2.5.26. It's the span of $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -1 \\ -1 \end{pmatrix}$; the dimension is 3.

2.5.27. (*a*) $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (*b*) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (*c*) $\begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

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2.5.28. First method: $\begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\3\\-4\\5\end{pmatrix}$; second method: $\begin{pmatrix}1\\0\\2\\1\end{pmatrix}, \begin{pmatrix}0\\3\\-8\\3\end{pmatrix}$. The first vectors are the same, while $\begin{pmatrix}2\\3\\-4\\5\end{pmatrix} = 2\begin{pmatrix}1\\0\\2\\1\end{pmatrix} + \begin{pmatrix}0\\3\\-8\\3\end{pmatrix}$; $\begin{pmatrix}0\\3\\-8\\3\end{pmatrix} = -2\begin{pmatrix}1\\0\\2\\1\end{pmatrix} + \begin{pmatrix}2\\3\\-4\\5\end{pmatrix}$.

2.5.29. Both sets are linearly independent and hence span a three-dimensional subspace of $\mathbb{R}^4$. Moreover, $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3, \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ all lie in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and hence, by Theorem 2.31(d) also form a basis for the subspace.

2.5.30.
(a) If $A = A^T$, then $\ker A = \{A\mathbf{x} = \mathbf{0}\} = \{A^T\mathbf{x} = \mathbf{0}\} = \operatorname{coker} A$, and $\operatorname{rng} A = \{A\mathbf{x}\} = \{A^T\mathbf{x}\} = \operatorname{corng} A$.
(b) $\ker A = \operatorname{coker} A$ has basis $(2,-1,1)^T$; $\operatorname{rng} A = \operatorname{corng} A$ has basis $(1,2,0)^T, (2,6,2)^T$.
(c) No. For instance, if A is any nonsingular matrix, then $\ker A = \operatorname{coker} A = \{\mathbf{0}\}$ and $\operatorname{rng} A = \operatorname{corng} A = \mathbb{R}^3$.

2.5.31.
(a) Yes. This is our method of constructing the basis for the range, and the proof is outlined in the text.
(b) No. For example, if $A = \begin{pmatrix}1&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$, then $U = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}$ and the first three rows of U form a basis for the three-dimensional $\operatorname{corng} U = \operatorname{corng} A$. but the first three rows of A only span a two-dimensional subspace.
(c) Yes, since $\ker U = \ker A$.
(d) No, since $\operatorname{coker} U \neq \operatorname{coker} A$ in general. For the example in part (b), $\operatorname{coker} A$ has basis $(-1,1,0,0)^T$ while $\operatorname{coker} A$ has basis $(0,0,0,1)^T$.

2.5.32. (a) Example: $\begin{pmatrix}0&0\\1&0\end{pmatrix}$. (b) No, since then the first r rows of U are linear combinations of the first r rows of A. Hence these rows span $\operatorname{corng} A$, which, by Theorem 2.31c, implies that they form a basis for the corange.

2.5.33. Examples: any symmetric matrix; any permutation matrix since the row echelon form is the identity. Yet another example is the complex matrix $\begin{pmatrix}0&0&1\\1&\mathrm{i}&\mathrm{i}\\0&\mathrm{i}&\mathrm{i}\end{pmatrix}$.

◇ 2.5.34. The rows $\mathbf{r}_1, \ldots, \mathbf{r}_m$ of A span the corange. Reordering the rows — in particular interchanging two — will not change the span. Also, multiplying any of the rows by nonzero scalars, $\widetilde{\mathbf{r}}_i = a_i\,\mathbf{r}_i$, for $a_i \neq 0$, will also span the same space, since

$$\mathbf{v} = \sum_{i=1}^{n} c_i\,\mathbf{r}_i = \sum_{i=1}^{n} \frac{c_i}{a_i}\,\widetilde{\mathbf{r}}_i.$$

2.5.35. We know $\operatorname{rng} A \subset \mathbb{R}^m$ is a subspace of dimension $r = \operatorname{rank} A$. In particular, $\operatorname{rng} A = \mathbb{R}^m$ if and only if it has dimension $m = \operatorname{rank} A$.

2.5.36. This is false. If $A = \begin{pmatrix}1&1\\1&1\end{pmatrix}$ then $\operatorname{rng} A$ is spanned by $\begin{pmatrix}1\\1\end{pmatrix}$ whereas the range of its row echelon form $U = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

♢ 2.5.37.

(a) Method 1: choose the nonzero rows in the row echelon form of A. Method 2: choose the columns of A^T that correspond to pivot columns of its row echelon form.

(b) Method 1: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}$. Method 2: $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ -7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$. Not the same.

♢ 2.5.38. If $\mathbf{v} \in \ker A$ then $A\mathbf{v} = \mathbf{0}$ and so $BA\mathbf{v} = B\mathbf{0} = \mathbf{0}$, so $\mathbf{v} \in \ker(BA)$. The first statement follows from setting $B = A$.

♢ 2.5.39. If $\mathbf{v} \in \operatorname{rng} AB$ then $\mathbf{v} = AB\mathbf{x}$ for some vector $\mathbf{x}$. But then $\mathbf{v} = A\mathbf{y}$ where $\mathbf{y} = B\mathbf{x}$, and so $\mathbf{v} \in \operatorname{rng} A$. The first statement follows from setting $B = A$.

2.5.40. First note that BA and AC also have size $m \times n$. To show $\operatorname{rank} A = \operatorname{rank} BA$, we prove that $\ker A = \ker BA$, and so $\operatorname{rank} A = n - \dim\ker A = n - \dim\ker BA = \operatorname{rank} BA$. Indeed, if $\mathbf{v} \in \ker A$, then $A\mathbf{v} = \mathbf{0}$ and hence $BA\mathbf{v} = \mathbf{0}$ so $\mathbf{v} \in \ker BA$. Conversely, if $\mathbf{v} \in \ker BA$ then $BA\mathbf{v} = \mathbf{0}$. Since B is nonsingular, this implies $A\mathbf{v} = \mathbf{0}$ and hence $\mathbf{v} \in \ker A$, proving the first result. To show $\operatorname{rank} A = \operatorname{rank} AC$, we prove that $\operatorname{rng} A = \operatorname{rng} AC$, and so $\operatorname{rank} A = \dim \operatorname{rng} A = \dim \operatorname{rng} AC = \operatorname{rank} AC$. Indeed, if $\mathbf{b} \in \operatorname{rng} AC$, then $\mathbf{b} = AC\mathbf{x}$ for some $\mathbf{x}$ and so $\mathbf{b} = A\mathbf{y}$ where $\mathbf{y} = C\mathbf{x}$, and so $\mathbf{b} \in \operatorname{rng} A$. Conversely, if $\mathbf{b} \in \operatorname{rng} A$ then $\mathbf{b} = A\mathbf{y}$ for some $\mathbf{y}$ and so $\mathbf{b} = AC\mathbf{x}$ where $\mathbf{x} = C^{-1}\mathbf{y}$, so $\mathbf{b} \in \operatorname{rng} AC$, proving the second result. The final equality is a consequence of the first two: $\operatorname{rank} A = \operatorname{rank} BA = \operatorname{rank}(BA)C$.

♢ 2.5.41. (a) Since they are spanned by the columns, the range of $(\, A \;\; B \,)$ contains the range of A. But since A is nonsingular, $\operatorname{rng} A = \mathbb{R}^n$, and so $\operatorname{rng}(\, A \;\; B \,) = \mathbb{R}^n$ also, which proves $\operatorname{rank}(\, A \;\; B \,) = n$. (b) Same argument, using the fact that the corange is spanned by the rows.

2.5.42. True if the matrices have the same size, but false in general.

♢ 2.5.43. Since we know $\dim \operatorname{rng} A = r$, it suffices to prove that $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent. Given

$$\mathbf{0} = c_1 \mathbf{w}_1 + \cdots + c_r \mathbf{w}_r = c_1 A\mathbf{v}_1 + \cdots + c_r A\mathbf{v}_r = A(c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r),$$

we deduce that $c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r \in \ker A$, and hence can be written as a linear combination of the kernel basis vectors:

$$c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r = c_{r+1} \mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n.$$

But $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, and so $c_1 = \cdots = c_r = c_{r+1} = \cdots = c_n = 0$, which proves linear independence of $\mathbf{w}_1, \ldots, \mathbf{w}_r$.

♢ 2.5.44.

(a) Since they have the same kernel, their ranks are the same. Choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{R}^n$ such that $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ form a basis for $\ker A = \ker B$. Then $\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_r = A\mathbf{v}_r$ form a basis for $\operatorname{rng} A$, while $\mathbf{y}_1 = B\mathbf{v}_1, \ldots, \mathbf{y}_r = B\mathbf{v}_r$ form a basis for $\operatorname{rng} B$. Let M be any nonsingular $m \times m$ matrix such that $M\mathbf{w}_j = \mathbf{y}_j$, $j = 1, \ldots, r$, which exists since both sets of vectors are linearly independent. We claim $MA = B$. Indeed, $MA\mathbf{v}_j = B\mathbf{v}_j$, $j = 1, \ldots, r$, by design, while $MA\mathbf{v}_j = \mathbf{0} = B\mathbf{v}_j$, $j = r+1, \ldots, n$, since these vectors lie in the kernel. Thus, the matrices agree on a basis of $\mathbb{R}^n$ which is enough to conclude that $MA = B$.

(b) If the systems have the same solutions $\mathbf{x}^\star + \mathbf{z}$ where $\mathbf{z} \in \ker A = \ker B$, then $B\mathbf{x} = MA\mathbf{x} = M\mathbf{b} = \mathbf{c}$. Since M can be written as a product of elementary matrices, we conclude that one can get from the augmented matrix $\big(\, A \mid \mathbf{b} \,\big)$ to the augmented matrix

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$\left(B \mid \mathbf{c} \right)$ by applying the elementary row operations that make up M.

♢ 2.5.45. (*a*) First, $W \subset \operatorname{rng} A$ since every $\mathbf{w} \in W$ can be written as $\mathbf{w} = A\mathbf{v}$ for some $\mathbf{v} \in V \subset \mathbb{R}^n$, and so $\mathbf{w} \in \operatorname{rng} A$. Second, if $\mathbf{w}_1 = A\mathbf{v}_1$ and $\mathbf{w}_2 = A\mathbf{v}_2$ are elements of W, then so is $c\mathbf{w}_1 + d\mathbf{w}_2 = A(c\mathbf{v}_1 + d\mathbf{v}_2)$ for any scalars c, d because $c\mathbf{v}_1 + d\mathbf{v}_2 \in V$, proving that W is a subspace. (*b*) First, using Exercise 2.4.25, $\dim W \le r = \dim \operatorname{rng} A$ since it is a subspace of the range. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ form a basis for V, so $\dim V = k$. Let $\mathbf{w} = A\mathbf{v} \in W$. We can write $\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_k \mathbf{v}_k$, and so, by linearity, $\mathbf{w} = c_1 A\mathbf{v}_1 + \cdots + c_k A\mathbf{v}_k$. Therefore, the k vectors $\mathbf{w}_1 = A\mathbf{v}_1, \ldots, \mathbf{w}_k = A\mathbf{v}_k$ span W, and therefore, by Proposition 2.33, $\dim W \le k$.

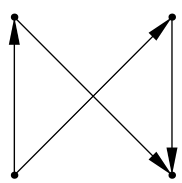
♢ 2.5.46.

(*a*) To have a left inverse requires an $n \times m$ matrix B such that $BA = \mathrm{I}$. Suppose $\dim \operatorname{rng} A = \operatorname{rank} A < n$. Then, according to Exercise 2.5.45, the subspace $W = \{ B\mathbf{v} \mid \mathbf{v} \in \operatorname{rng} A \}$ has $\dim W \le \dim \operatorname{rng} A < n$. On the other hand, $\mathbf{w} \in W$ if and only if $\mathbf{w} = B\mathbf{v}$ where $\mathbf{v} \in \operatorname{rng} A$, and so $\mathbf{v} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. But then $\mathbf{w} = B\mathbf{v} = BA\mathbf{x} = \mathbf{x}$, and therefore $W = \mathbb{R}^n$ since every vector $\mathbf{x} \in \mathbb{R}^n$ lies in it; thus, $\dim W = n$, contradicting the preceding result. We conclude that having a left inverse implies $\operatorname{rank} A = n$. (The rank can't be larger than n.)

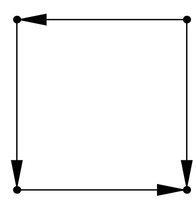
(*b*) To have a right inverse requires an $m \times n$ matrix C such that $AC = \mathrm{I}$. Suppose $\dim \operatorname{rng} A = \operatorname{rank} A < m$ and hence $\operatorname{rng} A \subsetneq \mathbb{R}^m$. Choose $\mathbf{y} \in \mathbb{R}^m \setminus \operatorname{rng} A$. Then $\mathbf{y} = AC\mathbf{y} = A\mathbf{x}$, where $\mathbf{x} = C\mathbf{y}$. Therefore, $\mathbf{y} \in \operatorname{rng} A$, which is a contradiction. We conclude that having a right inverse implies $\operatorname{rank} A = m$.

(*c*) By parts (a–*b*), having both inverses requires $m = \operatorname{rank} A = n$ and A must be square and nonsingular.

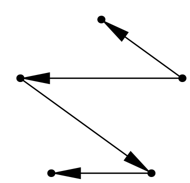
2.6.1. (*a*)



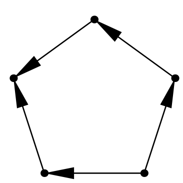
(*b*)



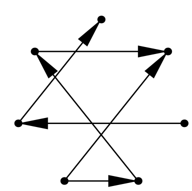
(*c*)



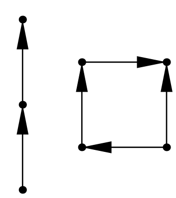
(*d*)



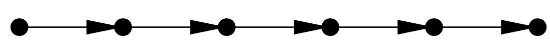
(*e*)



or, equivalently,



2.6.2. (*a*)



(*b*) $(1,1,1,1,1,1,1)^T$ is a basis for the kernel. The cokernel is trivial, containing only the zero vector, and so has no basis. (*c*) Zero.

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2.6.3. (a) $\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$; (b) $\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$;

(d) $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$; (e) $\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$;

(f) $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$.

2.6.4. (a) 1 circuit: $\begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$; (b) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$; (c) 2 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$;

(d) 3 circuits: $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; (e) 2 circuits: $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$;

(f) 3 circuits: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

♡ 2.6.5. (a) $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$; (b) rank $= 3$; (c) $\dim \operatorname{rng} A = \dim \operatorname{corng} A = 3$,

$\dim \ker A = 1$, $\dim \operatorname{coker} A = 2$; (d) kernel: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$; cokernel: $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$;

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(e) $b_1 - b_2 + b_4 = 0, \;\; b_1 - b_3 + b_5 = 0$; (f) example: $\mathbf{b} = \begin{pmatrix} 1\\1\\1\\0\\0 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} 1+t\\t\\t\\t \end{pmatrix}.$

♢ 2.6.6.
(a)

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Cokernel basis: $\mathbf{v}_1 = \begin{pmatrix} -1\\1\\0\\-1\\0\\1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1\\0\\1\\0\\-1\\0\\0\\1\\0\\0\\0\\0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0\\-1\\1\\0\\0\\0\\-1\\0\\1\\0\\0\\0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0\\0\\0\\-1\\1\\0\\0\\0\\0\\-1\\1\\0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 0\\0\\0\\0\\0\\-1\\1\\0\\0\\-1\\0\\1 \end{pmatrix}.$

These vectors represent the circuits around 5 of the cube's faces.

(b) Examples: $\begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\-1\\1\\0\\-1\\1 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 + \mathbf{v}_5, \quad \begin{pmatrix} 0\\1\\-1\\-1\\1\\1\\0\\-1\\0\\0\\0\\0 \end{pmatrix} = \mathbf{v}_1 - \mathbf{v}_2, \quad \begin{pmatrix} 0\\-1\\1\\1\\-1\\0\\-1\\0\\1\\1\\-1\\0 \end{pmatrix} = \mathbf{v}_3 - \mathbf{v}_4.$

♡ 2.6.7.

(a) Tetrahedron: $\begin{pmatrix} 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1 \end{pmatrix}$

number of circuits $= \dim \operatorname{coker} A = 3$, number of faces $= 4$;

(b) Octahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

number of circuits $= \dim \operatorname{coker} A = 7$, number of faces $= 8$.

(c) Dodecahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}$$

number of circuits $= \dim \operatorname{coker} A = 11$, number of faces $= 12$.

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(*d*) Icosahedron:

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}$$

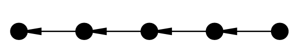
number of circuits $= \dim \operatorname{coker} A = 19$, number of faces $= 20$.

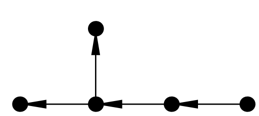
♡ 2.6.8.

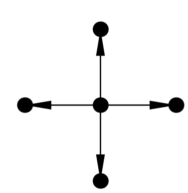
(a) (*i*) $\begin{pmatrix} -1 & 1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1 \end{pmatrix}$, (*ii*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$,

(*iii*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$, (*iv*) $\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

(*b*)







$\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

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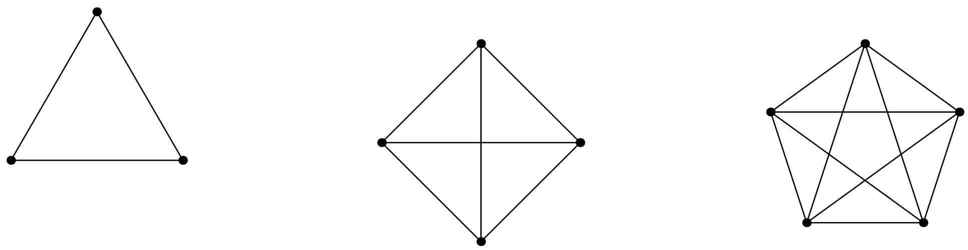
(*c*) Let m denote the number of edges. Since the graph is connected, its incidence matrix A has rank $n-1$. There are no circuits if and only if $\operatorname{coker} A = \{0\}$, which implies $0 = \dim \operatorname{coker} A = m - (n-1)$, and so $m = n - 1$.

♡ 2.6.9.

(*a*)

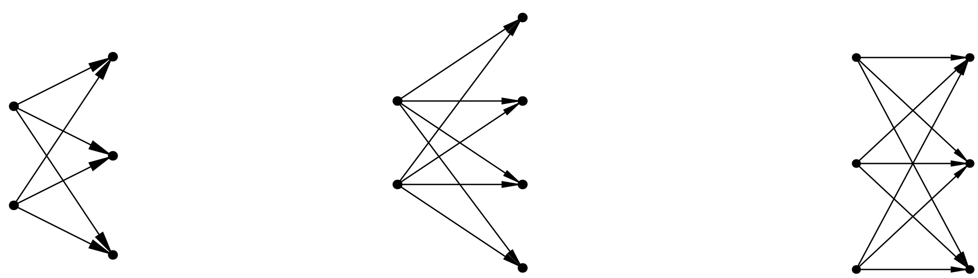


(*b*) $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$.

(*c*) $\frac{1}{2} n(n-1)$; (*d*) $\frac{1}{2}(n-1)(n-2)$.

♡ 2.6.10.

(*a*)



(*b*) $\begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$.

(*c*) $m\,n$; (*d*) $(m-1)(n-1)$.

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♡ 2.6.11.

(a) $A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$

(b) The vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis for $\ker A$.

(c) The entries of each $\mathbf{v}_i$ are indexed by the vertices. Thus the nonzero entries in $\mathbf{v}_1$ correspond to the vertices 1,2,4 in the first connected component, $\mathbf{v}_2$ to the vertices 3,6 in the second connected component, and $\mathbf{v}_3$ to the vertices 5,7,8 in the third connected component.

(d) Let A have k connected components. A basis for $\ker A$ consists of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ where $\mathbf{v}_i$ has entries equal to 1 if the vertex lies in the i^{th} connected component of the graph and 0 if it doesn't. To prove this, suppose $A\mathbf{v} = \mathbf{0}$. If edge $\#\ell$ connects vertex a to vertex b, then the ℓ^{th} component of the linear system is $v_a - v_b = 0$. Thus, $v_a = v_b$ whenever the vertices are connected by an edge. If two vertices are in the same connected component, then they can be connected by a path, and the values $v_a = v_b = \cdots$ at each vertex on the path must be equal. Thus, the values of v_a on all vertices in the connected component are equal, and hence $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ can be written as a linear combination of the basis vectors, with c_i being the common value of the entries v_a corresponding to vertices in the i^{th} connected component. Thus, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span the kernel. Moreover, since the coefficients c_i coincide with certain entries v_a of $\mathbf{v}$, the only linear combination giving the zero vector is when all c_i are zero, proving their linear independence.

◇ 2.6.12. If the incidence matrix has rank r, then # circuits

$$= \dim \operatorname{coker} A = n - r = \dim \ker A \geq 1,$$

since $\ker A$ always contains the vector $(1, 1, \ldots, 1)^T$.

2.6.13. Changing the direction of an edge is the same as multiplying the corresponding row of the incidence matrix by -1. The dimension of the cokernel, being the number of independent circuits, does not change. Each entry of a cokernel vector that corresponds to an edge that has been reversed is multiplied by -1. This can be realized by left multiplying the incidence matrix by a diagonal matrix whose diagonal entries are -1 is the corresponding edge has been reversed, and $+1$ if it is unchanged.

♡ 2.6.14.

(a) Note that P permutes the rows of A, and corresponds to a relabeling of the vertices of the digraph, while Q permutes its columns, and so corresponds to a relabeling of the edges.

(b) (i),(ii),(v) represent equivalent digraphs; none of the others are equivalent.

(c) $\mathbf{v} = (v_1, \ldots, v_m) \in \operatorname{coker} A$ if and only if $\widehat{\mathbf{v}} = P\mathbf{v} = (\, v_{\pi(1)} \, \cdots \, v_{\pi(m)} \,) \in \operatorname{coker} B$. Indeed, $\widehat{\mathbf{v}}^T B = (P\mathbf{v})^T PAQ = \mathbf{v}^T AQ = \mathbf{0}$ since, according to Exercise 1.6.14, $P^T = P^{-1}$ is the inverse of the permutation matrix P.

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2.6.15. False. For example, any two inequivalent trees, cf. Exercise 2.6.8, with the same number of nodes have incidence matrices of the same size, with trivial cokernels: $\operatorname{coker} A = \operatorname{coker} B = \{\mathbf{0}\}$. As another example, the incidence matrices

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

both have cokernel basis $(1,1,1,0,0)^T$, but do not represent equivalent digraphs.

2.6.16.

(a) If the first k vertices belong to one component and the last $n-k$ to the other, then there is no edge between the two sets of vertices and so the entries $a_{ij} = 0$ whenever $i = 1,\ldots,k$, $j = k+1,\ldots,n$, or when $i = k+1,\ldots,n$, $j = 1,\ldots,k$, which proves that A has the indicated block form.

(b) The graph consists of two disconnected triangles. If we use $1,2,3$ to label the vertices in one triangle and $4,5,6$ for those in the second, the resulting incidence matrix has the indicated block form $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$, with each block a 3×3 submatrix.