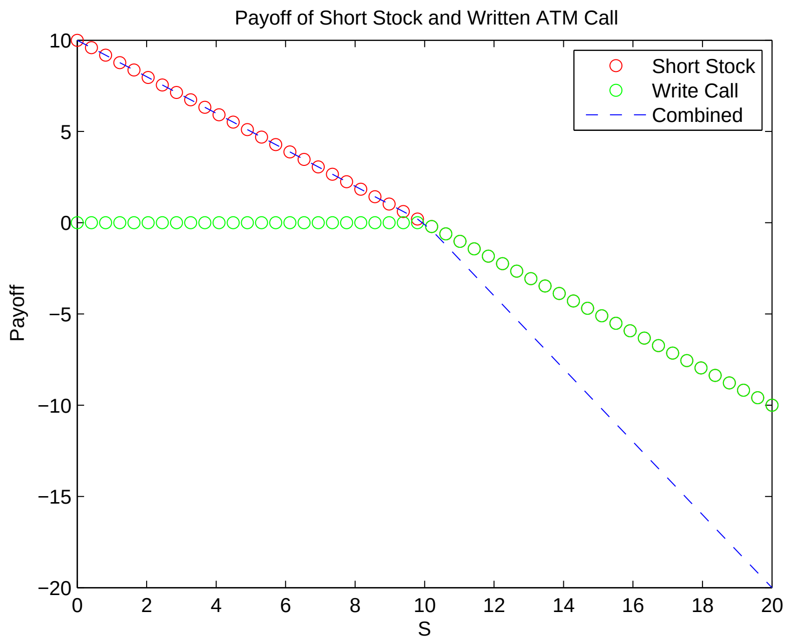
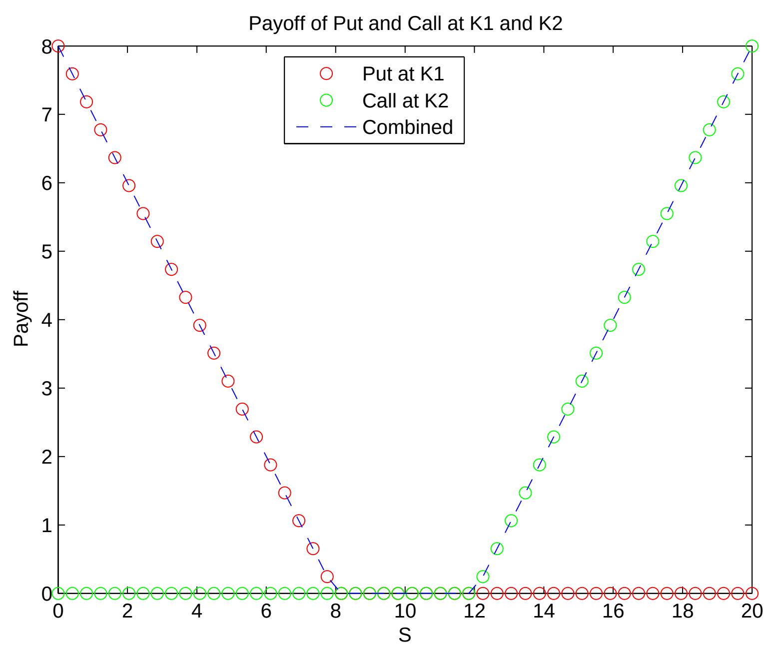
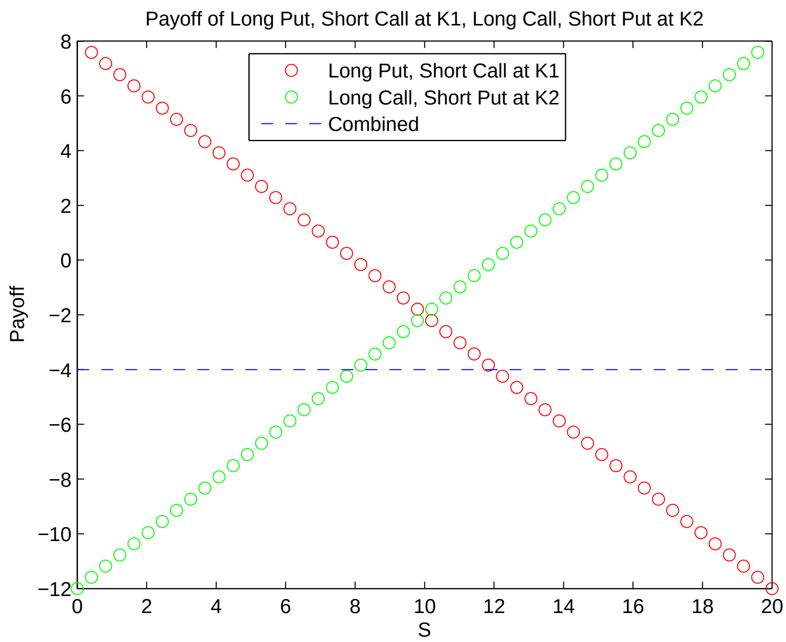
1 Chapter 1: Financial Derivatives

Problem 1

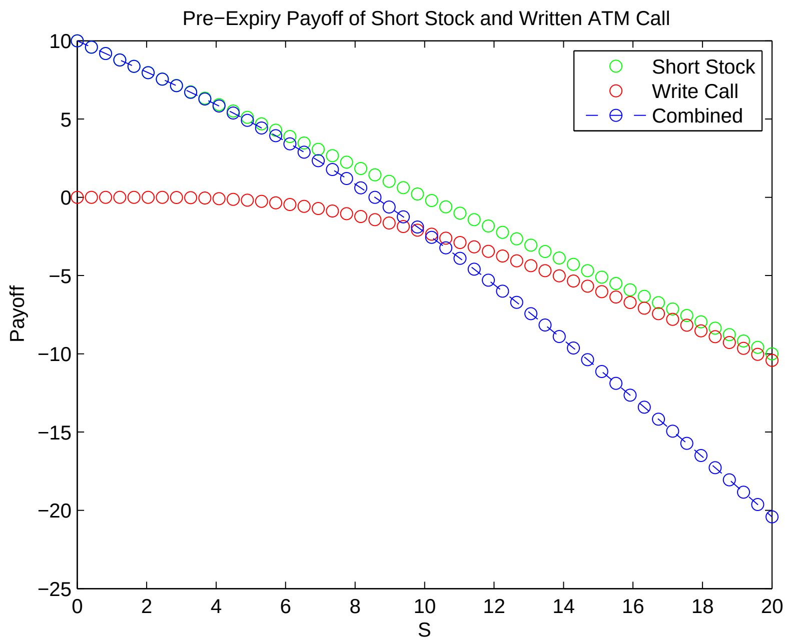
(a) The following plots show the respective payoff diagrams at expiration:

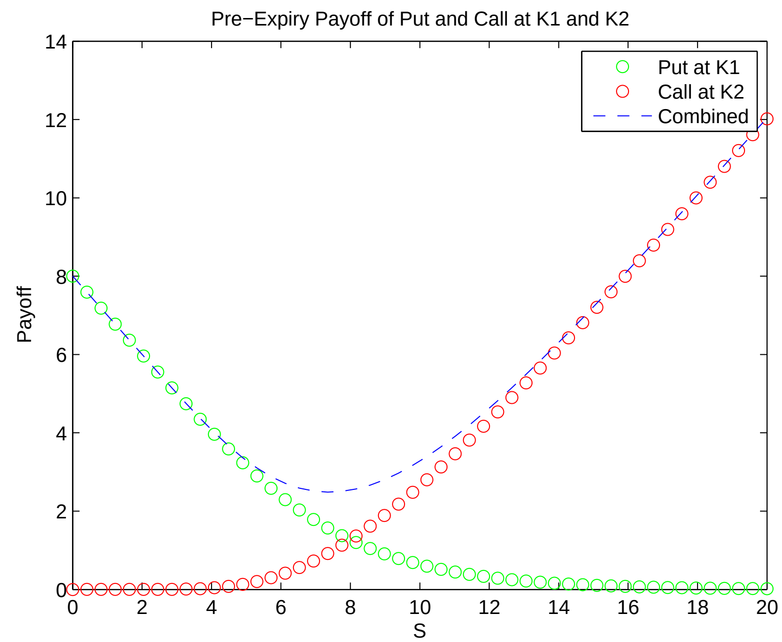


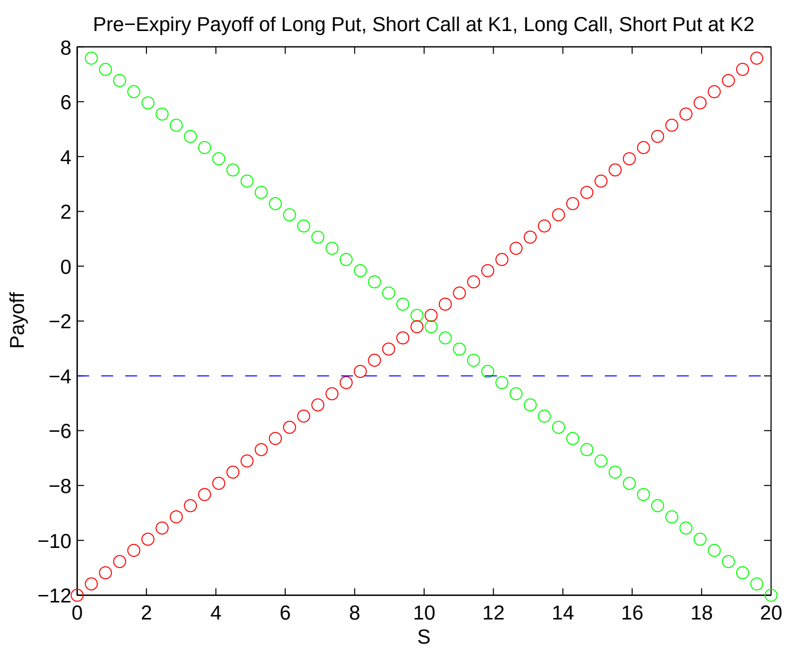




(b) The following plots show the respective payoff diagrams prior to expiration:







Problem 2

(a) Assuming a notional amount N. A cashflow takes place every 6 months, beginning in 12 months. The following table summarizes the cashflow:

Payoff	12 months	18 months	24 months
Floating Leg	N	$N\frac{L_{12}}{2}$	$N(1+\frac{L_{18}}{2})$
Fixed Leg	$-N$	$-N\frac{\kappa}{2}$	$-N(1+\frac{\kappa}{2})$

Here, $\kappa = 5\%$.
(b) We can replicate the above swap using FRA's by receiving the interest rate differential of the swap at 18 and 24 months. Note that an FRA is a forward contract in which one party pays a fixed interest rate in exchange for a floating rate at a specified future date. Consequently, in order to do replicate the swap, from the perspective the fixed payer, we would enter into two separate FRA contracts as the fixed payer at 18 and 24 months, each with a forward rate equal to κ. In this case we will receive the 6 month Libor and be make a fixed payment equal to κ.
(c) Assuming that interest rate caps and floors are both available, we can synthesize our swap using interest rate options. For example, a portfolio consisting of a long interest rate cap with a strike of κ and a long interest rate floor also struck at κ will have a payoff of $L - \kappa$. We notice that is equivalent to the interest rate differential of the swap, so in order to replicate the swap we must enter into these contracts at both 18 and 24 months.

Problem 3

(a) In the absence of arbitrage the futures price must be between:

$$S_t(1+r) \leq F_t \leq (S_t + c + s)(1+r) \tag{1}$$

If $F_t > (S_t + c + s)(1+r)$, then we can create an arbitrage portfolio consisting of the future, the underlying asset, wheat as well as borrowing. The arbitrage portfolio consists of the following:

Position	Initial Payoff at t	Terminal Payoff at T
Short futures	0	$F_t - S_T$
Borrow $S_t + c + s$	$+(S_t + c + S)$	$-(S_t + c + s)(1+r)$
Buy wheat and pay storage, insurance costs	$-(S_t + c + s)$	S_T
Total	0	$Ft - (St + c + s)(1+r)$

Since $F_t - (S_t + c + s)(1+r) > 0$ by assumption we have an arbitrage opportunity, and, consequently, $F_t \leq (S_t + c + s)(1+r)$ in the absence of arbitrage.

However, if $F_t < (S_t + c + s)(1+r)$, an arbitrage opportunity does necessarily exist. This is due to the fact that wheat cannot be viewed as an investment asset. Specifically, If one sells wheat, we cannot assume that we are entitled to receive the storage and insurance costs.

If the asset were of a financial nature or a commodity held for investment such as gold, then this asset could be sold and storage and insurance costs could be saved. These assets produce an exact no arbitrage price, $F_t = (S_t + c + s)(1+r)$.

However, returning to the case of wheat, if we consider the case when $S_t(1+r) > F_t$ we do have an arbitrage opportunity.

Position	Initial Payoff at t	Terminal Payoff at T
Buy futures	0	$S_T - F_t$
Invest S_t	$-S_t$	$+S_t(1+r)$
Sell wheat	$+S_t$	$-S_T$
Total	0	$S_t(1+r) - F_t$

Since $S_t(1+r) - F_t > 0$ by assumption, we have found an arbitrage opportunity. Thus, combining the two equations we obtain (**??**).
(b) To check if F_t is arbitrage free at $1500 we need to see if it is within the bounds obtained in part (a). That is:

$$S_t(1+r) \leq F_t \leq (S_t + c + s)(1+r) \tag{2}$$

Plugging in the appropriate values, we get:

$$1470(1+1.05) \leq 1500 \leq (1470 + 100/4 + 150/4)(1+0.05) \tag{3}$$

This violates the above inequality, as $1470(1.05) = 1,543.50 > 1500$. In order to construct the a ppropriate arbitrage strategy, we would follow the second strategy outlined in part (a).
(c) The profit p is equal to:
$p = 1,543.50 - 1,500 = \$43.50$.

Problem 4

(a) As all markets are liquid and there are no transaction costs or dividends, the forward price is uniquely determined by lack of arbitrage. That is:

$$F_t = S_t(1+r)(T-t) = 1.05S_t = \$105 \tag{4}$$

Here, we note that the delivery, in years, is equal to: $T - t = 1$.
(b) If the futures price is equal to 101, we can create an arbitrage portfolio in the following manner:

Position	Initial Payoff at t	Terminal Payoff at T
Long forward	0	$S_T - \$101$
Short stock	+$100	-$\$S_T$
Invest at risk - free rate	-$I00	+$105
Total	$0	$4

As the initial investment is 0 and the payoff is always positive we have found an arbitrage portfolio.

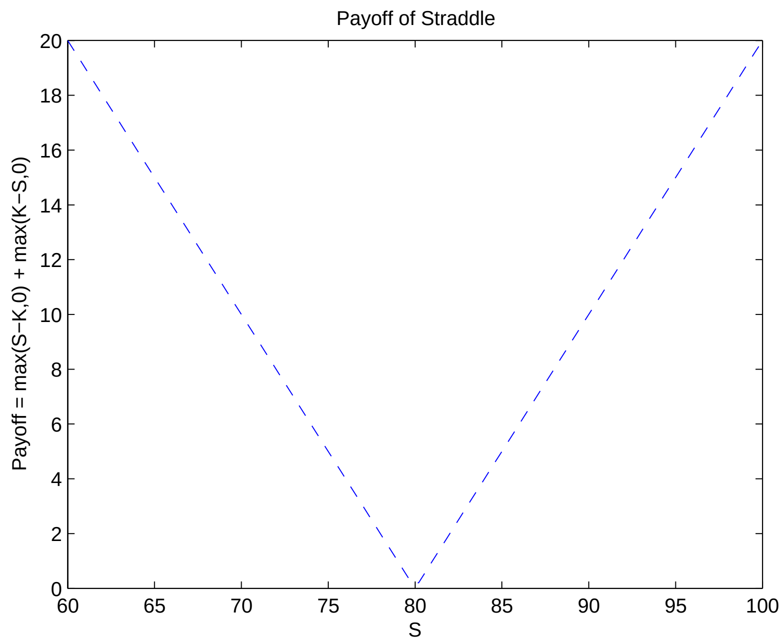
Alternatively, we can include the traded options to find another arbitrage portfolio in the following way:

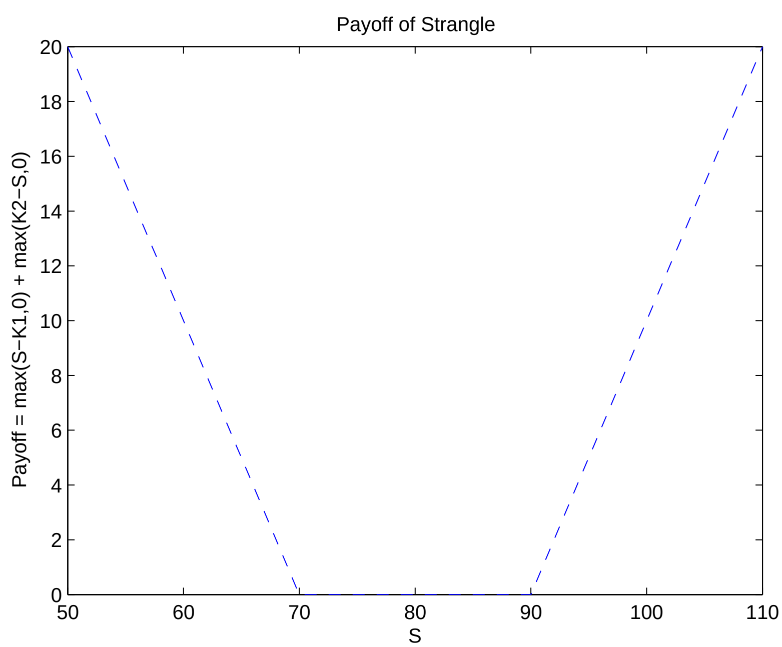
Position	Initial Payoff at t	Terminal Payoff at T
Short stock	+$l00	$-S_T$
Long Call	-$3.0	$\max(S_T - 100, 0)$
Short Put	+$3.5	$\min(S_T - 100, 0)$
Invest PV(100) at risk - free rate	-$\frac{100}{1.05}$	$100
Total	$5.26	$0

In this case, we can see that we receive a positive premium initially, and always have a terminal payoff of zero. Therefore, this is also an arbitrage opportunity.

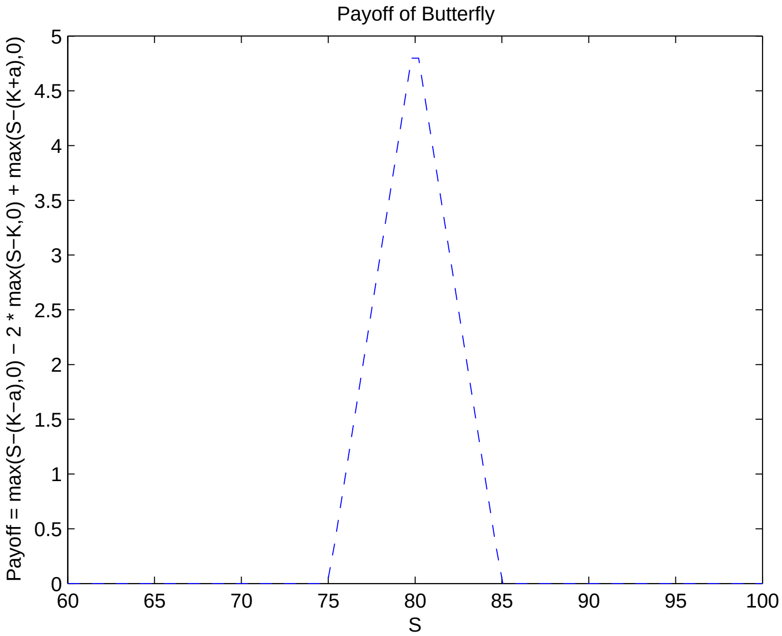
Problem 5

(a) An investor expecting high upcoming volatility could profit by buying a combination of a call and a put. If the strike of the call and the put is the same, this is called a straddle position. If the strike of the put is less than the strike of the call, then this strategy is called a strangle. The payoff diagrams for the straddle and strangle, respectively, are as follows:

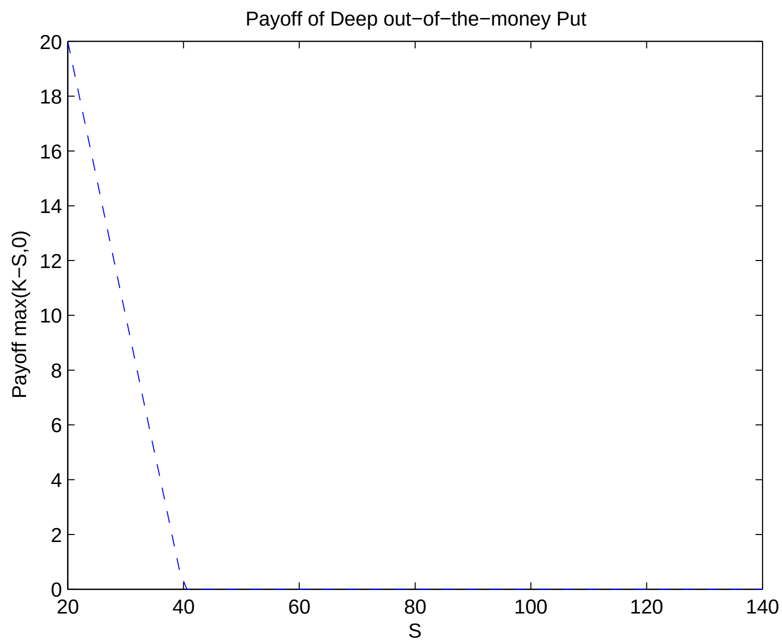




(b) An investor expecting low upcoming volatility could profit by selling 2 at-the-money calls and buying calls at $K - \epsilon$ and $K + \epsilon$ respectively. This strategy is often referred to as a butterfly. The resulting payoff diagram would be:

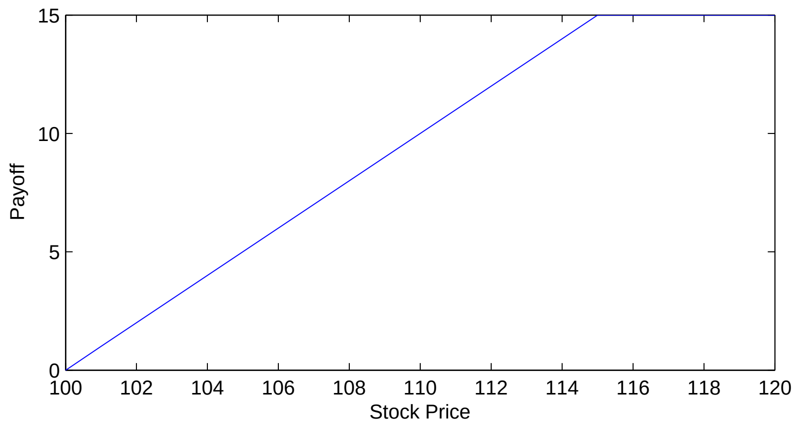


(c) An investor expecting a potential rare event negatively affecting the stock price, could express this position cheaply by buying a deep out-of-the-money put. The payoff diagram for this strategy would be:



Problem 6

A spread position is entered by buying and selling equal number of options of the same class on the same underlying security but with different strike prices. If a spread is designed to profit from a rise in the price of the underlying security, it is a bull spread. To be more specific, a bull call spread is constructed by buying a call option with a low exercise price (K), and selling another call option with a higher exercise price. For example, take an arbitrary stock X currently priced at \$100. Assume that the price of a call option with strike 100 is 3, and the price of a call option with strike 115 is 1. Then the payoff function of a portfolio consisting of longing a call with strike 100 and shorting a call with strike 115 at expiration is:




```
K1=100;
K2=115;
S=100:1:120
payoff= max(S-K1,0)-max(S-K2,0);
plot(S,payoff)
```

2 Chapter 2: A Primer on the Arbitrage Theorem

Problem 1

(a) In the risk-neutral martingale measure, we know that $\mathbb{E}(S_{t+\Delta}) = (1 + r\Delta)S_t$. Writing out the expectation in both states, we have:

$$(1 + r\Delta)S_t = uP^* + d(1 - P^*) \tag{5}$$

Solving for P^* we get:

$$\begin{aligned} P^* &= \tfrac{(1+r\Delta)S_t - d}{u-d} && (6)\\ &= \tfrac{280(1+0.05(0.25))-260}{320-260} && (7)\\ &= 0.3917 && (8) \end{aligned}$$

(b)

$$\begin{aligned} C_t &= \tfrac{1}{1+r\Delta}E^{P^*}[C_{t+\Delta}] && (9)\\ &= \tfrac{1}{1+r\Delta}E^{P^*}[(S_{t+\Delta} - K)^+] && (10)\\ &= \tfrac{1}{1+r\Delta}(320 - 280)P^* - 0(1 - P^*) && (11) \end{aligned}$$

Solving for C_t we get:

$$C_t = 15.47 \tag{12}$$

(c) In order to normalize by S_t, we consider the following equations for our two assets:

$$\begin{aligned} 1 &= (1 + r\Delta)\phi_u + (1 + r\Delta)\phi_d && (13)\\ S_t &= S^u_{t+\Delta}\phi_u + S^d_{t+\Delta}\phi_d && (14) \end{aligned}$$

In order to normalize, we divide the second equation by S_t:

$$\begin{aligned} 1 &= (1 + r\Delta)\phi_u + (1 + r\Delta)\phi_d && (15)\\ 1 &= \tfrac{S^u_{t+\Delta}}{S_t}\phi_u + \tfrac{S^d_{t+\Delta}}{S_t}\phi_d && (16) \end{aligned}$$

Next, substituting the values of r, Δ and S, we have:

$$\begin{aligned} 1 &= 1.0125\phi_u + 1.0125\phi_d && (17)\\ 1 &= \tfrac{320}{280}\phi_u + \tfrac{260}{280}\phi_d && (18) \end{aligned}$$

Solving the system of equations, we can see that $\phi_u = 0.3868$ and $\phi_d = 0.6008$. Having ϕ_u and ϕ_d, we can then solve for the new normalized probabilities:

$$\begin{aligned} p^u &= (1 + r\Delta)\phi_u && (19)\\ &= 1.0125(0.3868) = 0.3917 && (20)\\ p^d &= (1 + r\Delta)\phi_d && (21)\\ &= 1.0125(0.6008) = 0.6083 && (22) \end{aligned}$$

It should be noted that these probabilities are the same as those computed in part a.
(d) As the discounted stock price is a martingale under the risk - neutral measure calculated by the St normalization, the martingale condition is as follows:

$$\mathbb{E}(\frac{S_t}{1+r\Delta}|I_t) = \frac{1}{1+r\Delta}\{p^u S^u_{t+\Delta} + p^d S^d_{t+\Delta}\} \quad (23)$$

$$= \frac{1}{1.0125}\{0.3917(320) + 0.6083(260)\} \quad (24)$$

$$= 280 \quad (25)$$

$$= S_t \quad (26)$$

(e) As noted earlier, the probabilities in part c are equal to those obtained in part a. Therefore, the call price remains unchanged, that is:

$$C_t = 15.47 \quad (27)$$

(f) No, an option's price is dependent on the martingale measure, or set of risk neutral probabilities chosen. As an example, if we increase the price in the up state, then the risk neutral probabilites would change, as would the value of the option. An option's price, however, is independent of the numeraire or normalization used to compute the risk neutral probabilities.
(g) Both probability measures are constructed such that the asset is a martingale using a particular normalization. As a result, the option price is the same. As stated above, the option price is independent of the normalization used to compute the martingale measure.
(h) The risk premium incorporated in the call price will satisfy:

$$(1 + r + \text{risk premium}) = E^{true}[\frac{C(t+1)}{C(t)}] \quad (28)$$

This risk premium may be extremely difficult to calculate and is generally not used in the real world. That is, computing the probabilities under the true or empirical measure may be impossible. However, the choice of the risk neutral measure instead of the empirical measure overcomes this potential problem. In markets where a risk neutral probability measure does not exist, such as incomplete markets, then we may need to estimate the risk premium and use the empirical measure.

Problem 2

(a) The system of equations that governs asset A, B and C is:

$$\begin{bmatrix} 124 & 71 \\ 83 & 61 \\ 92 & 160 \end{bmatrix} \times \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} A_0 = 100 \\ B_0 = 70 \\ C_0 = 180 \end{bmatrix} \quad (29)$$

Solving this system of equations, yield:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0.1945 \\ 1.0072 \end{bmatrix} \quad (30)$$

Next, remembering that a system is arbitrage free if there exists a solution for ϕ_1, ϕ_2, such that:

1.) $\phi_1 > 0$, $\phi_2 > 0$

2.) $\phi_1 + \phi_2 = \frac{1}{1+r\Delta}$

Since condition 1 is satisfied as ϕ_1 and ϕ_2 are positive, and r is not specified, we conclude that the system would be arbitrage free should r be chosen to satisfy condition 2. If r is chosen differently, however, then the system would not be arbitrage-free.

(b) If there is no solution to the system of equations such that $\phi_1 > 0$ and $\phi_2 > 0$ then there is at least one negative state price. In this case, one would buy the asset which has a non-positive price and be assured of a positive payoff in the future.

(c) There are infinitely many choices of ϕ_1 and ϕ_2 such that the system is arbitrage free. For example, in the case where $\phi_1 = \phi_2 = \frac{1}{2}$, we would have:

$$\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 97.5 \\ 72.0 \\ 126.0 \end{bmatrix} \tag{31}$$

(d) The future's price is set such that the value of the contract, F_0, is worth zero today, that is:

$$0 = \frac{\phi_1}{1+r\Delta}(83 - K) + (1 - \frac{\phi_1}{1+r\Delta})(61 - K) \tag{32}$$

$$0 = p(83 - K) + (1 - p)(61 - K) \tag{33}$$

Solving for K assuming for simplicity that $r = 0$, yields:

$$K = 22\phi_1 + 61 \tag{34}$$

In the case where $\phi_1 = \phi_2 = \frac{1}{2}$ as before, we get $K = 72$.

(e) As the put option on C expires out-of-the-money in state 2, we are only concerned with the probability and payoff in state 1, where it expires in-the-money and will have payoff $(125 - 92)$. The options price will the simply be the discounted payoff multiplied by the probability of state 1, that is:

$$P_0 = (125 - 92)\phi_1 = 33\phi_1 \tag{35}$$

In the case where $\phi_1 = \frac{1}{2}$, we have $P_0 = 16.5$.

Problem 3

(a) The following three equation linear system captures the arbitrage free requirements of the three asset universe:

$$\begin{bmatrix} S^u_{t+\Delta} & S^d_{t+\Delta} \\ C^u_{t+\Delta} & C^d_{t+\Delta} \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} S_t \\ C_t \\ \frac{1}{1+r} \end{bmatrix} \tag{36}$$

(b) A two period binomial tree for S_t is as follows:

$S^{uu}_{t+2\Delta}$

$S^{u}_{t+\Delta}$

$S^{ud}_{t+2\Delta}$

S

$S^{du}_{t+2\Delta}$

$S^{d}_{t+\Delta}$

$S^{dd}_{t+2\Delta}$

(c) The three systems of equations, each consisting of three assets, are as follows: Beginning at $t = 0$, we have:

$$\begin{bmatrix} S^u_{t+\Delta} & S^d_{t+\Delta} \\ C^u_{t+\Delta} & C^d_{t+\Delta} \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} S_t \\ C_t \\ \frac{1}{1+r} \end{bmatrix} \tag{37}$$

Next, at $t = 1$, starting at $S^u_{t+\Delta}$, we have:

$$\begin{bmatrix} S^{uu}_{t+2\Delta} & S^{ud}_{t+2\Delta} \\ C^{uu}_{t+2\Delta} & C^{ud}_{t+2\Delta} \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} \phi^u_1 \\ \phi^u_2 \end{bmatrix} = \begin{bmatrix} S^u_{t+2\Delta} \\ C^u_{t+2\Delta} \\ \frac{1}{1+r} \end{bmatrix} \tag{38}$$

Finally, at $t = 1$, starting at $S^d_{t+\Delta}$, we have:

$$\begin{bmatrix} S^{du}_{t+2\Delta} & S^{dd}_{t+2\Delta} \\ C^{du}_{t+2\Delta} & C^{dd}_{t+2\Delta} \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} \phi^d_1 \\ \phi^d_2 \end{bmatrix} = \begin{bmatrix} S^d_{t+2\Delta} \\ C^d_{t+2\Delta} \\ \frac{1}{1+r} \end{bmatrix} \tag{39}$$

(d) Letting τ be the terminal time of the system. In order for the system to be consistent, all terminal nodes must be arbitrage free. This will by necessity make the intermediate nodes on the tree arbitrage - free.

A state price, ϕ_t, can be interpreted as the discounted risk - neutral probability of that state occurring, $\phi_t = \frac{p_i(0,t)}{B(0,t)}$. Here, $p_i(0, t)$ denotes the risk - neutral probability that state i occurs at time t conditional on some initial node. B(t) represents the discount factor until time t, that is: $B(t) = \frac{1}{(1+r)^t} > 0$.

Since the terminal nodes are arbitrage - free, by assumption, the values of $\phi_i(T)$ satisfy the following properties:

$$\phi_i(\tau) > 0 \tag{40}$$
$$p_i(0, \tau) > 0 \tag{41}$$
$$\sum \phi_i(\tau) = \frac{1}{B(\tau)} \tag{42}$$

We still need to show that ϕ_t also satisfies the two properties above for 0 ¡ t ¡ T and all states i which can occur at time t. As the tree is not recombining, the number of states depends on t. At time t, the sum extends from $i = 1$ to $i = 2^t$.

Since $p_i(0,\tau) > 0$ for all i, we know that all intermediate nodes are reachable from the initial node. If this was not true and an intermediate node was inaccessible, $p_i(0,t) = 0$, then there would also exist a terminal node with $p_i(0,\tau) = 0$, which would contract the properties above. Therefore, we assert that $\phi_i(t) > 0$ for all i and for all t.

The second property requires the sum of the state prices across the number of states be equal the discount factor for a given t. This assertion follows from

$$\sum p_i(0,t) = 1 \tag{43}$$

$$B(t)\sum \phi_i(t) = 1 \tag{44}$$

Therefore, we clearly have:

$$\sum \phi_i(t) = \frac{1}{B(t)} \tag{45}$$

As a result, we observe that the sum of the prices across all states at a given time t is equal to the discount factor. Consequently, all intermediate nodes are free of arbitrage and the system of equations is internally consistent.

Problem 4

(a) In order to find the annualized volatility, we can use the given probability of an up movement. That is, we know that the multiplicative up movement should approximate the up move in the case of lognormal volatility, as shown below:

$$uS = Se^{\sigma\sqrt{\Delta}} \tag{46}$$

Substituting $\Delta = \frac{1}{12}$ we can solve for σ:

$$\sigma = \frac{log(u)}{\sqrt{\Delta}} = 0.48 \tag{47}$$

(b) The following tree shows the 4 period binomial tree for the option in general. We note that u and d are specified such that the tree is recombining.

				u^4S
			u^3S	
		u^2S		u^2S
	uS		uS	
S		S		S
	dS		dS	
		d^2S		d^2S
			d^3S	
				d^4S

For example, if we set $S_0 = 50$ we get:

				87.45
			76.04	
		66.13		66.13
	57.50		57.50	
50		50		50
	43.48		43.48	
		37.80		37.80
			32.88	
				28.59

If we consider a call with a strike of $K = 50$, we get the following tree for the call price:

				37.45
			26.25	
		16.54		16.13
	1.76		7.71	
5.6		3.69		0
	9.82		0	
		0		0
			0	
				0

(c) The arbitrage price of the call at time $t = 0$ is the value on the initial node in the above call tree, that is $C_0 = \$5.60$.

Problem 5

(a) First, we need to compute the up and down multipliers such that we match the given volatility, keeping mind that each time period is one month. This gives:

$$u = e^{0.3\sqrt{\frac{1}{12}}} = 1.09 \tag{48}$$

Conversely, $d = \frac{1}{u} = 0.917$.

Next, we consider the tree for the stock, which is:

			132.26
		121.30	
	111.23		111.23
102		102	
	93.54		93.54
		85.78	
			78.66

Given that $K = 120$, the tree for the call option becomes:

			12.26
		6.135	
	3.06		0
1.53		0	
	0		0
		0	
			0

This can be found by iterating backward through the tree, beginning at option expiration.

In order to construct a portfolio that replicates the option, we do the following at each time step:

Time 0

Position	Portfolio Value
Borrow at the risk - free rate	-\$16.113
Long 0.173 shares	\$17.646
Total Value	\$1.53

Time 1 At $t = 1$, if the price move was down, the portfolio becomes worthless and no action is required until expiration. If the price move was up, the portfolio value increases to \$3.06. In order to continue replicating the option we need to adjust the portfolio in the following way:

Position	Portfolio Value
Borrow at the risk - free rate	-\$32.279
Long 0.3178 shares	\$35.348
Total Value	\$3.06

Time 2 At $t = 2$, if the price move was down, again the portfolio becomes worthless and no action is required until expiration. If the price move was up, the portfolio value increases to \$6.135. In order to continue replicating the option we need to re-balance in the following way:

Position	Portfolio Value
Borrow at the risk - free rate	-\$-64.5754
Long 0.583 shares	\$70.7116
Total Value	\$6.135

Time 3 At $t = 3$, if the price move was down, the portfolio is zero matching the payoff of the option. If the price move was up, the portfolio value increases to \$12.26, which also matches the options payoff. Therefore, we have replicated the options payoff in all states with a dynamic hedging strategy.

(b) The portfolio constructed above replicates the payoff of the option in every future state. As a result, lack of arbitrage requires that the price of the option matches the price of the replicating portfolio. Therefore, the price of the call must equal the initial value of the replicating portfolio, \$1.53.

(c) To hedge the position one must follow the steps detailed in part a, the only distinction being that each position should be multiplied by 100 to hedge the portfolio of 100 calls. For example, at $t = 0$ we would:

Position	Portfolio Value
Borrow at the risk - free rate	-\$1611.33
Long 17.3 shares	\$1764.6
Total Value	\$153.27

Continuing along the tree as in part a, we can build our dynamic hedge.

(d) If the call traded at \$5, we would sell the call and buy the replicating portfolio for the cost of \$1.53. As the payoff's are the same, this would result in a risk-free profit of \$3.47.

Problem 6

(a) If $\mu = r$, then the discounted expected value of $S_{t+\Delta}$ must be equal to S_t. That is:

$$E^p[S_{t+\Delta}|S_t] = S_t(1+r) \tag{49}$$

Further, we know that:

$$S_{t+\Delta} = S_t + rS_t + \sigma S_t \epsilon_t \tag{50}$$

Since ϵ takes on only values of 1 and -1, we can re-write this as:

$$E^p[S_{t+\Delta}|S_t] = S_t + rS_t + S_t\{\sigma p - \sigma(1-p)\} \tag{51}$$

Plugging in the right hand side of (**??**) we get:

$$S_t(1+r) = \quad S_t + rS_t + S_t\{\sigma p - \sigma(1-p)\} \tag{52}$$
$$0 = \quad \sigma p - \sigma(1-p) \tag{53}$$
$$2\sigma p = \quad \sigma \tag{54}$$

Therefore, $p = \frac{1}{2}$.

(b) No, if $p = \frac{1}{3}$ then the discounted stock price is not a martingale as we found in part a that this only occurred when $p = \frac{1}{2}$.

This can be verified by plugging in $p = \frac{1}{3}$ into the expectation for $E^p[S_{t+\Delta}|S_t]$ as follows:

$$E^p[S_{t+\Delta}|S_t] = S_t + rS_t + S_t\{\frac{1}{3}\sigma - (1 - \frac{1}{3})\sigma\} \tag{55}$$

(c) With the drift including a risk premium, the stock price no longer is a martingale under the risk - neutral measure. Hence p represents the physical, empirical, statistical, or true measure. Correspondingly, ϵ would represent the volatility under the same physical measure.

(d) We can only determine the value of p under the physical measure using statistical techniques. It cannot be calculated using a replication argument as in the case of the risk-neutral measure. Instead, we would need to infer probabilities from observed stock prices.

Problem 7

(a) We want to choose Δ so that there are 5 evenly spaced time intervals. Since there are 200 days in the interval, we have: $\Delta = \frac{\frac{200}{5}}{365}$, or 40 days.
(b) We want annualized volatility to equal 12%. Therefore:

$$
\begin{aligned}
u = & \; e^{0.12\sqrt{\frac{40}{365}}} && (56)\\
= & \; 1.0405 && (57)
\end{aligned}
$$

We would then set $d = \frac{1}{u} = 0.9611$.
(c) The implied probability of an up move can be computed as:

$$
\begin{aligned}
p = & \; \frac{1+r\Delta-d}{u-d} && (58)\\
= & \; \frac{1+0.0066-0.9611}{1.0405-0.9611} && (59)\\
= & \; 0.5727 && (60)
\end{aligned}
$$

(d) The tree for the stock prices can be found by multiplying by u and d, at each of the 5 time steps. This leads to the following tree:

					121.9725
				117.2221	
			112.6567		112.6567
		108.2692		108.2692	
	96.1054		104.0525		104.0525
100		100		100	
	104.0525		96.1054		96.1054
		92.3624		92.3624	
			88.7652		88.7652
				85.3081	
					81.9857

(e) In order to compute the call values at each node, we need to iterate backwards through the tree. At the last time step, $t = 5$, the value of the option is simply the call payoff, $(S - K)^+$.

Then at previous time steps we can price each node using p and the values of S_t in the next time period. For example, at time $t = 4$, for the upper most node we have:

$$
\begin{aligned}
C &= \frac{21.9725p - 12.6567(1-p)}{1+r\Delta} && (61)\\
&= 17.8743 && (62)
\end{aligned}
$$

All additional nodes can be obtained using this methodology. This leads to the following tree of call prices:

					21.9725
				17.8743	
			13.9570		12.6567
		10.5113		8.9215	
	7.6793		6.0548		4.0525
5.4703		4.0018		2.3057	
	2.5937		1.3118		0
		0.7464		0	
			0		0
				0	
					0

Problem 8

First, we can find u and d in order to match the volatility on the asset.

$$u = e^{\sigma\sqrt{\Delta}} = e^{(0.10)(1)} = 1.1052 \qquad (63)$$

In order to make the tree recombining, we set $d = \frac{1}{u} = 0.9048$.

This gives the following tree of asset prices:

			67.4929
		61.0701	
	55.2585		55.2585
50		50	
	45.2418		45.2418
		40.9365	
			37.0409

The risk-neutral probability of an up movement, p^u can be computed as:

$$p^u = \frac{(1 + r\Delta) - d}{u - d} = \frac{(1.01 - 0.9048)}{(1.1052 - 0.9048)} = 0.5249 \tag{64}$$

Now we can solve the binomial tree backward by iterating backward from the digital option expiration:

			1
		0.99	
	0.759		1
0.5215		0.5197	
	0.2701		0
		0	
			0

Therefore, the price of the digital option today is \$0.5215.

Problem 9

```
S=50;
K=55;
r=0.01;
sigma=0.1;
T=3;
n=3;
delta=T/n;
u=exp(sigma*delta);
d=1/u;
p=((1+r*delta)-d)/(u-d);
S1=zeros(1,n+1);
C=zeros(n+1,n+1);
for i=1:n+1
S1(1,i)=S*power(u,i-1)*power(d,n+1-i);
end
for i=1:n+1
    if(S1(1,i)>55)
   C(n+1,i)= 1;
    end
end
for i=n:-1:1
    for j=1:1:i
       C(i,j)=(C(i+1,j)*(1-p)+C(i+1,j+1)*p)*exp(-r*delta);
    end
end
C(1,1)
```