Extended Solutions for Instructors

for the Book

An Introduction to Partial Differential Equations

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Chapter 1

1.1 (a) Write $u_x = af'$, $u_y = bf'$. Therefore, a and b can be any constants such that $a + 3b = 0$.

1.3 (a) Integrate the first equation with respect to x to get $u(x, y) = x^3y + xy + F(y)$, where $F(y)$ is still undetermined. Differentiate this solution with respect to y and compare to the equation for u_y to conclude that F is a constant function. Finally, using the initial condition $u(0, 0) = 0$, obtain $F(y) = 0$.

(b) The compatibility condition $u_{xy} = u_{yx}$ does not hold. Therefore, there does not exist a function u satisfying both equations.

1.5 Differentiate $u = f(x + p(u)t)$ by t:

$$u_t = f'(x + p(u)t)\,(p(u) + tp'(u)u_t) \Rightarrow (1 - tf'p')u_t = pf'.$$

The expression $1 - tf'p'$ cannot vanish on a t-interval, otherwise, $pf' = 0$ there. But this is a contradiction, since if either p or f' vanishes in this interval, then $tf'p' = 0$ there. Therefore, we can write

$$u_t = \frac{pf'}{1 - tp'f'}.$$

Similarly,

$$u_x = \frac{f'}{1 - tp'f'},$$

and the claim follows.

(a) Substituting $p = k$ (for a constant k) into $u = f(x + p(u)t)$ provides the explicit solution $u(x, t) = f(x + kt)$, where f is any differentiable function.

(b), (c) Equations (b) and (c) do not have such explicit solutions. Nevertheless, if we select $f(s) = s$, we obtain that (b) is solved by $u = x + ut$ that can be written explicitly as $u(x, t) = x/(1 - t)$, which is well-defined if $t \neq 1$.

1.7 (a) Substitute $v(s, t) = u(x, y)$, and use the chain rule to get

$$u_x = v_s + v_t, \qquad u_y = -v_t,$$

and

$$u_{xx} = v_{ss} + v_{tt} + 2v_{st}, \qquad u_{xy} = -v_{tt} - v_{st}, \qquad u_{yy} = v_{tt}.$$

Therefore, $u_{xx} + 2u_{xy} + u_{yy} = v_{ss}$, and the equation becomes $v_{ss} = 0$.

(b) The general solution is $v = f(t) + sg(t)$, where f and g are arbitrary differentiable functions. Thus, $u(x, y) = f(x - y) + xg(x - y)$ is the desired general solution in the (x, y) coordinates.

(c) Proceeding similarly, we obtain for $v(s,t) = u(x,y)$:

$$\begin{aligned} u_x &= v_s + 2v_t, & u_y &= v_s, \\ u_{xx} &= v_{ss} + 4v_{tt} + 4v_{st}, & u_{yy} &= v_{tt}, & u_{xy} &= v_{ss} + 2v_{st}. \end{aligned}$$

Hence, $u_{xx} - 2u_{xy} + 5u_{yy} = 4(v_{ss} + v_{tt})$, and the equation is $v_{ss} + v_{tt} = 0$.