An Introduction to Mathematical Biology
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Answers to Selected Exercises and Supplementary Exercises

Chapter 1 Answers (* denotes supplementary exercises)

1 (b) first, nonlinear, autonomous

(d) second, nonlinear, nonautonomous

3 (b) $x_t = c_1 + c_2(-1)^t + c_3(-5)^t$

(d) $x_t = \left(\dfrac{2\sqrt{3}}{3}\right)^t \left[c_1 \sin\left(\dfrac{\pi t}{6}\right) + c_2 \cos\left(\dfrac{\pi t}{6}\right)\right]$

4 (a) Solution to 3 (d) $x_t = \left(\dfrac{2\sqrt{3}}{3}\right)^t \sqrt{3} \sin\left(\dfrac{\pi t}{6}\right)$

(b) Solution to 3 (b) $x_t = \dfrac{5}{12} + \dfrac{5}{8}(-1)^{t+1} + \dfrac{5}{24}(-5)^t$

5 (b) $x_t = c_1 2^t + c_2(-2)^t - 1 - 2t$

(d)* Solve $x_{t+1} - 5x_t = 5^{t+1}$. Solution: $x_t = c_1 5^t + t5^t$

(e)* Solve $x_{t+1} - x_t = 1 - 4t$

7 (a) $x_t = \dfrac{1}{\sqrt{5}}\lambda_1^{t+1} - \dfrac{1}{\sqrt{5}}\lambda_2^{t+1}$, where λ_1 and λ_2 are the roots of the characteristic equation, $\lambda_1 > \lambda_2$.

(b) Using the solution in (a) and the fact that $\lambda_1 > |\lambda_2|$ leads to $\lim_{t\to\infty} \dfrac{x_{t+1}}{x_t} = \lambda_1$.

(c)* Find the number of pairs of rabbits after one year ($t = 12$); after 5 years. The circumference of the earth is 24,902 miles. If the pairs of rabbits are lined end to end and they measure one foot in length, then, after 5 years, the pairs of rabbits would encircle the earth about 19,050 times.

8 Find the general solution, $x_t = c_1\lambda_1^t + c_2\lambda_2^t$, where $\lambda_1 > |\lambda_2|$. Then show x_{t+1}/x_t approaches λ_1 as $t \to \infty$.

10 (2) $Y(t+1) = BY(t)$, where $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & 0 & -a & 0 \end{pmatrix}$. Show $\det(B - \lambda I) = \lambda^4 + a\lambda^2 + b$.

11 (b) $Y(t+1) = AY(t)$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & -5 \end{pmatrix}$

14 $ab < 1$

15 (a) $X(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(-3)^t \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} + c_3(2)^t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

16 $A^t = \begin{pmatrix} 1 & 2^t - 1 \\ 0 & 2^t \end{pmatrix}$, $X(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 2^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

17 (a) $\lambda^3 - \dfrac{3a^2}{4}\lambda - \dfrac{a^3}{4} = 0$, $\lambda = a, \pm\dfrac{a}{2}$. $R_0 = \dfrac{a^2}{4}(a+3)$.

*Show $R_0 > 1$ iff $a > 1$.

(b) Apply Theorem 1.7.

19 (b) M_2 is reducible and imprimitive.

20 (a) Apply Theorem 1.7.

(b) $R_0 = s_1 b_2 + s_1 s_2 b_3 = 1 + 2s_2$ is never less than one and is greater than one when $s_2 > 0$.

*Let $b_2 = 2$, $b_3 = 4$ and $s_2 = 2$, then do part (b).

(c) $R_0 = 1 + f_2 p_1$.

21 Apply Theorem 1.5 or Theorem 1.7.

22 (b) $0 \le \alpha < 1$, $0 < \alpha_2 < 1$, $0 < \alpha_3 \le 1$, and $\gamma, \sigma > 0$.

(c) $R_0 = \alpha_1 \sigma\gamma + \alpha_2 \sigma^2 \gamma(1-\alpha_1) + \alpha_3 \sigma^3 (1-\alpha_1)(1-\alpha_2)$.

23 (b) $L^6 > 0$.

(c) In Example 1.21 when s_1 is increased to one, $\lambda \approx 0.965$. When p_7 is increased to 0.95, $\lambda \approx 1.002$.

25 (a) $\lambda_1 = \dfrac{1 - f + \sqrt{(1-f)^2 + 4\gamma f}}{2} > 0$, $\lambda_2 = \dfrac{1 - f - \sqrt{(1-f)^2 + 4\gamma f}}{2} < 0$.

(c) $\lim_{t\to\infty} R_t = \dfrac{R_0 + M_0}{1+f}$.

26* Suppose A is an $n \times n$ matrix with a zero row or a zero column. Show that A is reducible.

27* Suppose $A = (a_{ij})$ is an $n \times n$ irreducible matrix and $A = D + B$, where D is a diagonal matrix whose diagonal entries are equal to those of A, $D = \text{diag}(a_{11}, \ldots, a_{nn})$. Show that B is irreducible, i.e., the diagonal elements of an irreducible matrix do not affect irreducibility.

28* Suppose $A = (a_{ij})$ is an $n \times n$ nonnegative, irreducible matrix with one positive row, i.e., $a_{ij} > 0$ for some i and $j = 1, \ldots, n$. Show that A^2 has at least two positive rows, and in general A^k, $k \leq n$ has at least k positive rows. Conclude that A is primitive. (*Hint:* Use the results in Exercises 25 and 26.)