An Application of Residues

If f is a complex function defined at least for all z on the half-line $[0, \infty)$, then the Laplace transform of f is

$$\mathcal{L}[f](z) = \int_0^\infty e^{-zt} f(t)\, dt$$

for all z such that this integral converges. If $\mathcal{L}[f] = F$ then we write $f = \mathcal{L}^{-1}[F]$ for the inverse transform.

In Chapter Three we saw techniques for manipulating $\mathcal{L}$ and $\mathcal{L}^{-1}$. The following theorem provides a formula for the inverse Laplace transform of $F(s)$ in terms of residues of $e^{tz}F(z)$.

Theorem (Inverse Laplace Transform) Let F be differentiable for all z except for poles $z_1, \cdots, z_n$. Suppose for some real number σ, F is differentiable for all z with $\mathrm{Re}(z) \geq \sigma$. Suppose there are numbers M and R such that

$$|zF(z)| \leq M \text{ for } |z| \geq R.$$

Then

$$\mathcal{L}^{-1}[F(s)](t) = \sum_{j=1}^{n} \mathrm{Res}(e^{tz}F(z), z_j). \diamond$$

Because F is differentiable for $\mathrm{Re}(z) \geq \sigma$, $F'(z)$ exists at least for z to the right of the vertical line $x = \sigma$. The condition that $|zF(z)| \leq M$ for $|z| \geq R$ means that $zF(z)$ is bounded for all z on and outside some sufficiently large circle. This condition is satisfied by any rational function $p(z)/q(z)$ if the degree of $q(z)$ exceeds that of $p(z)$.

The theorem can be proved using an alternative version of the Cauchy integral given the module A Variation on Cauchy's Integral Formula. Following a sketch of the argument and two examples computing inverse Laplace transforms of functions, we will use the theorem to solve a problem of heat diffusion in a homogeneous solid cylinder.

Begin by writing

$$F(s) = -\frac{1}{2\pi i} \lim_{b\to\infty} \int_{\sigma-ib}^{\sigma+ib} \frac{F(z)}{z-s}\, dz.$$

Interchange $\mathcal{L}^{-1}$ and the integral (this is justified by hypotheses of the theorem) to compute

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)](t) &= \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\sigma-ib}^{\sigma+ib} \mathcal{L}^{-1}\left[\frac{F(z)}{s-z}\right] dz \\ &= \frac{1}{2\pi i} \lim_{b\to\infty} \int_{\sigma-ib}^{\sigma+ib} e^{tz}F(z)\, dz \\ &= \sum_p \mathrm{Res}(e^{tz}F(z), p), \end{aligned}$$

with this summation extending over all of the poles of $e^{tz}F(z)$. σ is chosen so that all of these poles are to the right of σ.

Example 1 Let a be a positive number. We will find the inverse Laplace transform of $F(z) = 1/(a^2 + z^2)$. We have several ways of doing this, but we want to illustrate the residue method.

$F(z)$ has simple poles at $\pm ai$. Write

$$e^{tz}F(z) = \frac{e^{tz}}{(z+ai)(z-ai)}$$

to compute the residues:

$$\mathrm{Res}(e^{tz}F(z), ai) = \frac{e^{ai}}{2ai} \text{ and } \mathrm{Res}(e^{tz}F(z), -ai) = \frac{e^{-ai}}{-2ai}.$$

Then

$$\mathcal{L}^{-1}[F](t) = \frac{1}{2ai}\left(e^{ai} - e^{-ai}\right) = \frac{1}{a}\sin(at). \diamond$$

Example 2 Let

$$F(z) = \frac{1}{(z^2-4)(z-1)^2}.$$

Then $F(z)$ has simple poles at ± 2 and a double pole at 1. Compute

$$\mathrm{Res}(e^{tz}F(z), 2) = \lim_{z\to 2} \frac{e^{tz}}{(z+2)(z-1)^2} = \frac{1}{4}e^{2t},$$

$$\mathrm{Res}(e^{tz}F(z), -2) = \lim_{z\to -2} \frac{e^{tz}}{(z-2)(z-1)^2} = -\frac{1}{36}e^{-2t},$$

and

$$\begin{aligned}\mathrm{Res}(e^{tz}F(z), 1) &= \lim_{z\to 1} \frac{d}{dz}((z^2-4)^{-1}e^{tz}) \\ &= \lim_{z\to 1}(-2z(z^2-4)^{-2}e^{tz} + te^{tz}(z^2-4)^{-1}) \\ &= -\frac{1}{3}te^t - \frac{2}{9}e^t.\end{aligned}$$

Then

$$\mathcal{L}^{-1}[F](t) = \frac{1}{4}e^{2t} - \frac{1}{36}e^{-2t} - \frac{1}{3}te^t - \frac{2}{9}e^t. \diamond$$

Problems

In each of Problems 1–10 use the theorem to find the inverse Laplace transform of the function.

1.

$$\frac{z^2}{(z-2)^2}$$

2.
$$\frac{z+3}{(z^3-1)(z+2)}$$

3.
$$\frac{z}{z^2+9}$$

4.
$$\frac{1}{(z+3)^2}$$

5.
$$\frac{1}{(z-5)^2(z+4)}$$

6.
$$\frac{1}{(z^2+9)(z-2)^2}$$

7.
$$\frac{1}{(z+5)^3}$$

8.
$$\frac{1}{z^3+8}$$

9.
$$\frac{1}{z^4+1}$$

10.
$$\frac{1}{e^z(z-1)}$$

11. How does the formula for $f(t)$ given in the theorem relate to Heaviside's formula?

Diffusion in a Cylinder

We will find the temperature distribution function for a homogeneous solid cylinder of radius R centered along the z - axis. This problem was solved in Section 8.3.2 using separation of variables.

We will assume facts about the Bessel functions $J_0(x)$ and $J_1(x)$, and also the modified Bessel function $I_0(x) = J_0(ix)$. Using these, we will use the Laplace transform to obtain the temperature distribution function.

Assuming angular independence, we will use cylindrical coordinates. The boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \text{ for } 0 \leq r \leq R, t > 0$$
$$u(r,0) = 0, u(R,t) = T_0.$$

Apply the Laplace transform with respect to t to this problem to obtain

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - sU(r,s) = 0.$$

This is a modified Bessel equation of order zero. A solution that isbounded at $r = 0$, the center of the cylinder, is given by

$$U(r,s) = cI_0(\sqrt{s}r).$$

Transform the condition $u(R,t) = T_0$ to obtain $U(R,s) = T_0/s$. Then

$$U(R,s) = cI_0(\sqrt{s}R) = \frac{T_0}{s}.$$

This means that

$$c = \frac{T_0}{sI_0(\sqrt{s}R)}.$$

The transform of the solution is therefore

$$U(r,s) = \frac{T_0 I_0(\sqrt{s}r)}{sI_0(\sqrt{s}R)}.$$

We must invert this to obtain $u(r,t)$. We need the singularities of

$$e^{tz}U(r,z) = e^{tz}\frac{I_0(\sqrt{z}r)}{zI_0(\sqrt{z}R)}.$$

Singularities of $e^{tz}U(r,z)$ occur at zeros of the denominator. There is also a simple pole at $z = 0$ because $I_0(0) = 1 \neq 0$. Further,

$$I_0(\sqrt{z}R) = J_0(i\sqrt{z}R) = 0$$

if $i\sqrt{z}R$ is a zero of J_0. These zeros are real, simple and nonzero, and if the positive zeros are labeled $j_1, j_2, \cdots$, then all the zeros are $\pm j_1, \pm j_2, \cdots$. Therefore $I_0(\sqrt{z}R) = 0$ if $\sqrt{z}R = \pm i j_n$ for some n. Then

$$z = -j_n^2/R^2.$$

Therefore $e^{tz}U(r,z)$ has simple poles at 0 and $-j_n^2/R^2$ for $n = 1, 2, \cdots$. Inverting $U(r,s)$ yields the solution

$$u(r,t) = \mathrm{Res}(e^{tz}U(r,z), z = 0) + \sum_{n=1}^{\infty} \mathrm{Res}(e^{tz}F(z), z = -j_n^2/R^2).$$

All that is left is to compute these residues. First,

$$\begin{aligned}
\mathrm{Res}(e^{tz}U(r,z), 0) &= \lim_{z\to 0} ze^{tz}\frac{I_0(\sqrt{z}r)}{zI_0(\sqrt{z}R)} \\
&= \lim_{z\to 0} e^{tz}\frac{I_0(\sqrt{z}r)}{I_0(\sqrt{z}R)} \\
&= \frac{I_0(0)}{I_0(0)} = 1.
\end{aligned}$$

For the residues at the other poles, use the fact that these poles are simple zeros of the denominator of a function of the form $g(z)/h(z)$, with

$$g(z) = \frac{e^{tz} I_0(\sqrt{z}r)}{z} \text{ and } h(z) = I_0(\sqrt{z}R).$$

Then

$$\begin{aligned}
&\operatorname{Res}(g(z)/h(z), -j_n^2/R^2) \\
&= \frac{e^{-j_n^2 t/R^2} I_0(j_n ri/R)}{-j_n^2/R^2} \left[\frac{1}{\frac{d}{dz} I_0(\sqrt{z}R)}\right]_{-j_n^2/R^2} \\
&= \frac{e^{-j_n^2 t/R^2} I_0(j_n ri/R)}{-j_n^2/R^2} \left[\frac{2\sqrt{z}}{R I_0'(\sqrt{z}R)}\right]_{z=-j_n^2/R^2} \\
&= \frac{-2Ri}{j_n} e^{-j_n^2 t/R^2} \frac{I_0(j_n ri/R)}{I_0'(j_n i)}.
\end{aligned}$$

Now use the facts that

$$I_0'(z) = i J_0'(iz)$$

and

$$J_0'(z) = -J_1(z) = J_1(-z)$$

to obtain

$$\begin{aligned}
&\operatorname{Res}(g(z)/h(z), j_n) \\
&= \frac{-2R}{j_n} \frac{J_0(j_n r/R)}{J_1(j_n)} e^{-j_n^2 t/R^2}.
\end{aligned}$$

The solution is

$$u(r,t) = T_0 \left(1 - 2\sum_{n=1}^{\infty} \frac{-2R}{j_n} \frac{J_0(j_n r/R)}{J_1(j_n)} e^{-j_n^2 t/R^2}\right).$$

Problem

1. Relate the solution of this diffusion problem for the cylinder with the solution obtained by separation of variables in Chapter Eight.