Competing Species Population Models

In a competing species model, two populations compete with each other for the same resource. We will develop two models for analyzing the behavior of these populations.

A Simple Competing Species Model

A relatively simple competing species model can be obtained by assuming that an increase in either population causes a reduction in the resource for both populations, hence should contribute to a decline of both populations. The effect of this interaction is modeled as the product of the populations, leading to the system

$$x'(t) = ax(t) - bx(t)y(t),$$
$$y'(t) = ky(t) - cx(t)y(t),$$

in which $x(t)$ and $y(t)$ are the populations at time t and the coefficients are positive constants. In matrix form,

$$\mathbf{X}' = \begin{pmatrix} ax - bxy \\ ky - cxy \end{pmatrix}.$$

The critical points are $(0, 0)$ and $(k/c, a/b)$.

Example 1 We will analyze the competing species model

$$\mathbf{X}' = \begin{pmatrix} 2x - 0.3xy \\ 4y - 0.7xy \end{pmatrix}.$$

The critical points are $(0, 0)$ and $(\frac{40}{7}, \frac{20}{3})$.

First look at $(0, 0)$. The matrix of the linearized system is

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

with eigenvalues $2, 4$. The origin is an unstable nodal source for this system. Figure 1 is a phase portrait showing trajectories moving out of and away from the origin.

At the other critical point, the matrix of the linearized system is

$$\mathbf{A}_{(40/7,20/3)} = \begin{pmatrix} 0 & -\frac{12}{7} \\ -\frac{14}{3} & 0 \end{pmatrix},$$

with eigenvalues $\pm 2\sqrt{2}$. This critical point is an unstable saddle point of the linearized model, hence also of the original competing species model. Figure 2 is a phase portrait of this competing species system, showing some trajectories

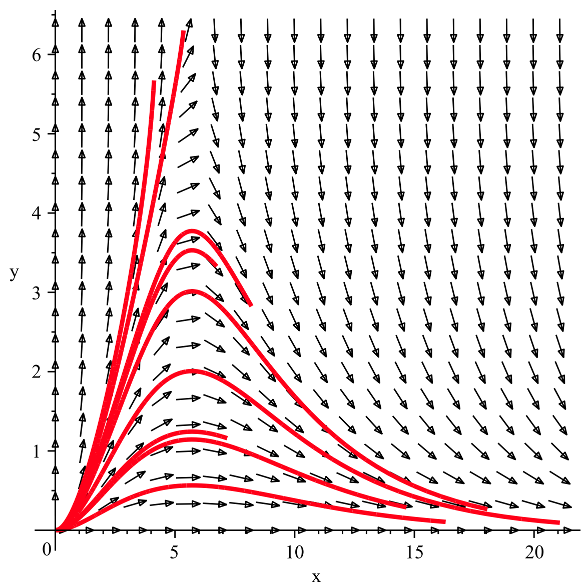

Figure 1: Trajectories near the origin in Example 1.

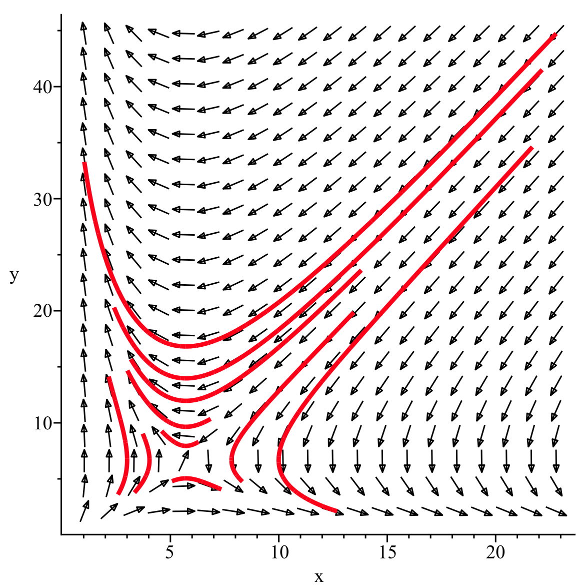

Figure 2: Trajectories near $(40/7, 20/3)$ in Example 1.

near the critical point $(\frac{40}{7}, \frac{20}{3})$. If the initial conditions are "near" this critical point, say

$$x(0) = 15, y(0) = 5$$

then the $x-$ population prospers and grows indefinitely as t increases, while the $y-$ population dies out. $\diamond$

This first competing species model can be reduced to a single differential equation in terms of x and y. For the system of Example 1, first write

$$\frac{dy}{dx} = \frac{4y - 0.7xy}{2x - 0.3xy} = \frac{y(40 - 7x)}{x(20 - 3y)},$$

and the variables separate to yield

$$\frac{20 - 3y}{y}\, dy = \frac{40 - 7x}{x}\, dx.$$

Integrate and rearrange terms to obtain the implicitly defined general solution

$$\frac{y^{20}}{x^{40}} e^{7x - 3y} = k.$$

Initial conditions $x(0)$ and $y(0)$ determine k and the trajectory through this point.

In general, for this competing species model, the asymptotes of the trajectories separate the first quadrant into four regions, labeled *1, 2, 3, 4* in Figure 3. If the initial population point $(x(0), y(0))$ is in region *1* or *4*, then the $x-$ population will increase in time while the $y-$ population shrinks to zero as $t \to \infty$. If the initial population point is in region *2* or *3*, then the $y-$ population survives and the $x-$ population becomes extinct.

An Extended Competing Species Model

The competing species model just considered leaves no room for compromise or diplomacy - one species survives, the other dies. We would like a more sophisticated model allowing a greater variety of outcomes (as occur in the real world).

One way to do this is to add a term to each equation that accounts for factors within each population that might limit its growth,independent of interactions with the other population, which are already taken into account by the xy terms. Assuming that such factors are proportional to the square of the population, we have the competing species model

$$\mathbf{X}' = \begin{pmatrix} Gx - Dx^2 - Cxy, \\ gy - dy^2 - cxy \end{pmatrix}.$$

Unlike the previous model, these equations do not admit exponential growth or decay in the absence of the other population. Think of D and d as the internal growth-limiting factors, while C and c are the competition factors.

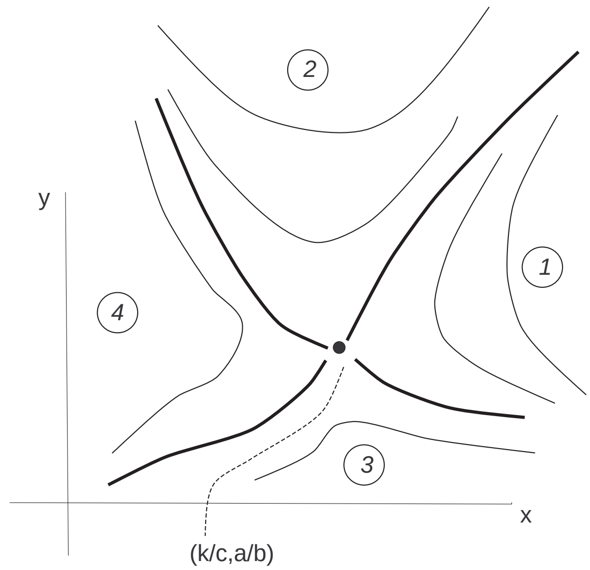

Figure 3: Phase portrait of a simple competing species model.

We will look at two examples before making a general analysis of this model.

Example 2 The model

$$\mathbf{X}' = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3}{5}x - \frac{1}{6}x^2 - \frac{1}{4}xy \\ y - \frac{1}{4}y^2 - \frac{1}{2}xy \end{pmatrix}$$

has four critical points in the first quadrant:

$$(0,0), (0,4), (\frac{18}{5}, 0), \text{ and } (\frac{6}{5}, \frac{8}{5}).$$

For behavior of trajectories near $(0,0)$, look at

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & 1 \end{pmatrix}.$$

This has eigenvalues 1 and $\frac{3}{5}$ so the origin is an unstable nodal source.

For $(0,4)$, form

$$\mathbf{A}_{(0,4)} = \begin{pmatrix} -\frac{2}{5} & 0 \\ -2 & -1 \end{pmatrix}.$$

This has eigenvalues $-\frac{2}{5}, -1$, so $(0,4)$ is an asymptotically stable nodal sink. Solutions beginning "near" $(0,4)$ tend toward $(0,4)$ as t increases. Because we are only interested in integer values of $x(t)$ and $y(t)$ as population counts, this

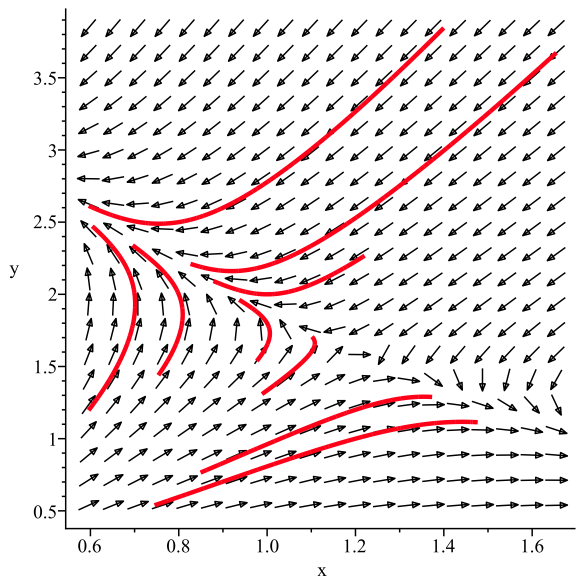

Figure 4: Phase portrait in Example 2.

is the obvious result that, if $x(0) = 0$, and $y(0) = 4$, then the populations tend toward $x = 0, y = 4$ as t increases.

For $(\frac{18}{5}, 0)$, compute

$$\mathbf{A}_{(18/5,0)} = \begin{pmatrix} -\frac{3}{5} & -\frac{9}{10} \\ 0 & -\frac{4}{5} \end{pmatrix}.$$

This has eigenvalues $-\frac{3}{5}, -\frac{4}{5}$ and this critical point is also an asymptotically stable nodal sink.

Finally, for the critical point $(\frac{6}{5}, \frac{8}{5})$, we find that

$$\mathbf{A}_{(6/5,8/5)} = \begin{pmatrix} -\frac{1}{5} & -\frac{3}{10} \\ -\frac{4}{5} & -\frac{7}{5} \end{pmatrix}$$

with eigenvalues $-\frac{4}{5}, \frac{1}{5}$. This critical point is an unstable saddle point.

Figure 4 shows a phase portrait for this system. Trajectories flow outward from the origin and, depending on where they start at time zero, flow toward the critical point $(0, 4)$ (so y survives and x becomes extinct), or toward $(\frac{18}{5}, 0)$ (so x survives and y becomes extinct). Some of these trajectories also suggest the behavior of the system near the saddle point $(\frac{6}{5}, \frac{8}{5})$. Because this point is unstable, it is possible to find initial points close to this critical point from which the $x-$ species survives and the $y-$ species does not, or from which the $y-$ species survives and the $x-$ species dies out. $\diamond$

Example 3 Contrast the outcomes of Example 2 with those of the model

$$\mathbf{X}' = \begin{pmatrix} x(3 - x - \frac{1}{4}y) \\ y(2 - \frac{1}{2}y - \frac{1}{6}x) \end{pmatrix}.$$

The critical points are

$$(0,0), (0,4), (3,0) \text{ and } (\frac{24}{11}, \frac{36}{11}).$$

Analyze each critical point as follows.

For $(0,0)$, compute

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

with eigenvalues $3, 2$. The origin is an unstable nodal source.

For $(0,4)$,

$$\mathbf{A}_{(0,4)} = \begin{pmatrix} 2 & 0 \\ -\frac{2}{3} & -2 \end{pmatrix},$$

with eigenvalues $-2, 2$. Then $(0,4)$ is an unstable saddle point.

For $(3,0)$, we have

$$\mathbf{A}_{(3,0)} = \begin{pmatrix} -3 & -\frac{3}{4} \\ 0 & \frac{3}{2} \end{pmatrix},$$

with eigenvalues $-3, \frac{3}{2}$. Then $(3,0)$ is also an unstable saddle point.

Finally, at $(\frac{24}{11}, \frac{36}{11})$,

$$\mathbf{A}_{(24/11,36/11)} = \begin{pmatrix} -\frac{24}{11} & -\frac{6}{11} \\ -\frac{6}{11} & -\frac{18}{11} \end{pmatrix},$$

with eigenvalues $\frac{-21 \pm 3\sqrt{5}}{11}$. These numbers are both negative, so this critical point is an asymptotically stable nodal sink. Figure 5 near $(6/5, 8/5)$ shows a phase portrait for this system with some trajectories near this critical point. $\diamond$

It is possible to carry out a general analysis for the competing species model

$$\mathbf{X}' = \begin{pmatrix} Gx - Dx^2 - Cxy, \\ gy - dy^2 - cxy \end{pmatrix}.$$

First look at the critical points from a geometric point of view. These critical points are simultaneous solutions of

$$\begin{aligned} x(G - Dx - Cy) &= 0, \\ y(g - cx - dy) &= 0, \end{aligned}$$

Solutions for x and y are coordinates of points of intersection of pairs of lines, namely the $x-$, axis, the $y-$ axis, and the lines

$$\begin{aligned} Dx + Cy &= G \\ cx + dy &= g. \end{aligned}$$

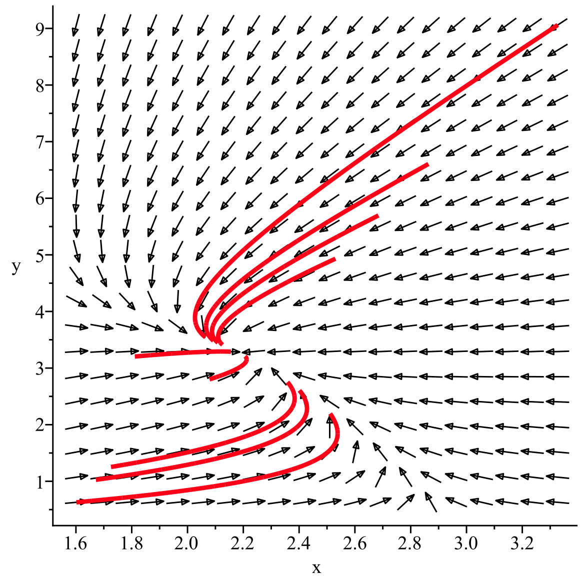

Figure 5: Phase portrait in Example 3.

Figure 6 shows the four possible relative positions of the last two lines in the first octant. The critical points are

$$(0,0), (\frac{G}{D},0), (0,\frac{g}{d}), \text{ and } \left(\frac{Gd-Cg}{dD-cD}, \frac{Dg-cG}{cD-cD}\right).$$

Now consider the critical points in turn.

$(0,0)$ - Write the system as

$$\mathbf{A}_{(0,0)} = \begin{pmatrix} G & 0 \\ 0 & K \end{pmatrix} \mathbf{X} + \begin{pmatrix} -Dx^2 - Cxy \\ -dy^2 - cxy \end{pmatrix}.$$

The matrix of the linear part has eigenvalues G, g, both positive. If these are unequal then the origin is an unstable node. If $G = g$ then the origin is an unstable node.

$(\frac{G}{D}, 0)$ - Now compute

$$\mathbf{A}_{(G/D,0)} = \begin{pmatrix} -G & -CG/D \\ 0 & -cG/D \end{pmatrix}$$

with eigenvalues $-g, g - \frac{cG}{D}$. Certainly $-G < 0$. If $\frac{g}{c} > \frac{G}{D}$ then the second eigenvalue is positive and this critical point is an unstable node. If $\frac{g}{c} < \frac{G}{D}$ then both eigenvalues are negative. If they are distinct, then the critical point is an asymptotically stable node. If they are equal, then the almost linear system has an asymptotically stable node or spiral point at $(\frac{G}{D}, 0)$.

$(0, \frac{g}{d})$ - Now we find that

$$\mathbf{A}_{(0,g/d)} = \begin{pmatrix} G - \frac{Cg}{d} & 0 \\ -\frac{cg}{d} & -g \end{pmatrix},$$

with eigenvalues $-g, G - \frac{Cg}{d}$. If $G/C > g/d$ then the second eigenvalue is positive and $(0, g/d)$ is an unstable node. If $G/C < g/d$ then both eigenvalues are negative. If these eigenvalues are distinct, then $(0, g/d)$ is an asymptotically stable node. If these eigenvalues are equal, then the almost linear system as an asymptotically stable node or spiral point at $(0, g/d)$.

$((Gd - Cg)/(dD - cC), (Dg - cG)/(dD - cD))$ - Denote this point $(\tilde{x}, \tilde{y})$. This is the point of intersection of the lines

$$Dx + Cy = G, cx + dy = g.$$

In the present context, look at the case that the point of intersection falls in the first quadrant (Figures 6(3) and Figure 6(4)). In Figure 6(3),

$$\frac{G}{D} > \frac{g}{c} \text{ and } \frac{g}{d} > \frac{G}{C},$$

while in Figure 6(4),

$$\frac{g}{c} > \frac{G}{D} \text{ and } \frac{G}{C} > \frac{g}{d}.$$

Now compute

$$\mathbf{A}_{(\tilde{x},\tilde{y})} = \begin{pmatrix} G - 2D\tilde{x} - C\tilde{y} & -C\tilde{x} \\ -c\tilde{y} & g - 2d\tilde{y} - cx\tilde{x} \end{pmatrix}.$$

But recall that

$$G = D\tilde{x} + C\tilde{y} \text{ and } g = c\tilde{x} + d\tilde{y}.$$

Then

$$\mathbf{A}_{(\tilde{x},\tilde{y})} = \begin{pmatrix} -D\tilde{x} & -C\tilde{x} \\ -c\tilde{y} & -d\tilde{y} \end{pmatrix},$$

with eigenvalues

$$\frac{1}{2}\left(-(D\tilde{x} + d\tilde{y}) \pm \sqrt{(D\tilde{x} + d\tilde{y})^2 - 4(Dd - Cc)\tilde{x}\tilde{y}}\right).$$

These eigenvalues can be written

$$\frac{1}{2}\left(-(D\tilde{x} + d\tilde{y}) \pm \sqrt{(D\tilde{x} - d\tilde{y})^2 + 4Cc\tilde{x}\tilde{y}}\right).$$

This formulation makes it clears that these eigenvalues are real. Two cases occur.

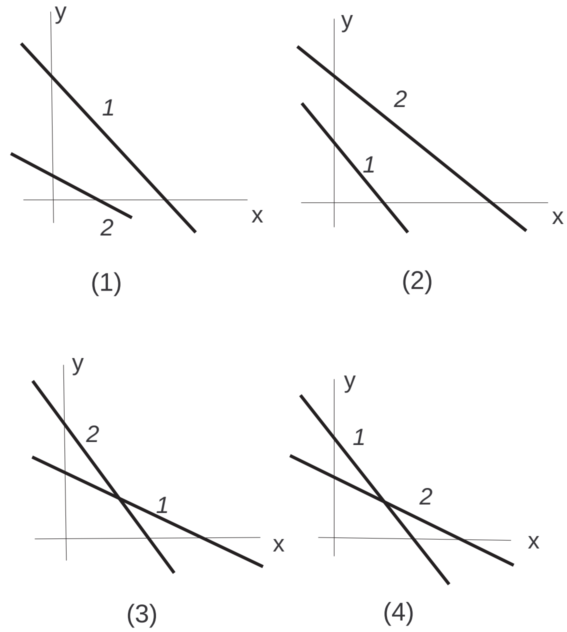

Figure 6: Relative positions of lines (*1*) $Dx + Cy = 0$ and (*2*) $cx + dy = 0$.

If $Dd = Cc < 0$, then one eigenvalue is positive and the other negative. In this case $(\tilde{x}, \tilde{y})$ is an unstable saddle point. If the population point $(x(0), y(0))$ starts near enough to this critical point, one population will die out with time and the other will survive. This case corresponds to Figure 6(3). The condition $Dd < Cd)$ can be interpreted to mean that the product of the internal limiting factors is less than the product of the competition factors. The competition factors tending to increase the populations are dominant in the model and only one population survives.

If $Dd - Cd > 0$ then both eigenvalues are negative and $(\tilde{x}, \tilde{y})$ is an asymptotically stable node (Figure 6(4)). If $(x(0), y(0)$ is sufficiently close to this node, trajectories through this initial point approach the node in the limit and both species survive (coexistence). Now $Cd < Dd$ and the competition factors are less important than internal limiting factors. With competition playing less of a role in the population, mutual survival can occur.

Problems

Each of Problems 1–6 deals with the simple competing species model treated first in this section. In each, (a) determine the critical points in the first octant and classify their type and stability properties, (b) draw a phase portrait for the system, (c) interpret the survival prospects for each species, as dictated by the model.

1. $x' = x - 4xy, y^{prime} = 3y - 6xy$
2. $x' = 3x - xy, y' = 4y - 10xy$
3. $x' = 3x - 2xy, y' = 6y - 2xy$
4. $x' = 8x - 3xy, y' = 2y - 7xy$
5. $x' = 3x - 7xy, y' = y - 4xy$
6. $x' = 4x - 9xy, y' = 5y - 2xy$

Problems 7–12 deal with the extended competing species model. In each, (a) determine the critical points in the first octant and the types and stability properties, (b) draw a phase portrait for the system, (c) interpret the survival prospects for each species, as dictated by the model.

7. $x' = 7x - 5x^2 - 2xy, y' = 3y - 4y^2 - xy$
8. $x' = 4x - 7x^2 - xy, y' = y - 2y^2 - 3xy$
9. $x' = 2x - 3x^2 - xy, y' = 3y - y^2 - 2xy$
10. $x' = x - 9x^2 - 3xy, y' = y - y^2 - xy$
11. $x' = 3x - x^2 - 2xy, y' = y - 2y^2 - xy$
12. $x' = x - 6x^2 - 3xy, y' = 5y - 2y^2 - 8xy$