Chapter 1
Section 1.2

1. (b) First order. y_1: $2\sin x\cos x \neq 9\sin 2x$ so No. y_2: $2(3\sin x)(3\cos x)$ does $= 9\sin 2x$ (because $2\sin x\cos x = \sin 2x$) so Yes. y_3: $2(e^x)(e^x) \neq 9\sin 2x$ so No.

(h) $y_1' + 2xy_1 - 1 = -2xAe^{-x^2}\int_0^x e^{t^2}dt + Ae^{-x^2}e^{x^2} + 2Axe^{-x^2}\int_0^x e^{t^2}dt - 1 = 0$ only if $A = 1$. Thus, in general, No.
$y_2' + 2xy_2 - 1 = -2xe^{-x^2}\int_a^x e^{t^2}dt + e^{-x^2}e^{x^2} + 2xe^{-x^2}\int_a^x e^{t^2}dt - 1$ does $= 0$ for all choices of a, so Yes.

3. Evaluating u_{xx}, u_{yy}, u_{zz}, we obtain
$u_{xx} + u_{yy} + u_{zz} = (c^2 - a^2 - b^2)\sin ax\sin by\sinh cz = 0$ if $c^2 = a^2 + b^2$ (or if $a = b = c = 0$ so that $\sin ax\sin by\sinh cz = 0$, but this is a subcase of $c^2 = a^2 + b^2$).

5. (b) $y' + 3y^2 = \lambda e^{\lambda x} + 3e^{2\lambda x} = e^{\lambda x}(\lambda + 3e^{\lambda x})$. The $e^{\lambda x}$ factor is not 0 for any x, let alone for all x. And for the second factor to be 0 for all x requires that $e^{\lambda x}$ is a constant and that, in turn, requires that $\lambda = 0$. But if $\lambda = 0$ then $\lambda + 3e^{\lambda x} = 0 + 3 \neq 0$. Thus, no such solutions.

(c) $y'' - 3y' + 2y = (\lambda^2 - 3\lambda + 2)e^{\lambda x} = 0$ if $\lambda^2 - 3\lambda + 2 = 0$, i.e., if $\lambda = 1$ or 2. Thus, e^x and e^{2x} are solutions.

6. (b) $y'' - y - x^2 = (-2 + A\sinh x + B\cosh x) - (-x^2 - 2 + A\sinh x + B\cosh x) - x^2$ does $= 0$.
$y(0) = -2 = -2 + B$ and $y'(0) = 0 = A$ give $A = B = 0$, so $y(x) = -x^2 - 2$.

7. (b) Nonlinear due to the yy' term
(d) Nonlinear due to the $\exp(y)$ term
(g) Nonlinear due to the yy''' term. All others linear.

8. $y'' \approx C$, since $y'^2 \ll 1$.

Section 1.3

3. (a) Since $\Delta W = w\Delta x = \mu\Delta s$, we see that $w = \mu\, ds/dx = \mu\sqrt{1+y'^2}$. Integrating (11a) and (11b),

$$\left.\begin{array}{l} T\cos\theta = A \\ T\sin\theta = \mu\int^x \sqrt{1+y'^2}\,dx + B \end{array}\right\} \text{so } \frac{T\sin\theta}{T\cos\theta} = \tan\theta = y' = \frac{\mu}{A}\int^x \sqrt{1+y'^2}\,dx + \frac{B}{A}$$

and d/dx gives $y'' = C\sqrt{1+y'^2}$.

Chapter 2
Section 2.2

2. (b) $y' + 4y = 8$, so (21) gives $y(x) = e^{-\int 4dx}\left(\int e^{\int 4dx}\, 8dx + C\right) = e^{-4x}\left(\frac{8e^{4x}}{4} + C\right) = 2 + Ce^{-4x}$. Or, by integrating factor method, consider $\sigma y' + 4\sigma y = \sigma 8$. For $\sigma y' + 4\sigma y$ to $= (\sigma y)' = \sigma y' + \sigma' y$ we need $\sigma' = 4\sigma$ so, from (7), $\sigma(x) = e^{4x}$. Thus, $(e^{4x}y)' = 8e^{4x}$, so $e^{4x}y = \int^x 8e^{4x}dx + C$ or

$y(x) = 2 + Ce^{-4x}$ again.

(e) $y(x) = e^{-\int -\tan x\,dx}\left(\int e^{\int -\tan x\,dx} 6dx + C\right) = e^{\int \frac{\sin x\,dx}{\cos x}}\left(\int e^{-\int \frac{\sin x}{\cos x}dx} 6dx + C\right)$
$= e^{-\int d(\cos x)/\cos x}\left(\int e^{\int d(\cos x)/\cos x} 6dx + C\right) = e^{-\ln|\cos x|}\left(\int e^{\ln|\cos x|} 6dx + C\right)$
$= \frac{1}{|\cos x|}\left(\int 6|\cos x|\,dx + C\right)$. Recall that the $\tan x$ in the ODE is defined only on $\dots, -3\pi/2 < x < -\pi/2, -\pi/2 < x < \pi/2, \pi/2 < x < 3\pi/2, \dots$ etc. On $-\pi/2 < x < \pi/2$, for ex., $\cos x > 0$ so $|\cos x| = \cos x$ and $y(x) = \frac{1}{\cos x}(6\sin x + C)$. On $\pi/2 < x < 3\pi/2$, for ex., $\cos x < 0$ so $|\cos x| = -\cos x$ and $y(x) = \frac{1}{-\cos x}\left(\int -6\cos x\,dx + C\right)$ $= \frac{1}{\cos x}(6\sin x - C)$, and so on, so on any of the stated x intervals the solution is $y(x) = \frac{1}{\cos x}(6\sin x + \text{“}A\text{”})$ where A is an arbitrary constant.

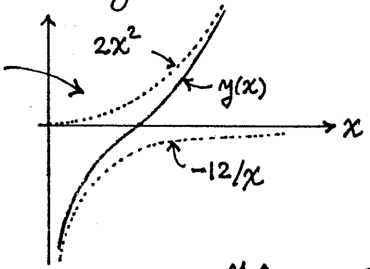
(f) $y(x) = e^{-\int 2dx/x}\left(\int e^{\int 2dx/x} x^2dx + C\right) = e^{-2\ln|x|}\left(\int e^{2\ln|x|} x^2 dx + C\right)$
$= \frac{1}{|x|^2}\left(\int |x|^2x^2dx + C\right) = \frac{1}{x^2}\left(\int x^4dx + C\right) = \frac{x^3}{5} + \frac{C}{x^2}$ for $0 < x < \infty$ or

for $-\infty < x < 0$.

3. (b) $y_h(x) = Ae^{-4x}$ so seek $y(x) = A(x)e^{-4x}$. Then $(A'e^{-4x} - 4\cancel{Ae^{-4x}}) + 4\cancel{Ae^{-4x}} = 8$ gives $A' = 8e^{4x}$, $A(x) = \int 8e^{4x}dx + C = 2e^{4x} + C$, so $y(x) = (2e^{4x} + C)e^{-4x} = 2 + Ce^{-4x}$, as in 2(b).

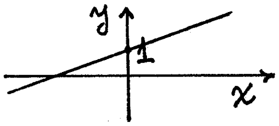
5. (b) $y(x) = 2x^2 + C/x$. $y(1) = 2 = 2 + C$ gives $C = 0$ so $y(x) = 2x^2$ on $-\infty < x < \infty$.

(c) $y(x) = 2x^2 + C/x$. $y(2) = 2 = 8 + C/2$ gives $C = -12$ so $y(x) = 2x^2 - \frac{12}{x}$ on $0 < x < \infty$.

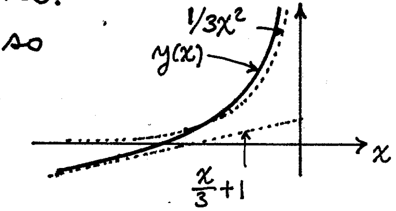


6. (21) gives general solution $y(x) = \frac{x}{3} + 1 + \frac{C}{x^2}$

(b) $y(0) = 1 = 0 + 1 + 0$ if we choose $C = 0$. Thus, $y(x) = \frac{x}{3} + 1$. (NOTE: If, instead, $y(0) = y_0$ where $y_0 \neq 1$, then there is no solution.) That solution holds on $-\infty < x < \infty$.

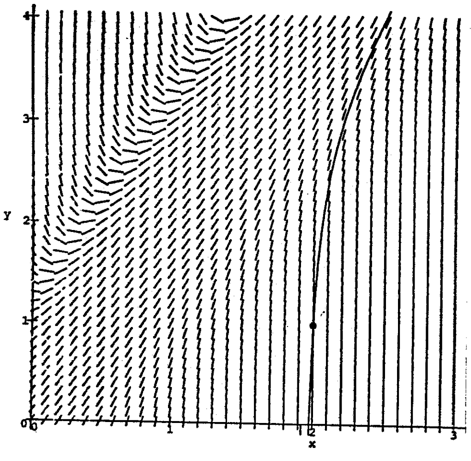


(c) $y(-1) = 1 = -\frac{1}{3} + 1 + C$ gives $C = \frac{1}{3}$, so $y(x) = \frac{x}{3} + 1 + \frac{1}{3x^2}$ on $-\infty < x < 0$.



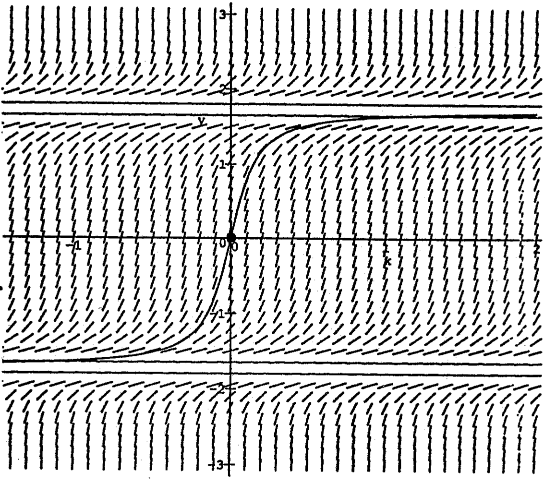
7. (b) Consider $x = x(y)$ rather than $y = y(x)$. Then $\frac{dx}{dy} = 6x + y^2$ or $\frac{dx}{dy} - 6x = y^2$, $x(y) = e^{-\int -6dy}\left(\int e^{\int -6dy} y^2dy + C\right)$
$= -\frac{1}{6}y^2 - \frac{1}{18}y - \frac{1}{108} + Ce^{6y}$

8. (a) Shown at the right is only the $0 < x < 3, 0 < y < 4$ part of the display, using the command phaseportrait (2+(2*x-y)^3, [x,y], x=-4..4, {[2,1]}, y=-4..4, grid=[40,40], stepsize=0.01, arrows=LINE); NOTE: The default grid is [20,20] and is too coarse so we use the Grid option **grid=[40,40]**. Also, the stepsize needs to be reduced sufficiently to get

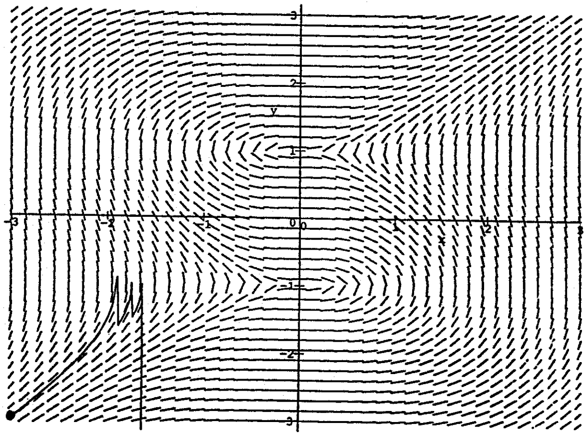


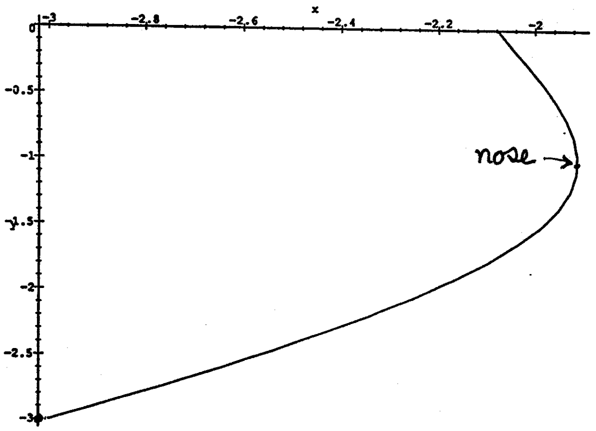
render the solution curve through [2,1] smooth so we used the additional option **stepsize** = 0.01. (For further discussion of the phaseportrait command see the Index.) Looking at the lineal element field (and peeking at the ODE) reveals the simple integral curve $y=2x$. The integral curve through [3,0], for instance, is almost vertical and bends to the right, eventually approaching $y=2x$.

(c) phaseportrait((3-y^2)^2, [x,y], x = -2..2, {[0,0]}, grid = [40,40], stepsize = 0.04, y=-3..3, arrows = LINE); gives the phaseportrait → shown at the right. We observe the integral curves $y = +\sqrt{3}$ and $y = -\sqrt{3}$.



(e) phaseportrait(x^2/(y^2-1), [x,y], x=-3..3, {[-3,-3]}, y=-3..3, grid=[40,40], stepsize=0.01, arrows = LINE); gives the portrait shown below left. To resolve the mysterious zigzags we reduced the stepsize to 0.01 but the zigzags persisted. It looks like the problem is that the integral curve rises from [-3,-3] reaches a vertical tangent at $y=-1$ (as can also be seen from the ODE) and then bends to the left, in which case a single-valued differentiable solution $y(x)$ would exist only up to the point of vertical tangency, the "nose" of the curve. NOTE: If we use separation of variables (not discussed until Sec. 2.4), we obtain, in implicit form, the solution $y^3-3y=x^3+9$. Next, the commands with(plots): and implicitplot(y^3-3*y = x^3+9, x = -3..0, y = -3..0, numpoints = 500); gives the integral curve plot shown below right, which plot substantiates the foregoing reasoning.

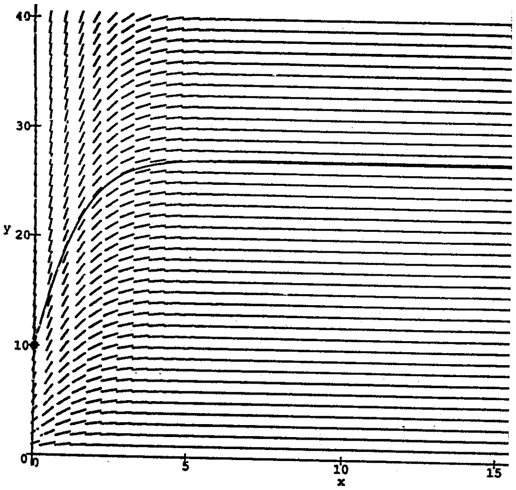




(g) $y' = e^{-x}y$. phaseportrait(exp(-x)*y, [x,y], x=0..20, {[0,10]}, grid = [40,40], y=0..40, stepsize = 0.04, arrows = LINE); gives the portrait shown on next page.

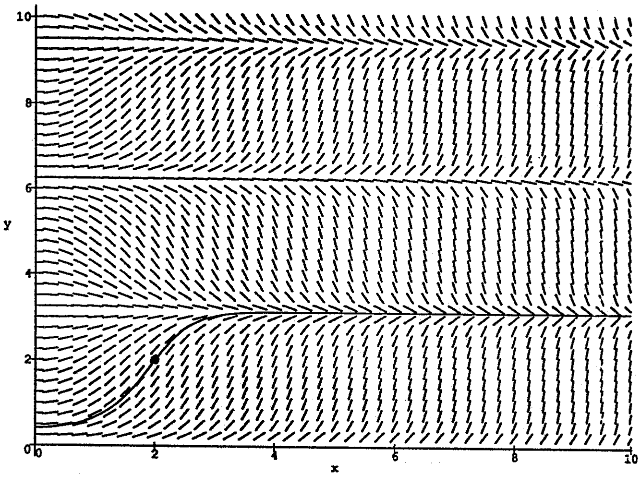
(h) phaseportrait(x*sin(y), [x,y], x=0..10, {[2,2]}, y=0..10, grid = [40,40], stepsize = 0.04, arrows = LINE); gives the portrait shown on next page.

(g) continued:



An exact integral curve is $y=0$.

(h) continued:



Exact integral curves are $y = n\pi$ $(n=0,\pm1,\pm2,\ldots)$

9. (b) $y'+py=qy^n$, $v=y^{1-n}$ $(n\neq 0,1)$. $v'=(1-n)y^{-n}y'$ so $y'=y^n v'/(1-n)$ and the ODE becomes $\frac{y^n v'}{1-n}+py=qy^n$ or, dividing by y^n, $v'+(1-n)pv=(q)(1-n)$.

10. (b) $n=2$, so $v'+\frac{2}{x}v=-x^2$, $v(x)=e^{-\int\frac{2}{x}dx}\left(\int e^{\int\frac{2}{x}dx}(-x^2)\,dx+C\right)$
$=\frac{1}{x^2}\left(\int(-x^4)\,dx+C\right)=-\frac{x^3}{5}+\frac{C}{x^2}$, so $y=\frac{1}{v}=\frac{5x^2}{A-x^5}$ (where $A=5C$).

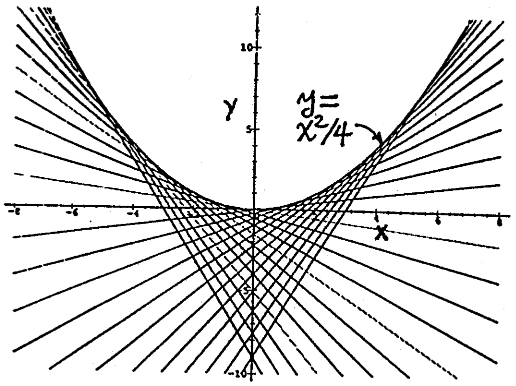
11. $y'=py^2+qy+r$, $y=Y+\frac{1}{u}$ gives $Y'-\frac{u'}{u^2}=p\left(Y+\frac{1}{u}\right)^2+q\left(Y+\frac{1}{u}\right)+r$. Using $Y'=pY^2+qY+r$ to cancel terms gives $-\frac{u'}{u^2}=2p\frac{Y}{u}+\frac{p}{u^2}+\frac{q}{u}$, or
$u'+(2pY+q)u=-p$.

12. (b) $y'=y^2-xy+1$ so $p=1, q=-x, r=1$ and (11.3) is $u'+xu=-1$,
$u=e^{-x^2/2}\left(\int e^{x^2/2}(-1)\,dx+C\right)$ or $u(x)=e^{-x^2/2}\left(C-\int_0^x e^{t^2/2}dt\right)$, say.
Thus, (11.2) gives $y(x)=x+e^{x^2/2}/\left(C-\int_0^x e^{-t^2}dt\right)$.
(e) Find $Y(x)=ax^b=x^2$. (f) Use $Y(x)=1$ or $Y(x)=2$ (h) $Y(x)=2$ or $Y(x)=0$

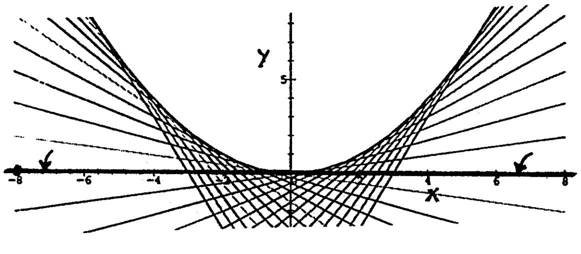
13. (c) (13.3) is $\frac{dx}{dp}-\frac{1+2p}{p-p-p^2}x=0$, $x'+\frac{1+2p}{p^2}x=0$, $x(p)=Ce^{-\int\left(\frac{1+2p}{p^2}\right)dp}$

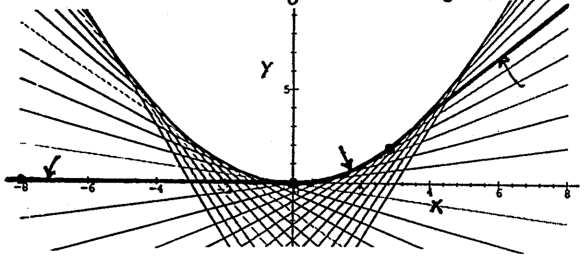
so the parametric solution is $x(p)=Ce^{1/p}/p^2$, $y(p)=x(p+p^2)$
$=Ce^{1/p}\left(1+\frac{1}{p}\right)$.
(d) Putting (13.4) into (13.1) gives $P_0x+g(P_0)=xf(P)+g(P)$, which is satisfied if $p=\text{constant}=p_0$, since $f(P_0)=P_0$.

14. (b) $f(p)=p$ in (13.2) gives $0=[xf'(p)+g'(p)]\frac{dp}{dx}$, which is satisfied by $p=\text{constant}\equiv C$ [hence (14.1) gives (14.2)] or by $xf'(p)+g'(p)=0$. Since $f(p)=p$, the latter gives $x=-g'(p)$
and (14.1) gives $y=xp+g(p)=-pg'(p)+g(p)$ $\Big\}*$
(c) In this case $g(p)=-p^2$ so $*$ gives $x=2p$, $y=2p^2-p^2=p^2$. In this case we are able to eliminate p between these two equations and obtain $y=x^2/4$.



(d) The point is that the Clairaut equation (14.1) admits both the family of straight-line solutions (14.2) and the additional solution (14.3). Geometrically, the integral curve given parametrically by (14.3) is an "envelope" of the family of straight lines; for the case in part (c), the envelope is the parabola $y=x^2/4$, as displayed at the right. Observe the breakdown in uniqueness which is in sharp contrast with the linear equation $y'+p(x)y=q(x)$, solutions of which are unique (subject to continuity conditions on $p(x)$ and $q(x)$; see Theorem 2.2.1, pg 26). For example, consider the solution(s) through the initial point $(-8,0)$. The solution curve through that point follows the x axis to $x=-\infty$. To the right, it follows the x axis to the origin, where it becomes tangent to the solution curve $y=x^2/4$. At that point it "has a choice": it can continue along the x axis to $x=+\infty$ or it can then move along the parabola $y=x^2/4$, getting off (or not) at any point along the straight line solution that is tangent to the parabola at that point, and proceeding along that line to $x=+\infty$. Two such solutions are shown below by the heavy lines.





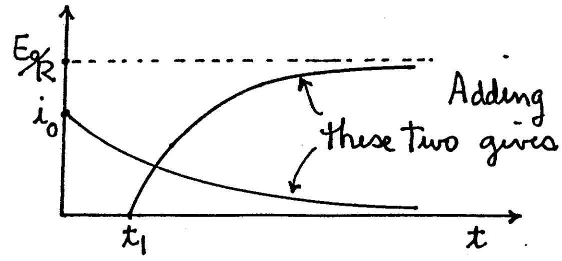
Section 2.3

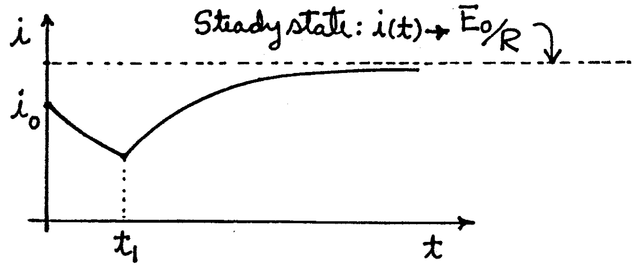
2. (b) $i(t) = e^{-\int_0^t \frac{R}{L}d\tau}\left(\int_0^t e^{\int_0^\tau \frac{R}{L}d\mu}\,\frac{E(\tau)}{L}\,d\tau+i_0\right)$

If $t<t_1$ then $E(\tau)=0$ in the integral, so $i(t)= e^{-Rt/L}(0+i_0)=i_0e^{-Rt/L}$.

If $t>t_1$ then

$i(t)= e^{-Rt/L}\left(\int_{t_1}^t e^{R\tau/L}\,\frac{E_0}{L}\,d\tau+i_0\right) = \frac{E_0}{R}\left(1-e^{-\frac{R}{L}(t-t_1)}\right)+i_0e^{-Rt/L}$





4: $i(t)=\frac{E_0\omega L}{R^2+(\omega L)^2}\left[e^{-Rt/L}+\frac{1}{\omega L}\underbrace{(R\sin\omega t-\omega L\cos\omega t)}_{*}\right]$. To change * from two terms to one, write $A\sin(\omega t-\phi) = A(\sin\omega t\cos\omega\phi - \sin\phi\cos\omega t)$. Identify (by comparing with $*$)

$$\left.\begin{aligned} A\sin\phi &= \omega L \\ A\cos\phi &= R \end{aligned}\right\}$$ Dividing gives $\tan\phi = \omega L/R$ or $\phi = \tan^{-1}(\omega L/R)$, and squaring and adding gives $A^2 = R^2 + (\omega L)^2$ so $A = \sqrt{R^2+(\omega L)^2}$.

Thus, $i(t) = \frac{E_0\omega L}{R^2+(\omega L)^2}\,{}^{D}e^{-Rt/L} + \frac{E_0}{\sqrt{R^2+(\omega L)^2}}\sin(\omega t-\phi)$.

6. $m(t) = m_0 e^{-kt}$ so $8 = 10e^{-60k}$ gives $-60k = \ln 0.8$, $k = 0.00372$ so $m(t) = 10e^{-0.00372t}$. $2 = 10e^{-0.00372t}$ gives $t = 432.6$ yrs, and $0.1 = 10\,e^{-0.00372t}$ gives $t = 1237.9$ yrs.

7. $m(t) = m_0 e^{-kt}$. $0.8\cancel{m}_0 = \cancel{m}_0 e^{-70k}$ gives $k = 0.003188$. Then, $0.5\cancel{m}_0 = \cancel{m}_0 e^{-0.003188T}$ gives $T = 217.4$ days.

12. (a) $mv' = mg - cv$; $v(0) = 0$. Then $v' + \frac{c}{m}v = g$ (first-order linear eqn.) so $v(t) = e^{-\int c\,dt/m}\left(\int e^{\int c\,dt/m} g\,dt + A\right) = e^{-ct/m}\left(g\int e^{ct/m}dt + A\right)$ $= \frac{mg}{c} + Ae^{-ct/m}$. Then $v(0) = 0 = \frac{mg}{c} + A$ gives $A = -\,mg/c$ and $v(t) = \frac{mg}{c}(1 - e^{-ct/m})$. As $t\to\infty$, $v(t) \to \frac{mg}{c}$ = "terminal velocity".

(b) $mv' = mg - cv^2$ is now a Riccati equation (see Exercise 11 in Sec. 2.2) with x, y changed to t, v, and $p(t) \overset{D}{=} -c/m$, $q(t) = 0$, $r(t) = g$. Observing the particular solution $\sqrt{mg/c}$, change dependent variable according to $v(t) = \sqrt{mg/c} + \frac{1}{u(t)}$. Then the ODE becomes $0 - \frac{u'}{u^2} = g - \frac{c}{m}\left(\sqrt{\frac{mg}{c}} + \frac{1}{u}\right)^2$

$= \cancel{g} - \frac{c}{m}\left(\frac{m\cancel{g}}{\cancel{c}} + 2\sqrt{\frac{mg}{c}}\frac{1}{u} + \frac{1}{u^2}\right)$ or $u' - 2\sqrt{\frac{gc}{m}}\,u = \frac{c}{m}$ with solution

$u(t) = -\frac{1}{2}\sqrt{\frac{c}{mg}} + Ae^{2\sqrt{gc/m}\,t}$. Then, $v(0) = 0 = \sqrt{\frac{mg}{c}} + \frac{1}{u(0)}$ gives $u(0) = -\sqrt{\frac{c}{mg}}$

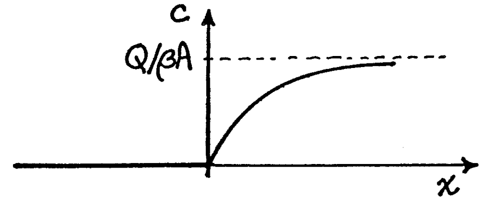
$= -\frac{1}{2}\sqrt{\frac{c}{mg}} + A$ gives $A = -\frac{1}{2}\sqrt{\frac{c}{mg}}$. Finally, $v(t) = \sqrt{\frac{mg}{c}} + \frac{1}{u(t)}$

$= \sqrt{\frac{mg}{c}} + \frac{1}{-\frac{1}{2}\sqrt{\frac{c}{mg}} - \frac{1}{2}\sqrt{\frac{c}{mg}}\,e^{2\sqrt{gc/m}\,t}} = \sqrt{\frac{mg}{c}}\left(1 - \frac{2}{1+e^{2\sqrt{gc/m}\,t}}\right)$ and the terminal velocity is $\sqrt{mg/c}$.

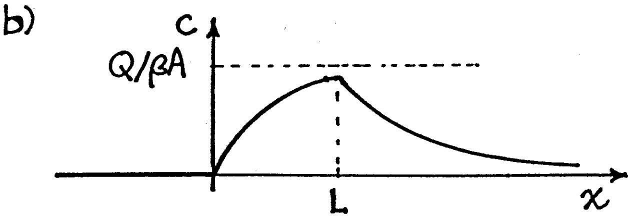
13. This problem is worked in the Answers to Selected Exercises. Here, we just wish to mention that to help the student feel more comfortable about the physical process of light extinction it might be useful to note the gradual extinction of light as we proceed deeper and deeper into the ocean.

14. NOTE: This problem is nice for use in class or lecture, especially in view of its environmental interest. Later on it will also make a nice example for the application of the Fourier transform, especially if the source is modeled as Q times a delta function at x = 0. The solution is given in the Answers to Selected Exercises, so here we will just give sketches of the results and (for possible class discussion) give a brief formal derivation of the governing ODE.

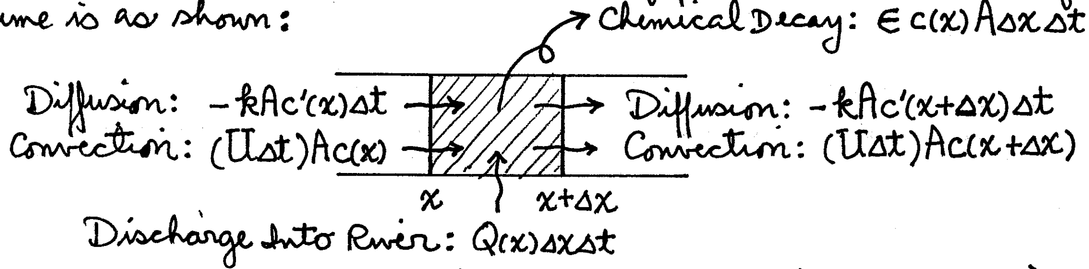
(a)



(b)



To derive the governing ODE carry out a mass balance for an arbitrary section of the river, between x and $x+\Delta x$. **Fick's law** of diffusion says that the flow of mass (of pollutant) across the "window" of area A at x is proportional to the area A and the concentration gradient $-c'(x)$ (minus because the flux will be from high concentration to low concentration, so $c'(x)>0$ will cause a flux, by diffusion, to the left and $c'(x)<0$ will cause a flux to the right) with a constant of proportionality k which is a diffusivity constant specific to the medium. Over a time Δt the movement of pollutant in and out of the control volume is as shown:



where the loss due to chemical decay is ϵ per unit mass per unit time and $Q(x)$ is the discharge into the river per unit x length per unit time. Now,

Decrease in mass of pollutant in control volume by decay, over time Δt = mass in − mass out,

so
$$\epsilon c(x)A\Delta x\Delta t=[-kAc'(x)\Delta t-(U\Delta t)Ac(x)]$$
$$-[-kAc'(x+\Delta x)\Delta t-(U\Delta t)Ac(x+\Delta x)+Q(x)\Delta x\Delta t]$$

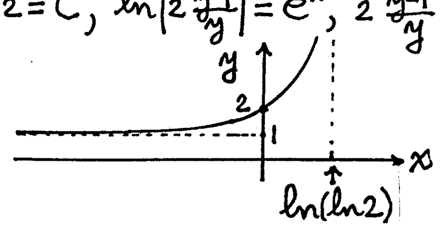
Dividing by $A\Delta x\Delta t$ and letting $\Delta x\to 0$ gives
$$kc''-Uc'-\boxed{\frac{\epsilon}{A}}c=-\frac{Q(x)}{A}$$
Let us call this $\beta \rightarrow$

15. (a) $\frac{du}{dt}+ku=kU$ gives $u(t)=e^{-\int k\,dt}\left(\int e^{\int k\,dt}kU\,dt+C\right)=U+Ce^{-kt}$.
$u(0)=u_0=U+C$ gives $C=u_0-U$, so $u(t)=u_0+U(1-e^{-kt})$.

16. (a) $S(t)=S_0(1+\frac{k}{n})^{nt}=S_0\left(1+\frac{1}{n/k}\right)^{(\frac{n}{k})kt}=S_0\left(1+\frac{1}{\text{“}m\text{”}}\right)^{mkt}\to S_0e^{kt}$ as $m\to\infty$.

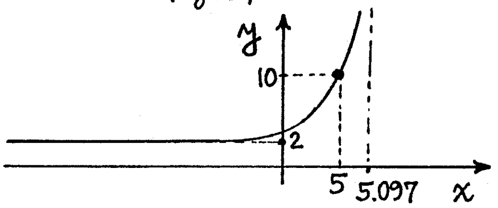
Section 2.4

1. (b) $y'=6x^2+5$, $\int dy=\int(6x^2+5)dx$, $y=2x^3+5x+C$, $y(0)=0=C$, $y(x)=2x^3+5x$
(c) $y'+4y=0$, $\int\frac{dy}{y}+4\int dx=0$, $\ln y+4x=A$, $y=e^{A-4x}=Ce^{-4x}$,
$y(-1)=0=Ce^4$ gives $C=0$, so $y(x)=0$.
(e) $y'=(y^2-y)e^x$, $\int\frac{dy}{y(y-1)}=\int e^x dx$, partial fractions → $-\int\frac{dy}{y}+\int\frac{dy}{y-1}=e^x+C$,
$\ln\left|\frac{y-1}{y}\right|=e^x+C$, $y(0)=2\to -\ln 2=C$, $\ln\left|2\frac{y-1}{y}\right|=e^x$, $2\frac{y-1}{y}=e^{e^x}$,
$y(x)=\frac{2}{2-e^{e^x}}$ on $-\infty<x<\ln(\ln 2)$.

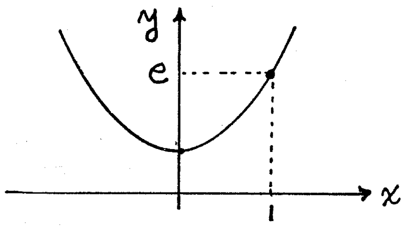


(f) $y' = y^2 + y - 6$, $\frac{dy}{(y-2)(y+3)} = dx$, $\frac{1}{5}\int\frac{dy}{y-2} - \frac{1}{5}\int\frac{dy}{y+3} = \int dx$, $\frac{1}{5}\ln\left|\frac{y-2}{y+3}\right| = x + C$,
$y(5) = 10$ gives $C = \frac{1}{5}\ln\frac{8}{13} - 5$, so
$\ln\left|\frac{13}{8}\frac{y-2}{y+3}\right| = 5x - 25$, $y(x) = \frac{26 + 24e^{5(x-5)}}{13 - 8e^{5(x-5)}}$

$\to +2$ as $x \to -\infty$ and $\to +\infty$ as $13 - 8e^{5(x-5)} \to 0$,
i.e, as $x \to \frac{1}{5}\ln\frac{13}{8} + 5 \approx 5.097$ from the left.



(h) $y' = 6\frac{y\ln y}{x}$, $\int\frac{dy}{y\ln y} = 6\int\frac{dx}{x}$. Let $\ln y = u$.
ln for convenience
Then $\ln(\ln y) = 6\ln x + \ln C$, $\ln(\ln y) = \ln Cx^6$,
$\ln y = Cx^6$, $y = e^{Cx^6}$. $y(1) = e = e^C \to C = 1$,
so $y(x) = e^{x^6}$.



2. (a) dsolve({diff(y(x), x) - 3*x^2 * exp(-y(x))=0, y(0)=0}, y(x)); gives
$y(x) = \ln(x^3 + 1)$.

3. $\frac{du}{dt} = k(U - u)$, $\frac{du}{u - U} = -k\,dt$, $\ln(u - U) = -kt + A$, $u(t) = U + e^{-kt+A} = U + Ce^{-kt}$.
$u(0) = u_0 = U + C$ gives $C = u_0 - U$ so $u(t) = U + (u_0 - U)e^{-kt}$.

5. $y' + py = qy^n$, where p & q are nonzero constants. $\frac{dy}{py - qy^n} = dx$.
Change variables by $v = y^{1-n}$ (consider $n \neq 0, 1$ here). Then $\int\frac{dv}{(1-n)(pv - q)}$
$= \int dx$ gives $\frac{1}{p(1-n)}\ln(v - \frac{q}{p}) = x + A$, $v - \frac{q}{p} = e^{p(1-n)(x+A)}$,
$y(x) = \left(\frac{q}{p} + Ce^{p(1-n)x}\right)^{1/1-n}$.

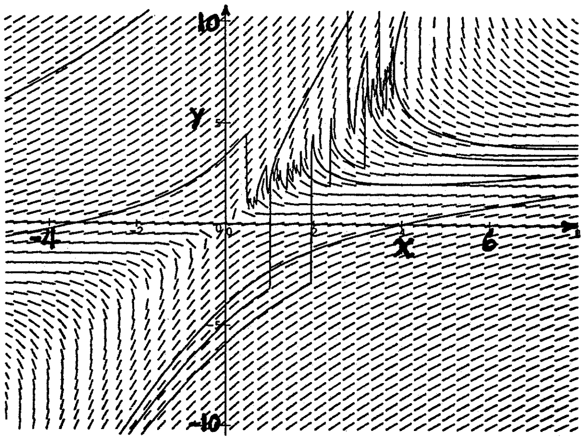
6. (b) $y' = (6x^2 + 1)/(y - 1)$, $(y - 1)dy = (6x^2 + 1)dx$, $y^2 - y = 2x^3 + x + C$. $y(0) = 4$ gives
$16 - 4 = 0 + 0 + C$ so $C = 12$, $y^2 - y - (2x^3 + x + 12) = 0$, $y = \frac{1 \pm \sqrt{8x^3 + 4x + 49}}{2}$.
Of these two solutions choose the + so $y(0) = 4$. Thus, $y(x) = [1 + \sqrt{8x^3 + 4x + 49}\,]/2$.

9. (a) $y' = \frac{y}{x}$ is separable, $y' = \sin(\frac{y}{x})$ is not.
(b) $y = vx$, $y' = v'x + v = f(v)$ gives $v' = \frac{f(v) - v}{x}$.

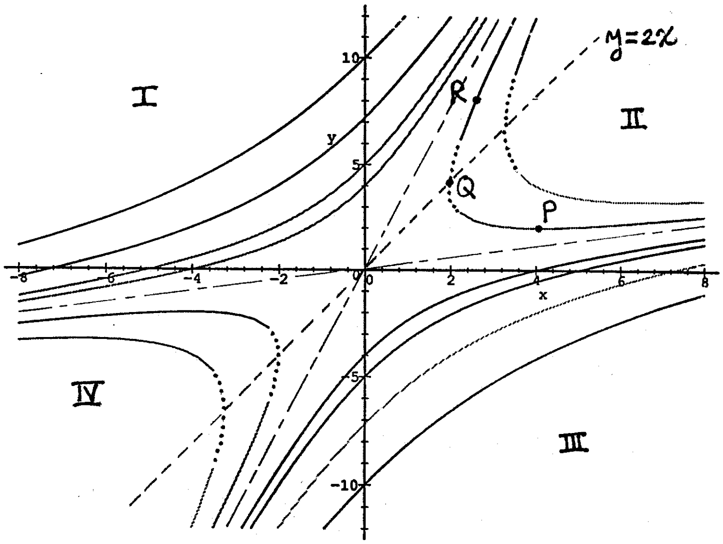
10. (b) $y' = \frac{2y - x}{y - 2x} = \frac{2v - 1}{v - 2} \equiv f(v)$, so $v' = \frac{\frac{2v-1}{v-2} - v}{x} = \frac{2v - 1 - v^2 + 2v}{x(v - 2)}$,

$\frac{(v - 2)dv}{v^2 - 4v + 1} = -\frac{dx}{x}$, $\frac{1}{2}\ln(v^2 - 4v + 1) = -\ln x + C$

so $v = (2x \pm \sqrt{3x^2 + C^2})/x$ and, since v is y/x, $y(x) = 2x \pm \sqrt{3x^2 + A}$. $(A \equiv C^2)$



To understand the ± choice we've used phaseportrait to show the direction field and integral curves through a few points: [4,0], [4,2], [4,4], [4,6], but we get some zig zag "garbage" – evidently where the integral curves have vertical tangents, namely, as seen from the ODE $y' = (2y - x)/(y - 2x)$, along the line $y = 2x$. Thus, instead, let us use implicitplot to plot the solution $y(x) = 2x \pm \sqrt{3x^2 + A}$ through the representative initial points [4,-2], [4,0], [4,2], [4,4],



[0,5],[0,10],[0,-5],[0,-10], [-4,2],[-4,0], [-4,-2], [-4,-4]. For each initial point we need to choose A and the + or - sign. Since $(y-2x)^2 = 3x^2+A$, $A=(y-2x)^2-3x^2$ and the points listed give the A values $A=52, 16, -12, -32, 25, 100$. For each of these use + and then -, giving 12 curves, as shown at the right. Even using the numpoints = 2500 in the command

```
with(plots):
implicitplot({y=2*x+sqrt(x^2+52),
y= 2*x-sqrt(x^2+52), and ten more
of these}, x=-8..8, y=-12..12, numpoints = 2500);
```

still there are gaps in the curves where the curve crosses the line $y=2x$. We have filled in those gaps by hand with dots. The two asymptotes $y \sim 2x \pm \sqrt{3}x = (2\pm\sqrt{3})x$ (shown as — — —) are important. In the regions I and III, between these asymptotes, through each initial point there exists a unique solution defined on $-\infty < x < \infty$, such as the integral curves through [0,10] and [0,-10]. But consider initial pts. in II and IV: through P there exists a unique solution over $x_Q < x < \infty$, through Q there is no solution ($y' = \infty$ there), and through R there exists a unique solution over $x_Q < x < \infty$. Similarly in IV.

NOTE: The preceding problem, 2.4/10b, or one like it, is recommended for discussion in class, even including the problems encountered with phaseportrait.

11. (c) With $x=u+h$, $y=v+k$ the equation $y' = (1-y)/(x+4y-3)$ becomes $\frac{dv}{du} = \frac{1-v-k}{u+h+4v+4k-3}$ so set $1-k=0$ and $h+4k-3=0$; hence, $k=1$ and $h=-1$. Then $\frac{dv}{du} = -\frac{v}{u+4v}$. With $w=\frac{v}{u}$, $v=uw$, the latter becomes

$$\frac{dv}{du} = u\frac{dw}{du} + w = -\frac{w}{1+4w} \quad \text{so} \quad u\frac{dw}{du} = -\frac{2w+4w^2}{1+4w} \quad \text{so} \quad \int \frac{1+4w}{2w(1+2w)}\,dw = -\int \frac{du}{u}$$

so $\frac{1}{2}\ln[w(1+2w)] = -\ln u + \boxed{\frac{1}{2}\ln C}$ ← for convenience

so $\ln[w(1+2w)] = \ln(C/u^2)$ so $w(1+2w) = \frac{C}{u^2}$. Putting back $w = v/u$, where $u=x+1$ and $v=y-1$, gives

$$2y^2+(x-3)y-x = A \qquad *$$

where A is an arbitrary constant. We can solve * for x as a single valued function of y or for y as a double valued function of x. The situation is similar to the one discussed in Exercise 10b and can be illuminated further using implicitplot.

(f) $y' = \frac{x+2y-1}{2x+4y-1}$. Let $x+2y = v$ so $\frac{dv}{dx} = 1+2\frac{dy}{dx} = 1+2\frac{v-1}{2v-1}$. Thus, $\frac{dv}{dx} = \frac{4v-3}{2v-1}$

$\int \frac{2v-1}{4v-3}\,dv = \int dx$ so $\frac{1}{2}v + \frac{1}{8}\ln(8v-6) = x + C$, or, $4(x+2y) + \ln(8x+16y-6) = 8x+A$ gives the solution in implicit form.

12. $dN/dt = kN^p$, $N^{-p}dN = k\,dt$, $\frac{N^{1-p}}{1-p} = kt + C \ (p \neq 1)$, $N(t) = [(1-p)kt+A]^{\frac{1}{1-p}}$.

For $p<1$, $N(t) \sim [(1-p)kt]^{\frac{1}{1-p}} = \alpha t^{\beta}$ where $\beta = \frac{1}{1-p} \rightarrow \begin{cases} 1 \text{ as } p \to 0 \\ \infty \text{ as } p \to 1 \end{cases}$

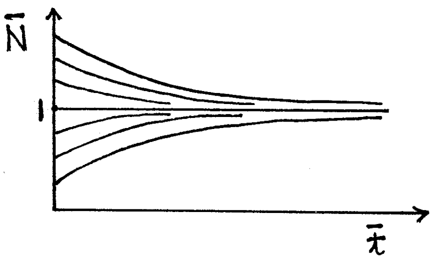
For $p>1$, $N(t) = \frac{1}{[A-(p-1)kt]^{\frac{1}{p-1}}} \rightarrow \infty$ as $t \rightarrow \frac{A}{(p-1)k}$, where A can be expressed in terms of N_0 since $N_0 = A^{\frac{1}{1-p}}$ gives $A = N_0^{1-p}$. Thus, $N(t) \to \infty$ as $t \to T$, where $T = 1/[(p-1)kN_0^{p-1}]$.

13. $dN/dt = (a-bN)N$, $N(0) = N_0$. With $\bar t = at$ and $\bar N = bN/a$,

$\frac{\frac{a}{b}d\bar N}{\frac{1}{a}d\bar t} = (a - b\frac{a}{b}\bar N)\frac{a}{b}\bar N$ or $\frac{d\bar N}{d\bar t} = (1-\bar N)\bar N$; $\frac{a}{b}\bar N(0) = N_0$ or $\bar N(0) = \frac{bN_0}{a} \equiv \beta$

$\frac{d\bar N}{\bar N(1-\bar N)} = d\bar t$, $\ln \bar N - \ln(\bar N - 1) = \bar t + A$, $\frac{\bar N}{\bar N - 1} = Ce^{\bar t}$, $\bar N(0) = \beta$ gives $C = \frac{\beta}{\beta-1}$.

Thus, $\bar N(t) = -\frac{Ce^{\bar t}}{1-Ce^{\bar t}} = -\frac{\frac{\beta}{\beta-1}e^{\bar t}}{1-\frac{\beta}{\beta-1}e^{\bar t}}$
$= \frac{\beta}{\beta + (1-\beta)e^{-\bar t}}$



14. Let $\mathcal{F}, \mathcal{L}, \mathcal{T}$ stand for force, length and time. By Newton's 2nd law, mass is not independent: mass $= \frac{\text{force}}{\text{accel}} = \frac{\mathcal{F}\mathcal{T}^2}{\mathcal{L}}$

Now,

Variable	Dimension	Parameter	Dimension
t	$\mathcal{T}$	m	$\mathcal{F}\mathcal{T}^2/\mathcal{L}$
x	$\mathcal{L}$	c	$\mathcal{F}\mathcal{T}/\mathcal{L}$
		k	$\mathcal{F}/\mathcal{L}$
		F	$\mathcal{F}$
		ω	$1/\mathcal{T}$
		x_0	$\mathcal{L}$
		x_0'	$\mathcal{L}/\mathcal{T}$

To nondimensionalize t we need a combination of the parameters that has units of $\mathcal{T}$, such as $1/\omega$, x_0/x_0', m/c, or c/k; the choice is not unique. Let us use $1/\omega$, say. That is, $\bar t \equiv \frac{t}{1/\omega} = \omega t$.

To nondimensionalize x we need a combination of the parameters that has units of $\mathcal{L}$, such as x_0, x_0'/ω, F/k, and so on. Let us use x_0, say: $\bar x \equiv \frac{x}{x_0}$.

Noting that $dt = \frac{1}{\omega}d\bar t$, the ODE becomes

$$m\frac{d}{\frac{1}{\omega}d\bar t}\frac{d}{\frac{1}{\omega}d\bar t}x_0\bar x(\bar t) + c\frac{d}{\frac{1}{\omega}d\bar t}x_0\bar x(\bar t) + kx_0\bar x(\bar t) = F\sin\bar t\,; \quad x_0\bar x(0) = x_0,$$
$$\frac{d}{\frac{1}{\omega}d\bar t}x_0\bar x(0) = x_0'$$

or $m\omega^2 x_0 \frac{d^2\bar{x}}{d\bar{t}^2} + c\omega x_0 \frac{d\bar{x}}{d\bar{t}} + k x_0 \bar{x} = F \sin\bar{t}$; $\bar{x}(0)=1$, $\bar{x}'(0) = \frac{x_0'}{\omega x_0}$,

or $$\frac{d^2\bar{x}}{d\bar{t}^2} + \underbrace{\left(\frac{c}{m\omega}\right)}_{\alpha} \frac{d\bar{x}}{d\bar{t}} + \underbrace{\left(\frac{k}{m\omega^2}\right)}_{\beta} \bar{x} = \underbrace{\left(\frac{F}{m\omega^2 x_0}\right)}_{\gamma} \sin\bar{t}; \quad \bar{x}(0)=1,\ \bar{x}'(0) = \underbrace{\left(\frac{x_0'}{\omega x_0}\right)}_{\delta}$$

Thus, the nondimensionalized system contains only <u>four</u> (nondimensional) parameters $\alpha, \beta, \gamma, \delta$ rather than the original seven (dimensional) parameters. How can we see that $\alpha, \beta, \gamma, \delta$ are nondimensional? The simplest way is to use the fact that all terms in the final equation (or, indeed, in <u>any</u> equation) must have the same units. Since $d^2\bar{x}/d\bar{t}^2$ is dimensionless the other terms must be too. Since $d\bar{x}/d\bar{t}$ is dimensionless α must be. Similarly for the other terms and initial conditions. As noted above, the nondimensionalization is not unique. However, the final number of nondimensional parameters <u>is</u> unique—i.e., independent of the choices made in the nondimensionalization.

Section 2.5

1. (b) $M_y = 0$, $N_x = 0$ ✓ $\frac{\partial F}{\partial x} = x^2 \rightarrow F(x,y) = \int x^2 \partial x = \frac{x^3}{3} + A(y)$

$\frac{\partial F}{\partial y} = y^2 = 0 + A'(y)$ so $A(y) = \int y^2 dy = \frac{y^3}{3} + C$

so $F(x,y) = \frac{x^3}{3} + \frac{y^3}{3} + C =$ constant gives $x^3 + y^3 = B$, say.

Then, $y(9) = -1$ gives $9^3 - 1 = B$ so $B = 728$, so $x^3 + y^3 = 728$.

(f) $M_z = 1$, $N_y = 1$ ✓ $\frac{\partial F}{\partial y} = e^y + z \rightarrow F(y,z) = \int (e^y + z) \partial y = e^y + yz + A(z)$

$\frac{\partial F}{\partial z} = y - \sin z = y + A'(z)$ so $A(z) = -\int \sin z\, dz = \cos z + C$

so $F(y,z) = e^y + yz + \cos z + C =$ const. gives $e^y + yz + \cos z = B$, say.

Then, $z(0) = 0$ gives $e^0 + 0 + \cos 0 = B$ gives $B = 1$, so $e^y + yz + \cos z = 1$.

(h) $M_y = \cos y + \cos x$, $N_x = \cos x + \cos y$ ✓

$\partial F/\partial x = \sin y + y\cos x \rightarrow F(x,y) = \int(\sin y + y \cos x)\partial x = x \sin y + y \sin x + A(y)$

$\partial F/\partial y = \cancel{\sin x} + x\cancel{\cos y} = x\cancel{\cos y} + \cancel{\sin x} + A'(y)$ so $A(y) = \int 0\, dy = C$

so $F(x,y) = x \sin y + y \sin x + C =$ const. gives $x \sin y + y \sin x = B$, say.

Then, $y(2) = 3$ gives $2 \sin 3 + 3 \sin 2 = B$, so $x \sin y + y \sin x = 2\sin 3 + 3 \sin 2$.

4. $M_y = b$, $N_x = A$, so the equation will be exact if $A = b$.

5. (b) $M = y$, $N = x \ln x$, $M_y \neq N_x$. $\frac{M_y - N_x}{N} = \frac{1 - \ln x - 1}{x \ln x} = -\frac{1}{x} =$ fn of x alone,

so $\sigma(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$. Thus, scale the ODE as $\frac{y}{x} dx + \ln x\, dy = 0$.

$\partial F/\partial x = \frac{y}{x} \rightarrow F(x,y) = \int \frac{y}{x} \partial x = y \ln x + A(y)$

$\partial F/\partial y = \cancel{\ln x} = \cancel{\ln x} + A'(y)$ so $A(y) = C$. Thus, $F(x,y) = y \ln x + C =$ const.

gives $y \ln x = B$ or $y(x) = B/\ln x$.

(e) $M = 1$, $N = x$, $M_y \neq N_x$. $\frac{M_y - N_x}{N} = \frac{0-1}{x} =$ fn of x alone, so $\sigma(x) = e^{\int -\frac{1}{x} dx} = \frac{1}{x}$

Thus, scale the ODE as $\frac{1}{x} dx + dy = 0$.

$\partial F/\partial x = \frac{1}{x} \to F(x,y) = \int \frac{1}{x}\partial x = \ln x + A(y)$

$\partial F/\partial y = 1 = 0 + A'(y)$ so $A(y) = y + C$

Thus, $F(x,y) = \ln x + y + C = \text{const.}$ gives $\ln x + y = B$, or, $y(x) = -\ln x + B$.

(h) Here, "y" is z. $M = 1-x-z, N = 1, M_z \neq N_y$. $\frac{M_z - N_x}{N} = \frac{-1-0}{1} = -1 =$ fn. of x alone so $\sigma(x) = e^{\int -dx} = e^{-x}$. Thus, scale the ODE as

$$e^{-x}(1-x-z)dx + e^{-x}dz = 0$$

$\partial F/\partial x = e^{-x}(1-x-z) \to F(x,z) = \int e^{-x}(1-x-z)\partial x = e^{-x}(x+z) + A(z)$

$\partial F/\partial z = \cancel{e^{-x}} = \cancel{e^{-x}} + A'(z)$ so $A(z) = C$. Thus, $F(x,z) = e^{-x}(x+z) + C = \text{const.}$ gives $e^{-x}(x+z) = B$ or, if we wish, $z(x) = Be^{x} - x$

6. $\underbrace{e^{\int p dx}(py-q)}_{M} dx + \underbrace{e^{\int p dx}}_{N} dy = 0$; $M_y = pe^{\int p dx}$, $N_x = pe^{\int p dx}$ ✓

$\partial F/\partial x = e^{\int p dx}(py-q) \to F(x,y) = \int e^{\int p dx}(py-q)\partial x + A(y)$

$\partial F/\partial y = e^{\int p dx} = \underbrace{\int p e^{\int p dx} dx}_{\uparrow \text{ This is } d(e^{\int p dx})} + A'(y)$, so this integral gives $e^{\int p dx}$, which cancels with the like term on the left, giving $0 = A'(y)$, $A(y) = \text{const.}$

Thus, $F(x,y) = \int e^{\int p dx}(py-q)dx + \text{const.} = \text{const.}$ gives

$$y\underbrace{\int p e^{\int p dx} dx}_{\uparrow \text{ this} = e^{\int p dx}\text{, as noted above}} - \int e^{\int p dx} q\,dx = C$$

Thus, $ye^{\int p dx} = \int e^{\int p dx} q\,dx + C$ or $y(x) = e^{-\int p dx}\left(\int e^{\int p dx} q\,dx + C\right)$

NOTE: Observing that $\int p e^{\int p dx} dx = \int d(e^{\int p dx}) = e^{\int p dx}$ is tricky. If we reverse the order the solution is simpler:

$\partial F/\partial y = e^{\int p dx} \to F(x,y) = ye^{\int p dx} + B(x)$

$\partial F/\partial x = e^{\int p dx}(\cancel{py} - q) = \cancel{ype^{\int p dx}} + B'(x)$ gives $B(x) = -\int e^{\int p dx} q\,dx + \text{const.}$

so $F(x,y) = \text{const.}$ gives $ye^{\int p dx} - \int e^{\int p dx} q\,dx + \text{const.} = \text{const.}$, which gives the same result, but more easily.

7. (b) $(M_y - N_x)/N = (3x+4y-6x-4y)/(3x^2+4xy) \neq$ fn. of x alone,

(")/M = (")$/(3xy+2y^2) \neq$ " " y ", so $\sigma(x)$ and $\sigma(y)$ do not exist. Try $\sigma = x^a y^b$: $\underbrace{x^a y^b(3xy+2y^2)}_{\text{new } M}dx + \underbrace{x^a y^b(3x^2+4xy)}_{\text{new } N}dy = 0$

Set $M_y = N_x$, i.e., $3x^{a+1}(b+1)y^b + 2x^a(b+2)y^{b+1} = 3(a+2)x^{a+1}y^b + 4(a+1)x^a y^{b+1}$ which can be satisfied by setting $3(b+1) = 3(a+2)$ and $2(b+2) = 4(a+1)$, i.e., $a=1$ and $b=2$. Then our exact equation is

$$(3x^2y^3 + 2xy^4)dx + (3x^3y^2 + 4x^2y^3)dy = 0$$

$\partial F/\partial x = 3x^2y^3 + 2xy^4 \to F = \int(3x^2y^3 + 2xy^4)\partial x = x^3y^3 + x^2y^4 + A(y)$

$\partial F/\partial y = 3\cancel{x^3y^2} + 4\cancel{x^2}y^3 = 3\cancel{x^3y^2} + 4\cancel{x^2}y^3 + A'(y) \to A(y) = \text{const.}$

so $F(x,y) = \text{Const.}$ gives the solution $x^3y^3 + x^2y^4 = C$.

8. The idea is that $f(x)dx + g(y)dy = 0$ is exact, for any functions $f(x)$ and $g(y)$. Thus, $h(y)dx + i(x)dy = 0$ can be made exact, easily, by dividing by $i(x)$ and $h(y)$, to obtain $\frac{1}{i(x)}dx + \frac{1}{h(y)}dy = 0$. That is, $\sigma(x,y) = 1/[i(x)h(y)]$.

(b) Thus, $e^{-3x}dx - y^{-2}dy = 0$. We can say $\partial F/\partial x = e^{-3x}$ so $F = \int e^{-3x}\partial x =$ etc and $\partial F/\partial y = -y^{-2}$ so ... etc, but it is simpler (and equivalent) to merely integrate: $\int e^{-3x}dx - \int y^{-2}dy = 0$, $\frac{e^{-3x}}{-3} + \frac{1}{y} = C$, or, $y(x) = 1/(C + \frac{1}{3}e^{-3x})$.

(c) $\cot x\,dx - e^{-2y}dy = 0$, $\int \cos x\,dx/\sin x - \int e^{-2y}dy =$ const., $\ln(\sin x) + \frac{1}{2}e^{-2y} = C$ or, $y(x) = -\frac{1}{2}\ln[A - 2\ln(\sin x)]$ (2C → A, for convenience)

9\. (b) $\underbrace{(2r\sin\theta+1)}_{M(r,\theta)}dr + \underbrace{r^2\cos\theta}_{N(r,\theta)}d\theta = 0$, $M_\theta = 2r\cos\theta = N_r$ so exact.

$\partial F/\partial r = 2r\sin\theta + 1 \to F(r,\theta) = \int(2r\sin\theta+1)\partial r = r^2\sin\theta + r + A(\theta)$

$\partial F/\partial\theta = r^2\cancel{\cos\theta} = r^2\cancel{\cos\theta} + A'(\theta)$ gives $A(\theta) =$ const., so $F(r,\theta) =$ const. gives the solution $r^2\sin\theta + r = C$ (could solve for $r(\theta)$ or $\theta(r)$, if desired).

(c) $(2xy - e^y)dx + x(x - e^y)dy = 0$, $M_y = 2x - e^y = N_x$, so exact.

$\partial F/\partial x = 2xy - e^y \to F(x,y) = \int(2xy - e^y)\partial x = x^2y - xe^y + A(y)$

$\partial F/\partial y = x^2 - xe^y = x^2 - xe^y + A'(y)$ gives $A(y) =$ const, so $x^2y - xe^y = C$.

10\. $\sigma = 1$ (or any nonzero constant)

11\. (b) Not necessarily. For ex. if $M(x,y) = e^{xy}$ and $M(y,x) = e^{yx}$, then $M_y(x,y) = xe^{xy}$ whereas $M_x(y,x) = ye^{xy} \neq xe^{xy}$.

12\. $F(a,b) = C$, so particular solution is $F(x,y) = F(a,b)$.

13\. Does $(M+P)_y = (N+Q)_x$? Yes, because it gives $\cancel{M_y} + \cancel{P_y} = \cancel{N_x} + \cancel{Q_x}$ or $0=0$ ✓