SECOND EDITION

ADVANCED ENGINEERING ELECTROMAGNETICS

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SOLUTIONS MANUAL

CHAPTER 1

1.1 $\nabla \times \underline{\mathcal{H}} = \underline{\mathcal{J}}_{ic} + \frac{\partial \underline{\mathcal{D}}}{\partial t}$

Taking the divergence of both sides

$$\nabla \cdot (\nabla \times \underline{\mathcal{H}}) = \nabla \cdot \underline{\mathcal{J}}_{ic} + \nabla \cdot \frac{\partial \underline{\mathcal{D}}}{\partial t} = \nabla \cdot \underline{\mathcal{J}}_{ic} + \frac{\partial}{\partial t} \nabla \cdot \underline{\mathcal{D}}$$

Using the vector identity of

$\nabla \cdot (\nabla \times \underline{A} = 0)$ and (1-3) we can write that

$$0 = \nabla \cdot \underline{\mathcal{J}}_{ic} + \frac{\partial}{\partial t}(q_{ve}) \Rightarrow \boxed{\nabla \cdot \underline{\mathcal{J}}_{ic} = -\frac{\partial q_{ve}}{\partial t}}$$

1.2 $\nabla \times \underline{\mathcal{E}} = -\underline{\mathcal{M}}_i - \frac{\partial \underline{\mathcal{B}}}{\partial t}$

Taking a surface integral of both sides, we can write that

$$\iint_S (\nabla \times \underline{\mathcal{E}}) \cdot d\underline{s} = -\iint_S \underline{\mathcal{M}}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{\mathcal{B}} \cdot d\underline{s}$$

Applying Stokes' theorem of (1-7) to the left side of the equation above leads to

$$\oint_C \underline{\mathcal{E}} \cdot d\underline{\ell} = -\iint_S \underline{\mathcal{M}}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{\mathcal{B}} \cdot d\underline{s}$$

Using the same procedure, we can write that

$$\oint_C \underline{\mathcal{H}} \cdot d\underline{\ell} = \iint_S \underline{\mathcal{J}}_i \cdot d\underline{s} + \iint_S \underline{\mathcal{J}}_c \cdot d\underline{s} + \frac{\partial}{\partial t} \iint_S \underline{\mathcal{D}} \cdot d\underline{s}$$

For the remaining three equations of Table 1-1 we proceed as follows:

$$\nabla \cdot \underline{\mathcal{D}} = q_{ve}$$

Taking a volume integral of both sides, we can write that

$$\iiint_V \nabla \cdot \mathcal{D} \, dv = \iiint_V q_{ve} ds = Q_e$$

1.2 cont'd

Applying the divergence theorem of (1-8) on the left side of the equation above leads to

$$\oiint_S \underline{D} \cdot d\underline{s} = Q_e$$

Using the same procedure, we can write that

$$\oiint_S \underline{B} \cdot d\underline{s} = Q_m$$

and

$$\oiint_S \underline{J}_{ic} \cdot d\underline{s} = -\frac{\partial}{\partial t}\iiint_V q_{ve}\, dv = -\frac{\partial Q_e}{\partial t}$$

1.3

(a) $\underline{D} = \hat{a}_x(3+x)$

$$Q_e = \oiint_S \underline{D}\cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=0}\cdot(-\hat{a}_x\, dy\, dz) + \int_0^1\int_0^1 \hat{a}_x(3+x)\Big|_{x=1}\cdot dy\, dz = -3+4 = 1$$

(b) $\underline{D} = \hat{a}_y(4+y^2)$

$$Q_e = \oiint_S \underline{D}\cdot d\underline{s} = \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=0}\cdot(-\hat{a}_y\, dx\, dz) + \int_0^1\int_0^1 \hat{a}_y(4+y^2)\Big|_{y=1}\cdot \hat{a}_y\, dx\, dz = -4+5 = 1$$

1.4 $\underline{D}_2 = 6\hat{a}_x + 3\hat{a}_z$, $\chi_{e2} = \varepsilon_{sr2} - 1 = 2.56 - 1 = 1.56$

(a) $$\underline{E}_2 = \frac{\underline{D}_2}{\varepsilon_{s2}} = \frac{1}{2.56\varepsilon_0}(6\hat{a}_x + 3\hat{a}_z) = \frac{1}{\varepsilon_0}\left(\frac{6}{2.56}\hat{a}_x + \frac{3}{2.56}\hat{a}_z\right)$$

$$\underline{E}_2 = \frac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)$$

(b) $$\underline{P}_2 = \varepsilon_0 \chi_e \underline{E}_2 = \varepsilon_0\left[1.56\frac{1}{\varepsilon_0}(2.34\hat{a}_x + 1.1718\hat{a}_z)\right]$$

$$\underline{P}_2 = 3.65\hat{a}_x + 1.8289\hat{a}_z$$

(c) $\underline{E}_{1x} = \underline{E}_{2x}$ Continuity of tangential components of $\underline{E}$-field

$$\frac{D_{1x}}{\varepsilon_0} = \frac{D_{2x}}{\varepsilon_{r2}\varepsilon_0} \Rightarrow D_{1x} = \frac{D_{2x}}{\varepsilon_{r2}} = \frac{6}{2.56} = 2.344$$

$\hat{n}\cdot(\underline{D}_2 - \underline{D}_1) = q_{es}$ Discontinuity of normal components of $\underline{D}$ density.

$\hat{n} = \hat{a}_z$: $D_{2z} - D_{1z} = q_{es} = 0.2 \Rightarrow D_{1z} = D_{2z} - q_{es} = 3 - 0.2 = 2.8$

$$D_{1z} = 2.8$$

$$\underline{D}_1 = 2.344\hat{a}_x + 2.8\hat{a}_z$$

cont'd

$\boxed{1.4 \text{ cont'd}}$

(d) $\varepsilon_1 \underline{E}_1 = \underline{D}_1 \Rightarrow \varepsilon_o \underline{E}_1 = \underline{D}_1 \Rightarrow \underline{E}_1 = \frac{1}{\varepsilon_o}\underline{D}_1 = \frac{1}{\varepsilon_o}\left(2.34\hat{a}_x + 2.8\hat{a}_z\right)$

$$\underline{E}_1 = \frac{1}{\varepsilon_o}\left(2.34\hat{a}_x + 2.8\hat{a}_z\right)$$

(e) $\chi_{e1} = \varepsilon_{sr1} - 1 = 1 - 1 = 0$

$$\underline{P}_1 = \varepsilon_o \chi_{e1} \underline{E}_1 = 0$$

$\boxed{1.5}$ $\underline{H}_1 = 3\hat{a}_x + \hat{a}_z 9, \quad \mu_2 = 4\mu_o$

(a) $\underline{B}_1 = \mu_o \underline{H}_1 = \mu_o(3\hat{a}_x + 9\hat{a}_z)$

(b) $\underline{M}_1 = \chi_{m1}\underline{H}_1, \quad \chi_{m1} = \mu_{sr} - 1 = 1 - 1 = 0$

$$\underline{M}_1 = 0$$

(c) $H_{2x} = H_{1x} = 3$ Continuity of tangential $\underline{H}$-field.

<u>Continuity of normal $\underline{B}$ density</u>

$$B_{2z} = B_{1z} = 9\mu_o \Rightarrow \mu_2 H_{2z} = 9\mu_o \Rightarrow 4\mu_o H_{2z} = 9\mu_o$$

$$H_{2z} = \frac{9}{4} = 2.25$$

$$\underline{H}_2 = 3\hat{a}_x + 2.25\hat{a}_z$$

(d) $\underline{B}_2 = \mu_2 H_2 = 4\mu_o\left(3\hat{a}_x + \frac{9}{4}\hat{a}_z\right) = \mu_o\left(12\hat{a}_x + 9\hat{a}_z\right)$

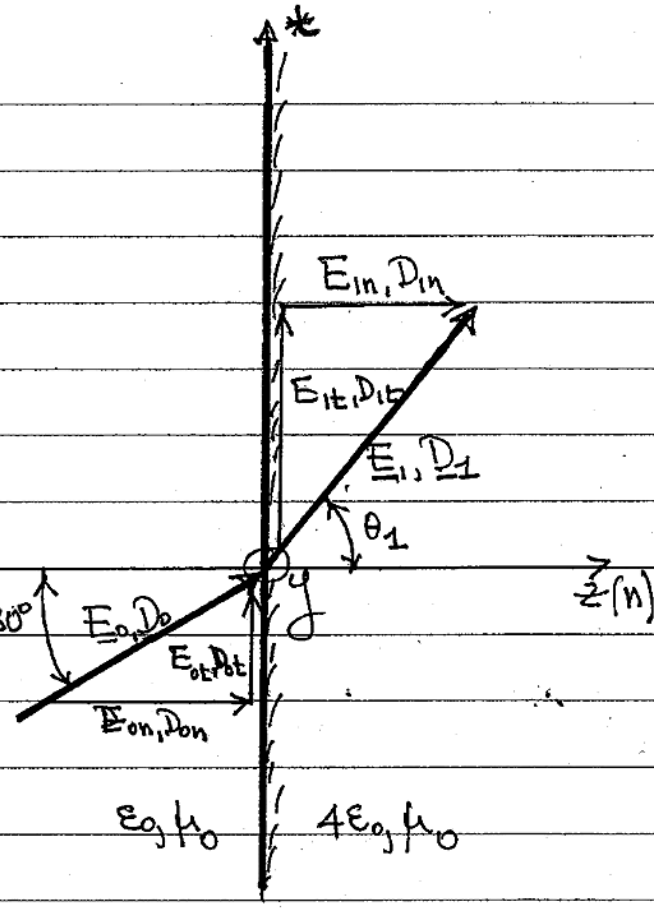
(e) $\underline{M}_2 = \chi_{m2}\underline{H}_2 \qquad \chi_{m2} = \mu_{2r} - 1 = 4 - 1 = 3$

$$\underline{M}_2 = 3\left(3\hat{a}_x + 2.25\hat{a}_z\right) = 9\hat{a}_x + 6.75\hat{a}_z$$

$\boxed{1.6}$ $D_0 = \varepsilon_0 E_0$

(a) $E_{0n} = E_0 \cos 30^0 = 0.866 E_0$

$E_{0t} = E_0 \sin 30^0 = 0.5 E_0$



$D_{0n} = \varepsilon_0 E_{0n} = 0.866 \varepsilon_0 E_0$

$D_{0t} = \varepsilon_0 E_{0t} = 0.5 \varepsilon_0 E_0$

From B.Cs.

$E_{1t} = E_{0t} = 0.5 E_0$

$D_{1t} = \varepsilon_1 E_{1t} = 0.5(4)\varepsilon_0 E_0$

$D_{1t} = 2\varepsilon_0 E_0$

$D_{1n} = D_{0n} = 0.866 \varepsilon_0 E_0$

$E_{1n} = \dfrac{D_{1n}}{\varepsilon_1} = \dfrac{0.866\varepsilon_0 E_0}{4\varepsilon_0}$

$E_{1n} = 0.2165 E_0$

$\boxed{E_1 = (0.5\hat{a}_t + 0.2165\hat{a}_n)E_0}$

$E_1 = \sqrt{(E_{1t})^2 + (E_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2}\,E_0 = \sqrt{0.25 + 0.046875}\,E_0$

$E_1 = \sqrt{0.296875}\,E_0 = 0.54486 E_0$

$\boxed{E_1 = 0.54486 E_0}$

$\boxed{D_1 = (2\hat{a}_t + 0.866\hat{a}_n)\varepsilon_0 E_0}$

$D_1 = \sqrt{(D_{1t})^2 + (D_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\,\varepsilon_0 E_0 = \sqrt{4 + 0.749956}\,\varepsilon_0 E_0$

$D_1 = \sqrt{4.749956}\,\varepsilon_0 E_0 = 2.17944 \varepsilon_0 E_0 = 0.54486(4\varepsilon_0)E_0$

$\boxed{D_1 = 2.17944 \varepsilon_0 E_0 = 0.54486(4\varepsilon_0)E_0}$

(b) $\theta_1 = \tan^{-1}\left(\dfrac{E_{1t}}{E_{1n}}\right) = \tan^{-1}\left(\dfrac{0.5E_0}{0.2165E_0}\right) = \tan^{-1}(2.309) = \boxed{66.587^0}$

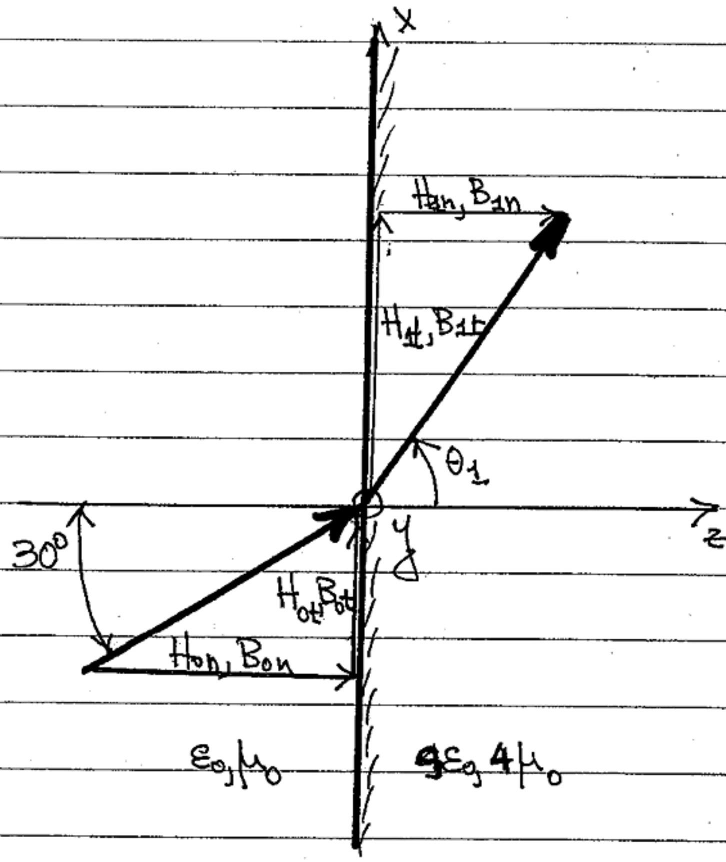
$\theta_1 = \tan^{-1}\left(\dfrac{D_{1t}}{D_{1n}}\right) = \tan^{-1}\left(\dfrac{2\varepsilon_0 E_0}{0.8666\varepsilon_0 E_0}\right) = \tan^{-1}\left(\dfrac{2}{0.86666}\right) = \tan^{-1}(2.308) = 66.571^0$

$\boxed{\theta_1 = 66.571^0}$

$\boxed{1.7}$ $B_o = \mu_o H_o$

$H_{on} = H_o \cos 30^\circ = 0.866 H_o$

$H_{ot} = H_o \sin 30^\circ = 0.5 H_o$



$B_{on} = \mu_o H_{on} = 0.866 \mu_o H_o$

$B_{ot} = \mu_o H_{ot} = 0.5 \mu_o H_o$

From B.Cs.

$H_{1t} = H_{ot} = 0.5 H_o$

$B_{1t} = \mu_1 H_{1t} = 0.5\, 4\mu_o H_o$

$B_{1t} = 2\mu_o H_o$

$B_{1n} = B_{on} = 0.866 \mu_o H_o$

$H_{1n} = \frac{B_{1n}}{\mu_1} = \frac{0.866\mu_o H_o}{4\mu_o}$

$H_{1n} = 0.2165 H_o$

$\boxed{H_1 = (0.5\hat{a}_t + 0.2165\hat{a}_n) H_o}$

$H_1 = \sqrt{(H_{1t})^2 + (H_{1n})^2} = \sqrt{(0.5)^2 + (0.2165)^2}\, H_o = \sqrt{0.296875}\, H_o$

$\boxed{H_1 = 0.5448\,6 H_o}$

$\underline{B_1} = \boxed{(2\hat{a}_t + 0.866\hat{a}_n)\mu_o H_o}$

$B_1 = \sqrt{(B_{1t})^2 + (B_{1n})^2} = \sqrt{(2)^2 + (0.866)^2}\, \mu_o H_o = \sqrt{4.751}\ \mu_o H_o$

$\boxed{B_1 = 2.18 \mu_o H_o = 0.545 (4\mu_o) H_o}$

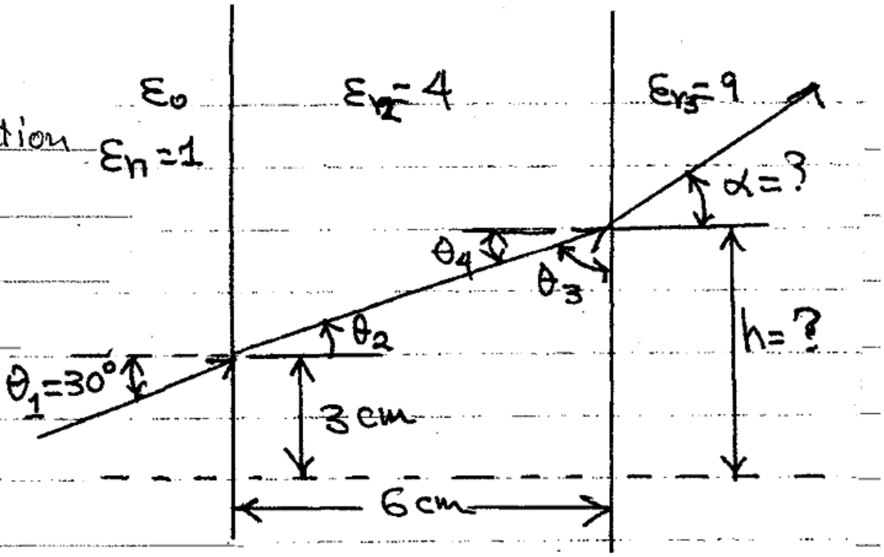
b. $\theta_1 = \tan^{-1}\left(\frac{H_{1t}}{H_{1n}}\right) = \tan^{-1}\left(\frac{0.5 H_o}{0.2165 H_o}\right) = \tan^{-1}(2.309) = 66.587^\circ$

$\boxed{\theta_1 = 66.587^\circ}$

$\theta_1 = \tan^{-1}\left(\frac{B_{1t}}{B_{1n}}\right) = \tan^{-1}\left(\frac{2\mu_o H_o}{0.8666\mu_o H_o}\right) = \tan^{-1}(2.308) = 66.571^\circ$

$\boxed{\theta_1 = 66.571^\circ}$

1.8 Snell's Law of Refraction



(a) $\beta_1\sin\theta_1=\beta_2\sin\theta_2$

$\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1=\omega\sqrt{\mu_2\varepsilon_2}\sin\theta_2$

Since $\mu_1=\mu_2=\mu_0$

$\sqrt{\varepsilon_1}\sin\theta_1=\sqrt{\varepsilon_2}\sin\theta_2$

$\sin\theta_2=\sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_1$

$=\sqrt{\frac{1}{\varepsilon_2/\varepsilon_1}}\sin\theta_1$

$\sin\theta_2=\sqrt{\frac{1}{4}}\sin\theta_1=\frac{1}{2}\sin(30^\circ)=\frac{1}{2}\left(\frac{1}{2}\right)=\frac{1}{4}$

$\theta_2=\sin^{-1}\left(\frac{1}{4}\right)=14.4775^\circ=\theta_4$

$\beta_2\sin\theta_4=\beta_3\sin\alpha\Rightarrow\omega\sqrt{\mu_2\varepsilon_2}\sin\theta_4=\omega\sqrt{\mu_3\varepsilon_3}\sin\alpha$

Since $\mu_2=\mu_3=\mu_0$:

$\sqrt{\varepsilon_3}\sin\alpha=\sqrt{\varepsilon_2}\sin\theta_4$

$\alpha=\sin^{-1}\left[\sqrt{\frac{\varepsilon_2}{\varepsilon_3}}\sin\theta_4\right]=\sin^{-1}\left(\sqrt{\frac{4}{9}}\sin\theta_4\right)=\sin^{-1}\left[\frac{2}{3}\sin(14.4775^\circ)\right]$

$\alpha=\sin^{-1}\left[\frac{2}{3}(0.25)\right]=\sin^{-1}(0.1667)$

$$\boxed{\alpha=9.594^\circ}$$

(b) $\tan\theta_2=\left(\frac{h-3}{6}\right)=\tan(14.4775^\circ)$

$\frac{h-3}{6}=(0.2582)\Rightarrow h-3=6(0.2582)$

$h=3+6(0.2582)=4.5492\text{ cm}$

$$\boxed{h=4.5492\text{ cm}}$$

1.9 $\underline{D}=\varepsilon_0\underline{E}$, $Q_e=\oiint_S\underline{D}\cdot d\underline{s}$

$$Q_e=\oiint_S\underline{D}\cdot d\underline{s}=\varepsilon_0\iint_S\hat{a}_z E_z\Big|_{z=0}\cdot(-\hat{a}_z d\underline{s})+\varepsilon_0\iint_S\hat{a}_z E_z\Big|_{z=h}\cdot\hat{a}_z d\underline{s}$$

$$Q_e=-\varepsilon_0\iint_S\left[-\frac{c}{h}-\frac{bh^2}{6\varepsilon_0}\right]ds+\varepsilon_0\iint_S\left[-\frac{c}{h}+2\frac{bh^2}{6\varepsilon_0}\right]ds=\frac{3h^2b}{6}(\pi a^2)=\frac{\pi}{2}b(ha)^2$$

$\boxed{1.10}$ $\varepsilon_r = 4$, $\mu_r = 9$, $a = 4$ cm

$$\underline{H} = 3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z$$

(a) $\underline{B} = \mu\underline{H} = \mu_r\mu_o\underline{H} = 9\mu_o(3\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z) = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 72\hat{a}_z)$

$$\underline{B} = = \mu_o(27\hat{a}_\rho + 54\hat{a}_\phi + 27\hat{a}_z)$$

(b) $\underline{H}_o = \hat{a}_\rho H_{\rho o} + \hat{a}_\phi H_{\phi o} + \hat{a}_z H_{zo}$

$$H_{\phi o} = H_\phi = 6$$

$$H_{zo} = H_z = 8$$

$$B_{\rho o} = B_\rho = \mu_o H_{\rho o} = \mu H_\rho = \mu_r \mu_o H_\rho$$

$$H_{\rho o} = \mu_r H_\rho = 9(3) = 27$$

$$\underline{H}_o = (27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

(c)

$$\underline{B}_o = \mu_o \underline{H}_o = \mu_o(27\hat{a}_\rho + 6\hat{a}_\phi + 8\hat{a}_z)$$

$\boxed{1.11}$ $\nabla \cdot \underline{E} = 0$ for a source-free and homogeneous medium.

Thus

$$\nabla \cdot \underline{E} = \left[\hat{a}_x \frac{\partial}{\partial x} + \hat{a}_y \frac{\partial}{\partial y} + \hat{a}_z \frac{\partial}{\partial z}\right] \cdot \left[\hat{a}_x A(x+y) + \hat{a}_y B(x-y)\right] \cos(\omega t) = 0$$

$$A\frac{\partial}{\partial x}(x+y) + B\frac{\partial}{\partial y}(x-y) = A(1) + B(-1) = 0 \Rightarrow A = B$$

Also

$$\underline{E} = \hat{a}_x A(x+y) + \hat{a}_y B(x-y)$$

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x+y) & B(x-y) & 0 \end{vmatrix}$$

$$\underline{H} = -\frac{1}{j\omega\mu}\left[\hat{a}_x(0) + \hat{a}_y(0) + \hat{a}_z(B-A)\right]$$

$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E} \Rightarrow \underline{E} = 0 \Rightarrow A = B = 0$$

1.12

$$\underline{\mathcal{B}} = \hat{a}_z \frac{10^{-12}}{1+25\rho}\cos(1500\pi t)$$

a. $$\Psi_m = \iint_{S_o} \underline{\mathcal{B}}\cdot d\underline{s} = \int_0^{2\pi}\int_0^a \hat{a}_z\mathcal{B}_z\cdot\hat{a}_z\,\rho\,d\rho\,d\phi = \int_0^{2\pi}\int_0^a \mathcal{B}_z\,\rho\,d\rho\,d\phi$$

$$= 2\pi\int_0^a \mathcal{B}_z\,\rho\,d\rho = 2\pi\times10^{-12}\cos(1500\pi t)\int_0^a \frac{\rho}{1+25\rho}\,d\rho$$

Using the integral

$$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}\left[a+bu-a\ln(a+bu)\right]$$

we can write the flux as

$$\Psi_m = 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{(25)^2}\left[1+25\rho-\ln(1+25\rho)\right]_0^a\right\}$$

$$= 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{625}\left[25a-\ln(1+25a)\right]\right\}$$

$$\Psi_m = 2\pi\times10^{-12}\cos(1500\pi t)\left\{\frac{1}{625}\left[2.5-\ln(3.5)\right]\right\} = 1.2539\times10^{-14}\cos(1500\pi t)$$

b. $$\oint_C \underline{\mathcal{E}}\cdot d\underline{\ell} = -\frac{\partial\psi}{\partial t}$$

$$\int_0^{2\pi}(\hat{a}_\phi\mathcal{E}_\phi)\cdot\hat{a}_\phi\,\rho\,d\phi = -\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\rho}\underline{\mathcal{B}}\cdot d\underline{s} = -\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{\rho}(\hat{a}_z\mathcal{B}_z)\cdot\hat{a}_z\,\rho\,d\rho\,d\phi$$

$$2\pi\rho\,\mathcal{E}_\phi = -2\pi\frac{\partial}{\partial t}\int_0^{\rho}\mathcal{B}_z\,\rho\,d\rho = 2\pi\times10^{-12}(1500\pi)\sin(1500\pi t)\int_0^{\rho}\frac{\rho}{1+25\rho}\,d\rho$$

$$\rho\,\mathcal{E}_\phi = 1500\pi\times10^{-12}\sin(1500\pi t)\left\{\frac{1}{(25)^2}\left[1+25\rho-\ln(1+25\rho)\right]_0^{\rho}\right\}$$

$$\mathcal{E}_\phi = \frac{7.5398\times10^{-12}}{\rho}\left[25\rho-\ln(1+25\rho)\right]\sin(1500\pi t)$$

<u>To check:</u> Use Maxwell's equation $\nabla\times\underline{\mathcal{E}} = -\frac{\partial\underline{\mathcal{B}}}{\partial t}$

$$\hat{a}\ \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\,\mathcal{E}_\phi) = -\frac{\partial}{\partial t}\left[\hat{a}_z\frac{10^{-12}}{1+25\rho}\cos(1500\pi t)\right]$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{7.5398\times10^{-12}\left[25\rho-\ln(1+25\rho)\right]\sin(1500\pi t)\right\} = 1500\pi\times10^{-12}\sin(1500\pi t)\frac{1}{1+25\rho}$$

$$\frac{4.712\times10^{-9}}{1+25\rho}\sin(1500\pi t) = \frac{4.712\times10^{-9}}{1+25\rho}\sin(1500\pi t)$$

QED

$\boxed{1.13}$ $\underline{\mathcal{B}} = \hat{a}_x B_x \cos(2y)\sin(\omega t - \pi z) + \hat{a}_y B_y \cos(2x)\cos(\omega t - \pi z)$

$$\nabla\times\underline{\mathcal{H}} = \cancel{\underline{\mathcal{J}}_i}^{\,0} + \cancel{\underline{\mathcal{J}}_c}^{\,0} + \underline{\mathcal{J}}_d = \underline{\mathcal{J}}_d = \frac{\partial\underline{\mathcal{D}}}{\partial t}$$

$$\underline{\mathcal{J}}_d = \nabla\times\underline{\mathcal{H}} = \frac{1}{\mu_0}\nabla\times\underline{\mathcal{B}} = \frac{1}{\mu_0}\left\{-\hat{a}_x\frac{\partial\mathcal{B}_y}{\partial z} + \hat{a}_y\frac{\partial\mathcal{B}_x}{\partial z} + \hat{a}_z\left[\frac{\partial\mathcal{B}_y}{\partial x} - \frac{\partial\mathcal{B}_x}{\partial y}\right]\right\}$$

$$= \frac{1}{\mu_0}\Big\{-\hat{a}_x \pi B_y \cos(2x)\sin(\omega t - \pi z) - \hat{a}_y \pi B_x \cos(2y)\cos(\omega t - \pi z)$$
$$+\hat{a}_z\left[-2B_y \sin(2x)\cos(\omega t - \pi z) + 2B_x \sin(2y)\sin(\omega t - \pi z)\right]\Big\}$$

$$\underline{\mathcal{J}}_d = -\hat{a}_x 2.5\times10^6 B_y \cos(2x)\sin(\omega t - \pi z) - \hat{a}_y 2.5\times10^6 B_x \cos(2y)\cos(\omega t - \pi z)$$
$$+\hat{a}_z\left[-1.59\times10^6 B_y \sin(2x)\cos(\omega t - \pi z) + 1.59\times10^6 B_x \sin(2y)\sin(\omega t - \pi z)\right]$$

$\boxed{1.14}$ $\underline{\mathcal{J}}_d = \hat{a}_x yz + \hat{a}_y y^2 + \hat{a}_z xyz$, $\quad \mathcal{I}_d = \oiint_S \underline{\mathcal{J}}_d \cdot d\underline{s}$

$$\mathcal{I}_d = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\hat{a}_x yz)\Big|_{x=-1/2}\cdot(-\hat{a}_x\,dy\,dz) + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\hat{a}_x yz)\Big|_{x=1/2}\cdot(\hat{a}_x\,dy\,dz)$$
$$+\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\hat{a}_y y^2)\Big|_{y=-1/2}\cdot(-\hat{a}_y\,dx\,dz) + \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\hat{a}_y y^2)\Big|_{y=1/2}\cdot(\hat{a}_y\,dx\,dz)$$
$$+\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\hat{a}_z xyz)\Big|_{z=-1/2}\cdot(-\hat{a}_z\,dx\,dy) + \int_{-1/2}^{1/2}\int_{1/2}^{1/2}(\hat{a}_z xyz)\Big|_{z=1/2}\cdot(\hat{a}_z\,dx\,dy)$$

$$\mathcal{I}_d = 0 + 0 + 0 = 0$$

$\boxed{1.15}$ $\underline{\mathcal{D}} = \hat{a}_r \dfrac{10^{-9}}{4\pi}\dfrac{1}{r^2}\cos(\omega t - \beta r) = \mathrm{Re}\left[\hat{a}_r \dfrac{10^{-9}}{4\pi r^2} e^{j(\omega t - \beta r)}\right] = \mathrm{Re}\left[\underline{D}e^{j\omega t}\right]$

where $\underline{D} = \hat{a}_r \dfrac{10^{-9}}{4\pi r^2} e^{-j\beta r}$, $\quad \beta = \omega\sqrt{\mu_0\varepsilon_0}$

$$Q_e = \oiint_S \underline{D}\cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi}(\hat{a}_r D_r)\cdot\hat{a}_r r^2 \sin\theta\,d\theta\,d\phi = \frac{10^{-9}}{4\pi}e^{-j\beta r}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi$$

$$Q_e = \frac{10^{-9}}{4\pi}e^{-j\beta r}\,2\pi(-\cos\theta)\Big|_0^{\pi} = 10^{-9}e^{-j\beta r}$$

$|Q_e| = 10^{-9}$ Coulombs

$\boxed{1.16}$ $\mathcal{E} = Re\left[\underline{E}(r,\theta)e^{j\omega t}\right] = Re\left[\hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r} e^{j\omega t}\right] = \hat{a}_\phi E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$

where $\underline{E}(r,\theta) = \hat{a}_\phi E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$

Using Maxwell's equation

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_r \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi \sin\theta) - \hat{a}_\theta \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right]$$

$$= -\hat{a}_r \frac{2E_o}{j\omega\mu_o}\cos\theta \frac{e^{-j\beta_o r}}{r^2} - \hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o \sin\theta \frac{e^{-j\beta_o r}}{r}$$

$$\underline{H} \simeq -\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o\sin\theta \frac{e^{-j\beta_o r}}{r} \Rightarrow \mathcal{H} = Re\left[\underline{H}e^{j\omega t}\right] \simeq -\hat{a}_\theta\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o \sin\theta \frac{\cos(\omega t - \beta_o r)}{r}$$

$\boxed{1.17}$ $v(t) = 10\cos(\omega t)$, $\mathcal{J}_c = \sigma\mathcal{E}$, $\mathcal{J}_d = \frac{\partial \mathcal{D}}{\partial t} = \varepsilon\frac{\partial \mathcal{E}}{\partial t}$

$$\mathcal{E}(t) = \frac{10}{2\times 10^{-2}}\cos(\omega t) = 500\cos(\omega t),\quad \frac{\partial\mathcal{E}}{\partial t} = -500\omega\sin(\omega t)$$

a. $f = 1$ MHz

$$|\mathcal{J}_c|_{max} = |\sigma\mathcal{E}|_{max} = (3.7\times 10^{-4})500 = 0.185$$

$$|\mathcal{J}_d|_{max} = \left|\varepsilon\frac{\partial\mathcal{E}}{\partial t}\right|_{max} = |-500\omega\varepsilon\sin(\omega t)|_{max} = \frac{2.56}{36} = 0.07111$$

b. $f = 100$ MHz

$$|\mathcal{J}_c|_{max} = |\sigma\mathcal{E}|_{max} = (3.7\times 10^{-4})500 = 0.185$$

$$|\mathcal{J}_d|_{max} = \left|\varepsilon\frac{\partial\mathcal{E}}{\partial t}\right|_{max} = \frac{2.56(100)}{36} = 7.111$$

$\boxed{1.18}$ $\mathcal{E} = [\hat{a}_y 5 + \hat{a}_z 10]\cos(\omega t - \beta x) = Re\left[(\hat{a}_y 5 + \hat{a}_z 10)e^{j(\omega t - \beta x)}\right] = Re\left[\underline{E}e^{j\omega t}\right]$

where $\underline{E} = (\hat{a}_y 5 + \hat{a}_z 10)e^{-j\beta x}$

a. $\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = \hat{a}_y\left(-\frac{\partial E_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x}\right) = 4.244\times 10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)e^{-j\beta x}$

$$\mathcal{H} = Re\left[\underline{H}e^{j\omega t}\right] = 4.244\times 10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t - \beta x)$$

(continued)

$\boxed{1.18 \text{ cont'd.}}$ The tangential components of the electric and magnetic fields must be <u>continuous</u> across the boundaries.

The normal components of the electric and magnetic fields must be <u>discontinuous</u> across the boundaries.

b. $\mathcal{E}_z^o(y=h^+) = 10\cos(\omega t - \beta x)$; $\mathcal{E}_y^o(y=h^+) = \frac{\varepsilon}{\varepsilon_o}(5)\cos(\omega t-\beta x) = 12.8\cos(\omega t - \beta x)$

$$\underline{\mathcal{E}}^o(y=h^+) = (\hat{a}_y\, 12.8 + \hat{a}_z 10)\cos(\omega t - \beta x)$$

$$\mathcal{H}_z^o(y=h^+) = 4.244\times10^{-3}(5)\cos(\omega t-\beta x);\ \mathcal{H}_y^o(y=h^+) = 4.244\times10^{-3}(-10)\cos(\omega t - \beta x)$$

$$\underline{\mathcal{H}}(y=h^+) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t - \beta x)$$

In a similar manner

$$\underline{\mathcal{E}}^o(y=-h^+) = (\hat{a}_y\, 12.8 + \hat{a}_z 10)\cos(\omega t - \beta x)$$

$$\underline{\mathcal{H}}(y=-h^-) = 4.244\times10^{-3}(-\hat{a}_y 10 + \hat{a}_z 5)\cos(\omega t - \beta x)$$

$\boxed{1.19}$

$$\underline{\mathcal{J}} = \hat{a}_z\, 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)$$

$$\underline{\mathcal{J}}(y=0.25\times10^{-3}) = \hat{a}_z\, 10^4 e^{-10^6(2.5\times10^{-4})}\cos(2\pi\times10^9 t) = \hat{a}_z 10^4 e^{-250}\cos(2\pi\times10^9 t) \simeq 0$$

$$I = \iint_S \underline{\mathcal{J}}\cdot d\underline{s} \simeq 2\int_0^{2.5\times10^{-4}}\int_0^{5\times10^{-3}} \left[\hat{a}_z 10^4 e^{-10^6 y}\cos(2\pi\times10^9 t)\right]\cdot \hat{a}_z\, dx\, dy$$

$$I \simeq 2(5\times10^{-3})(10^4)\cos(2\pi\times10^9 t)\int_0^{2.5\times10^{-4}} e^{-10^6 y}\, dy = 10^{-4}\cos(2\pi\times10^9 t)$$

$\boxed{1.20}$ a. $\underline{\mathcal{E}} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta_z z) = \mathrm{Re}\left[\hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)e^{j(\omega t - \beta_z z)}\right] = \mathrm{Re}\left[\underline{E}\, e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_y E_o \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = \hat{a}_x\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial z} - \hat{a}_z\frac{1}{j\omega\mu_o}\frac{\partial E_y}{\partial x}$$

$$= -\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} + \hat{a}_z j\frac{E_o}{\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\underline{\mathcal{H}} = \mathrm{Re}\left[\underline{H}\, e^{j\omega t}\right] = -\hat{a}_x\frac{\beta_z}{\omega\mu_o}E_o\sin\left(\frac{\pi}{a}x\right)\cos(\omega t - \beta_z z) + \hat{a}_z\frac{E_o}{\omega\mu_o}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\cos\left(\omega t + \frac{\pi}{2} - \beta_z z\right)$$

b. Using $\nabla\times\underline{H} = j\omega\varepsilon_o\underline{E} \Rightarrow \hat{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) = j\omega\varepsilon_o E_o\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$

$$\hat{a}_y\, j\frac{E_o}{\omega\mu_o}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right]\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} = j\omega\varepsilon_o E_o \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

$$\omega\varepsilon_o = \frac{1}{\omega\mu_o}\left[\beta_z^2 + \left(\frac{\pi}{a}\right)^2\right] \Rightarrow \beta_z = \pm\sqrt{\omega^2\mu_o\varepsilon_o - \left(\frac{\pi}{a}\right)^2}$$

$\boxed{1.21}$ $\underline{\mathcal{E}} = \hat{a}_\rho\left(\frac{100}{\rho}\right)\cos(10^8t-\beta z) = \text{Re}\left[\hat{a}_\rho\frac{100}{\rho}e^{j(10^8t-\beta z)}\right] = \text{Re}\left[\underline{E}\,e^{j\omega t}\right]$

where $\underline{E} = \hat{a}_\rho\frac{100}{\rho}e^{-j\beta z}$

a. $\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left(\hat{a}_\phi\frac{\partial E_\rho}{\partial z}\right) = \hat{a}_\phi\frac{\beta}{\omega\mu}\,\frac{100}{\rho}e^{-j\beta z}$

$\underline{\mathcal{H}} = \text{Re}\left[\underline{H}e^{j\omega t}\right] = \text{Re}\left[\underline{H}e^{j10^8t}\right] = \hat{a}_\phi\frac{\beta}{\omega\mu}\left(\frac{100}{\rho}\right)\cos(10^8t-\beta z)$

b. $\nabla\times\underline{H} = j\omega\varepsilon\,\underline{E} \Rightarrow -\hat{a}_\rho\frac{\partial H_\phi}{\partial z} = \hat{a}_\rho j\omega\varepsilon\,\frac{100}{\rho}e^{-j\beta z}$

$$\hat{a}_\rho j\frac{\beta^2}{\omega\mu_0}\left(\frac{100}{\rho}\right)e^{-j\beta z} = \hat{a}_\rho j\omega\varepsilon\left(\frac{100}{\rho}\right)e^{-j\beta z}$$

$$\omega\varepsilon = \frac{\beta^2}{\omega\mu_0} \Rightarrow \beta^2 = \omega^2\mu_0\varepsilon$$

c. $\underline{\mathcal{J}}_d = \varepsilon\frac{\partial\underline{\mathcal{E}}}{\partial t} = \varepsilon\frac{\partial}{\partial t}\left[\hat{a}_\rho\frac{100}{\rho}\cos(10^8t-\beta z)\right] = -\hat{a}_\rho\frac{10^{10}\varepsilon}{\rho}\sin(10^8t-\beta z)$

$= -\hat{a}_\rho\varepsilon_r\frac{8.854\times10^{-2}}{\rho}\sin(10^8t-\beta z) = -\hat{a}_\rho\frac{2.25(8.854\times10^{-2})}{\rho}\sin(10^8t-\beta z)$

$\underline{\mathcal{J}}_d = -\hat{a}_\rho\frac{0.1992}{\rho}\sin(10^8t-\beta z)$

$\boxed{1.22}$ $\underline{\mathcal{H}} = \hat{a}_\phi\frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8t) = \text{Re}\left[\hat{a}_\phi\frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)e^{j4\pi\times10^8t}\right] = \text{Re}\left[\underline{H}e^{j4\pi\times10^8t}\right]$

where $\underline{H} = \hat{a}_\phi\frac{2}{\rho}\cos\left(\frac{\pi}{\ell}z\right)$, $\omega = 4\pi\times10^8$

a. $\underline{E} = \frac{1}{j\omega\varepsilon_0}\nabla\times\underline{H} = -\hat{a}_\rho\frac{1}{j\omega\varepsilon_0}\frac{\partial H_\phi}{\partial z} = -\hat{a}_\rho j\frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)$

$\underline{\mathcal{E}} = \text{Re}\left[\underline{E}e^{j\omega t}\right] = \hat{a}_\rho\frac{1}{\omega\varepsilon_0}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)\sin(4\pi\times10^8t)$

b. $\underline{\mathcal{J}}_s = \hat{n}\times(\underline{\mathcal{H}}_2-\underline{\mathcal{H}}_1)$: At $\rho=a$: $\underline{\mathcal{J}}_s = \hat{a}_\rho\times\left(\hat{a}_\phi\mathcal{H}_\phi\right)\Big|_{\rho=a} = \hat{a}_z\frac{2}{a}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8t)$

At $\rho=b$: $\underline{\mathcal{J}}_s = -\hat{a}_\rho\times\left(\hat{a}_\phi\mathcal{H}_\phi\right)\Big|_{\rho=b} = -\hat{a}_z\frac{2}{b}\cos\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8t)$

c. $\underline{\mathcal{J}}_d = \varepsilon_0\frac{\partial\mathcal{E}_0}{\partial t} = \hat{a}_\rho\frac{4\pi\times10^8}{\omega}\left(\frac{\pi}{\ell}\right)\frac{2}{\rho}\sin\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8t) = \hat{a}_\rho\frac{\pi}{\ell}\left(\frac{2}{\rho}\right)\sin\left(\frac{\pi}{\ell}z\right)\cos(4\pi\times10^8t)$

d. $I_d = \int_0^\ell\int_0^{2\pi}\underline{\mathcal{J}}_d\cdot d\underline{s} = \int_0^\ell\int_0^{2\pi}\left(\hat{a}_\rho\mathcal{J}_d\right)\cdot\hat{a}_\rho\,\rho\,d\phi\,dz = \frac{\pi}{\ell}(4\pi)\left[-\frac{\ell}{\pi}\cos\left(\frac{\pi}{\ell}z\right)\right]_0^\ell\cos(4\pi\times10^8t)$

$I_d = 8\pi\cos(4\pi\times10^8t)$

$\boxed{1.23}$ $\nabla \times \underline{\mathcal{E}} = -\underline{\mathcal{M}}_i - \frac{\partial \mathcal{B}}{\partial t}$

Defining

$$\underline{\mathcal{E}} = Re\left[\underline{E} e^{j\omega t}\right]$$
$$\underline{\mathcal{M}}_i = Re\left[\underline{M}_i e^{j\omega t}\right]$$
$$\underline{\mathcal{B}} = Re\left[\underline{B} e^{j\omega t}\right]$$

Thus

$$\nabla \times \left(Re\left[\underline{E} e^{j\omega t}\right]\right) = -Re\left[\underline{M}_i e^{j\omega t}\right] - \frac{\partial}{\partial t}\left(Re\left[\underline{B} e^{j\omega t}\right]\right)$$

Interchanging differentiation with the Real part leads to

$$Re\left[\nabla \times \left(\underline{E} e^{j\omega t}\right)\right] = Re\left[-\underline{M}_i e^{j\omega t}\right] + Re\left[-\frac{\partial}{\partial t}\left(\underline{B} e^{j\omega t}\right)\right]$$

or

$$Re\left[\left(\nabla \times \underline{E}\right) e^{j\omega t}\right] = Re\left[-\underline{M}_i e^{j\omega t}\right] + Re\left[-j\omega \underline{B} e^{j\omega t}\right]$$

<u>Lemma:</u> If $\underline{A}$ and $\underline{B}$ are complex quantities and

$$Re\left[\underline{A} e^{j\omega t}\right] = Re\left[\underline{B} e^{j\omega t}\right] \quad \text{for all } t$$

then

$$\underline{A} = \underline{B}$$

Using this lemma, we can write that

$$\nabla \times \underline{E} = -\underline{M}_i - j\omega \underline{B}$$

The same procedure can be used for all the other differential form equations.

For the integral form

$$\oint_C \underline{\mathcal{E}} \cdot d\underline{\ell} = -\iint_S \underline{\mathcal{M}}_i \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S \underline{\mathcal{B}} \cdot d\underline{s}$$

Using the above definitions, we can write the integral form as

$$\oint_C Re\left[\underline{E} e^{j\omega t}\right] \cdot d\underline{\ell} = -\iint_S Re\left[\underline{M}_i e^{j\omega t}\right] \cdot d\underline{s} - \frac{\partial}{\partial t} \iint_S Re\left[\underline{B} e^{j\omega t}\right] \cdot d\underline{s}$$

$$Re\left\{\left[\oint \underline{E} \cdot d\underline{\ell}\right] e^{j\omega t}\right\} = Re\left\{\left[-\iint_S \underline{M}_i \cdot d\underline{s}\right] e^{j\omega t}\right\} + Re\left\{\left[-j\omega \iint_S \underline{B} \cdot d\underline{s}\right] e^{j\omega t}\right\}$$

Using the above lemma leads to

$$\oint_C \underline{E} \cdot d\underline{\ell} = -\iint_S \underline{M}_i \cdot d\underline{s} - j\omega \iint_S \underline{B} \cdot d\underline{s}$$

The same procedure can be used for all the other integral form equations.

1.24

$$\underline{\mathcal{E}} = \mathrm{Re}\left[\underline{E}\,e^{j\omega t}\right] = \mathrm{Re}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t)\right]$$

$$= \mathrm{Re}\left[(\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t) + j(\underline{E}_R\sin\omega t + \underline{E}_x\cos\omega t)\right]$$

$$\underline{\mathcal{E}} = (\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)$$

Similarly

$$\underline{\mathcal{E}} = \frac{1}{2}\left[\underline{E}e^{j\omega t} + (\underline{E}e^{j\omega t})^*\right] = \frac{1}{2}\left[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$$

$$= \frac{1}{2}\left[(\underline{E}_R + j\underline{E}_x)(\cos\omega t + j\sin\omega t) + (\underline{E}_R - j\underline{E}_x)(\cos\omega t - j\sin\omega t)\right]$$

$$\underline{\mathcal{E}} = \frac{1}{2}\left[2(\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)\right] = (\underline{E}_R\cos\omega t - \underline{E}_x\sin\omega t)$$

1.25

(a) $$\underline{\mathcal{E}}(z,t) = \hat{a}_x \mathcal{E}_0 \sin\left[(\omega t - \beta_0 z) - \frac{\pi}{2}\right]$$

$$= \hat{a}_x E_0\left[\sin(\omega t - \beta_0 z)\underbrace{\cos\left(-\frac{\pi}{2}\right)}_{0} + \cos(\omega t - \beta_0 z)\underbrace{\sin\left(-\frac{\pi}{2}\right)}_{-1}\right]$$

$$= -\hat{a}_x E_0 \cos(\omega t - \beta_0 z) = -\hat{a}_x E_0\,\mathrm{Re}\left[e^{j(\omega t - \beta_0 z)}\right]$$

$$\underline{\mathcal{E}}(z,t) = -\hat{a}_x E_0\,\mathrm{Re}\left[e^{j\omega t}e^{-j\beta_0 z}\right] = \mathrm{Re}\left[\underbrace{-\hat{a}_x E_0 e^{-j\beta_0 z}}_{E_x(z)} e^{j\omega t}\right]$$

$$\underline{E}(z) = -\hat{a}_x E_0 e^{-j\beta_0 z}$$

(b) $$\nabla\times\underline{E} = -j\omega\mu_0\underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\nabla\times\left[-\hat{a}_x E_0 e^{-j\beta_0 z}\right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_x(0) + \hat{a}_y\left(\frac{\partial E_x}{\partial z}\right) + \hat{a}_z\underbrace{\left(-\frac{\partial E_x}{\partial y}\right)}_{0}\right]$$

$$\underline{H} = -\frac{1}{j\omega\mu_0}\left[\hat{a}_y\left(\frac{\partial E_x}{\partial z}\right)\right] = -\hat{a}_y\frac{1}{j\omega\mu_0}\frac{\partial}{\partial z}\left[-E_0 e^{-j\beta_0 z}\right] = -\hat{a}_y E_0\frac{\beta_0}{\omega\mu_0}e^{-j\beta_0 z}$$

$$\underline{H} = -\hat{a}_y E_0\frac{\omega\sqrt{\mu_0\varepsilon_0}}{\omega\mu_0}e^{-j\beta_0 z} = -\hat{a}_y E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 z} = -\hat{a}_y\frac{E_0}{\eta_0}e^{-j\beta_0 z},\quad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

(c) $$\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \frac{1}{2}\mathrm{Re}\left[\left(-\hat{a}_x E_0 e^{-j\beta_0 z}\right)\times\left(-\hat{a}_y E_0\sqrt{\frac{\varepsilon_0}{\mu_0}}e^{-j\beta_0 z}\right)^*\right]$$

$$\underline{S}_{ave} = \hat{a}_z\frac{1}{2}|E_0|^2\sqrt{\frac{\varepsilon_0}{\mu_0}} = \hat{a}_z\frac{1}{2\eta_0}|E_0|^2$$

1.26 $\underline{H} = \hat{a}_\phi H_o \dfrac{e^{-j\beta_o \rho}}{\sqrt{\rho}}$

$$\underline{E} = \frac{1}{j\omega\varepsilon_o}\nabla\times\underline{H} = \frac{1}{j\omega\varepsilon_o}\left[\hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \frac{H_o}{j\omega\varepsilon_o}\frac{\hat{a}_z}{\rho}\frac{\partial}{\partial\rho}\left[\rho^{1/2}e^{-j\beta_o\rho}\right]$$

$$= \hat{a}_z \frac{H_o}{j\omega\varepsilon_o}\frac{1}{\rho}\left[\rho^{1/2}\left(-j\beta_o e^{-j\beta_o\rho}\right) + \frac{1}{2}\frac{1}{\rho^{1/2}}e^{-j\beta_o\rho}\right]$$

$$= \hat{a}_z H_o\left[-\frac{\beta_o}{\omega\varepsilon_o}\frac{e^{-j\beta_o\rho}}{\sqrt{\rho}} + \frac{e^{-j\beta_o\rho}}{j2\omega\varepsilon_o(\rho)^{3/2}}\right] \xrightarrow{\rho \to \text{large}} -\hat{a}_z H_o \frac{\beta_o}{\omega\varepsilon_o}\frac{e^{-j\beta_o\rho}}{\sqrt{\rho}}$$

$$\underline{E} = -\hat{a}_z H_o \frac{\omega\sqrt{\mu_o\varepsilon_o}}{\omega\varepsilon_o}\frac{e^{-j\beta_o\rho}}{\sqrt{\rho}} = -\hat{a}_z H_o\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{e^{-j\beta_o\rho}}{\sqrt{\rho}}$$

1.27 $\underline{E} = \hat{a}_r E_r(r,\theta) + \hat{a}_\theta E_\theta(r,\theta)$, $E_r = E_o\dfrac{\cos\theta}{r^2}\left(1+\dfrac{1}{j\beta_o r}\right)e^{-j\beta_o r}$, $E_\theta = jE_o\dfrac{\beta_o\sin\theta}{2r}\left[1+\dfrac{1}{j\beta_o r}-\dfrac{1}{(\beta_o r)^2}\right]e^{-j\beta_o r}$

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left\{\frac{\hat{a}_\phi}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]\right\}$$

Expanding using the above electric field components leads to

$$\underline{H} = \hat{a}_\phi j\sqrt{\frac{\varepsilon_o}{\mu_o}}\frac{\beta_o\sin\theta}{2r}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r}$$

or $\quad H_r = H_\theta = 0$

$$H_\phi = j\sqrt{\frac{\varepsilon_o}{\mu_o}}\frac{\beta_o\sin\theta}{2r}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r}$$

1.28 $\underline{E} = \hat{a}_\phi E_o \frac{\sin\theta}{r}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r} = \hat{a}_\phi E_\phi(r,\theta)$

$$\underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left\{\frac{\hat{a}_r}{r\sin\theta}\frac{\partial}{\partial\theta}\left(E_\phi\sin\theta\right)+\frac{\hat{a}_\theta}{r}\left[-\frac{\partial}{\partial r}\left(rE_\phi\right)\right]\right\}$$

Using the above electric field component leads to

$\underline{H} = \hat{a}_r H_r + \hat{a}_\theta H_\theta$ where $H_r = j\,\frac{2E_o\cos\theta}{\omega\mu_o r^2}\left(1+\frac{1}{j\beta_o r}\right)e^{-j\beta_o r}$

$$H_\theta = -\frac{E_o\sin\theta}{\eta r}\left[1+\frac{1}{j\beta_o r}-\frac{1}{(\beta_o r)^2}\right]e^{-j\beta_o r}$$

1.29 $\underline{E} = \hat{a}_z(1+j)\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$

Using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_o\underline{H}$ leads to

$$\underline{H} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_x\frac{\partial E_z}{\partial y}-\hat{a}_y\frac{\partial E_z}{\partial x}\right] = -\frac{(1+j)}{j\omega\mu_o}\left[\hat{a}_x\left(\frac{\pi}{b}\right)\cos\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right)-\hat{a}_y\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{b}y\right)\cos\left(\frac{\pi}{a}x\right)\right]$$

Now using Maxwell's equation of $\nabla\times\underline{H} = \underline{J}_c + j\omega\varepsilon\,\underline{E} = \sigma\underline{E} + j\omega\varepsilon\,\underline{E}$ leads to

$$\nabla\times\underline{H} = -\hat{a}_z\frac{(1+j)}{j\omega\mu_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right) = \hat{a}_z(\sigma+j\omega\varepsilon)(1+j)\sin\left(\frac{\pi}{b}y\right)\sin\left(\frac{\pi}{a}x\right)$$

Equating both sides, we can write that

$$(\sigma+j\omega\varepsilon) = -\frac{1}{j\omega\mu_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right] \Rightarrow \quad \sigma = 0$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = \frac{1}{\omega^2\mu_o\varepsilon_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]$$

1.30 $\underline{E}^i = \hat{a}_x e^{-j\beta_o z} \Rightarrow \underline{H}^i = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E}^i = -\hat{a}_y\frac{1}{j\omega\mu_o}\frac{\partial E_x^i}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}\,e^{-j\beta_o z}$

$\underline{E}^r = -\hat{a}_x e^{+j\beta_o z} \Rightarrow \underline{H}^r = -\hat{a}_y\frac{1}{j\omega\mu_o}\frac{\partial E_x^r}{\partial z} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}\,e^{+j\beta_o z}$

$$\underline{J}_s = \hat{n}\times(\underline{H}_2-\underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y\left(H^i+H^r\right)\Big|_{z=0} = \hat{a}_x\left(H^i+H^r\right)\Big|_{z=0} = \hat{a}_x\sqrt{\frac{\varepsilon_o}{\mu_o}}\left(e^{-j\beta_o z}+e^{+j\beta_o z}\right)\Big|_{z=0}$$

$$= \hat{a}_x 2\sqrt{\frac{\varepsilon_o}{\mu_o}} = \hat{a}_x\frac{2}{377} = \hat{a}_x\,5.3\times10^{-3}\ \text{A/m}$$

$\boxed{1.31}$ $\underline{E}^i = \hat{a}_y E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$, $\underline{E}^r = \hat{a}_y E_o \Gamma_h e^{-j\beta_o(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{E}^i + \underline{E}^r\right)^{tan}_{z=0} = 0 = \hat{a}_y E_o (1+\Gamma_h) e^{-j\beta_o x\sin\theta_i}$

which is satisfied provided $(1+\Gamma_h)=0 \Rightarrow \Gamma_h = -1$

$\boxed{1.32}$ Using Maxwell's equation of $\nabla \times \underline{E} = -j\omega\mu_o \underline{H}$, the magnetic

a. field components corresponding to the electric fields of Problem 1.31 can be written as

$$\underline{H}^i = \frac{E_o}{\sqrt{\mu_o/\varepsilon_o}}\left(-\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{H}^r = -\frac{E_o}{\sqrt{\mu_o/\varepsilon_o}}\left(\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i\right) e^{-j\beta_o(x\sin\theta_i - z\cos\theta_i)}$$

b.

$$\underline{J}_s = \hat{n}\times\left(\underline{H}_2 - \underline{H}_1\right)\Big|_{z=0} = -\hat{a}_z \times \left(\underline{H}^i + \underline{H}^r\right)\Big|_{z=0} = -\hat{a}_z \times \left[\hat{a}_x(H^i + H^r) + \hat{a}_z(H^i + H^r)\right]_{z=0}$$

$$\underline{J}_s = -\hat{a}_z \times \hat{a}_x (H^i + H^r) = \hat{a}_y \frac{2E_o}{\sqrt{\mu_o/\varepsilon_o}}\cos\theta_i \, e^{-j\beta_o x\sin\theta_i}$$

$\boxed{1.33}$ $\underline{E}^i = (\hat{a}_x\cos\theta_i - \hat{a}_z\sin\theta_i) E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$

$\underline{E}^r = (\hat{a}_x\cos\theta_i + \hat{a}_z\sin\theta_i)\Gamma_e e^{-j\beta_o(x\sin\theta_i - z\cos\theta_i)}$

Along the interface $(z=0)$ $\left(\underline{E}^i + \underline{E}^r\right)^{tan}_{z=0} = 0 = \hat{a}_x\cos\theta_i E_o (1+\Gamma_e) e^{-j\beta_o x\sin\theta_i}$

which is satisfied provided $(1+\Gamma_e)=0 \Rightarrow \Gamma_e = -1$

$\boxed{1.34}$ Along the interface the normal components of the electric flux density must be continuous; that is

$$\hat{n}\cdot(\underline{D}_2 - \underline{D}_1) = \hat{n}\cdot\underline{D}_2 = -\hat{n}\cdot\hat{a}_z\varepsilon_o E_o\sin\theta_i(1+\Gamma_e)e^{-j\beta_o x\sin\theta_i} = 0$$

$$= -\hat{a}_z\cdot\hat{a}_z\varepsilon_o E_o\sin\theta_i(1+\Gamma_e)e^{-j\beta_o x\sin\theta_i} = 0$$

which is satisfied provided

$$(1+\Gamma_e) = 0 \Rightarrow \Gamma_e = -1$$

$\boxed{1.35}$ Given the electric fields of Problem 1.33, the corresponding

a. magnetic field components can be found using Maxwell's equation of

$$\nabla \times \underline{E} = -j\omega\mu_o \underline{H} \Rightarrow \underline{H} = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_o}\left[\hat{a}_x \frac{\partial E_z}{\partial y} + \hat{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{a}_z \frac{\partial E_x}{\partial y}\right]$$

For the incident field

$$E_x^i = \cos\theta_i E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

$$E_y^i = 0$$

$$E_z^i = -\sin\theta_i E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

Using these

$$\frac{\partial E_x^i}{\partial y} = 0 \quad , \quad \frac{\partial E_z^i}{\partial y} = 0$$

However

$$\frac{\partial E_x^i}{\partial z} = -j\beta_o \cos^2\theta_i E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

$$\frac{\partial E_z^i}{\partial x} = +j\beta_o \sin^2\theta_i E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

Thus we can write the incident magnetic field as

$$\underline{H}^i = -\frac{E_o}{j\omega\mu_o}\hat{a}_y(-j\beta_o)(\cos^2\theta_i + \sin^2\theta_i)e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

$$= \hat{a}_y\frac{\beta_o}{\omega\mu_o}E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o e^{-j\beta_o(x\sin\theta_i + z\cos\theta_i)}$$

Using the same procedure we can write the reflected magnetic field as

$$\underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o\Gamma_e e^{-j\beta_o(x\sin\theta_i - z\cos\theta_i)} = \hat{a}_y\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o e^{-j\beta_o(x\sin\theta_i - z\cos\theta_i)}$$

b. $$\underline{J}_s = \hat{n}\times(\underline{H}_2 - \underline{H}_1) = \hat{n}\times\underline{H}_2 = -\hat{a}_z\times\hat{a}_y(H^i + H^r)\Big|_{z=0} = \hat{a}_x 2E_o\sqrt{\frac{\varepsilon_o}{\mu_o}}e^{-j\beta_o x\sin\theta_i}$$

$\boxed{1.36}$ To determine the coefficients Γ_o and T_o we apply the boundary conditions along the interface at $z=0$. To do this we first find the corresponding magnetic field components. This is accomplished using Maxwell's equation of $\nabla\times\underline{E} = -j\omega\mu_o\underline{H}_o \Rightarrow \underline{H}_o = -\frac{1}{j\omega\mu_o}\nabla\times\underline{E}$. Doing

(continued)

$\boxed{1.36 \text{ cont'd}}$ this for each component leads to

$$\underline{H}^i = \hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0 e^{-j\beta_0 z}, \quad \underline{H}^r = -\hat{a}_y\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0\Gamma_0 e^{+j\beta_0 z}, \quad \underline{H}^t = \hat{a}_y\sqrt{\frac{\varepsilon}{\mu_0}}E_0 T_0 e^{-j\beta z}$$

Applying the boundary conditions on the continuity of the tangential electric and magnetic fields along the interface at $z=0$ leads to

$1 + \Gamma_0 = T_0$ from continuity of the electric fields

$1 - \Gamma_0 = \sqrt{\varepsilon_r}\,T_0 = \sqrt{81}\,T_0$ from continuity of the magnetic fields

Solving these two equations, we find that

$$T_0 = \frac{2}{1+\sqrt{\varepsilon_r}} = \frac{2}{1+\sqrt{81}} = \frac{2}{1+9} = \frac{1}{5} = 0.2$$

$$\Gamma_0 = T_0 - 1 = \frac{1}{5} - 1 = -\frac{4}{5} = -0.8$$

$\boxed{1.37}$ The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

<u>Electric Fields</u>: $\left(\underline{E}^i + \underline{E}^r\right)^{\tan}_{z=0} = \left(\underline{E}^t\right)^{\tan}_{z=0}$

$$1 + \Gamma_h = T_h$$

<u>Magnetic Fields</u>; $\left(\underline{H}^i + \underline{H}^r\right)^{\tan}_{z=0} = \left(\underline{H}^t\right)^{\tan}_{z=0}$

$$\cos\theta_i\left(-1 + \Gamma_h\right)\sqrt{\varepsilon_0} = -\sqrt{1 - \frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}\,\sqrt{\varepsilon}\;T_h$$

Solving these two equations leads to

$$\Gamma_h = \frac{\cos\theta_i - \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1 - \frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1 - \frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

$$T_h = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon}{\varepsilon_0}}\sqrt{1 - \frac{\varepsilon_0}{\varepsilon}\sin^2\theta_i}}$$

$\boxed{1.38}$ The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

<u>Electric Flux Density</u>: $(\underline{D}^i+\underline{D}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied.

<u>Magnetic Flux Density</u>: $(\underline{B}^i+\underline{B}^r)^{nor}_{z=0} = (\underline{B}^t)^{nor}_{z=0}$

$$\sin\theta_i\,\mu_o\sqrt{\varepsilon_o}\,(1+\Gamma_h) = \sqrt{\frac{\varepsilon_o}{\varepsilon}}\sin\theta_i\,\mu_o\sqrt{\varepsilon}\,T_h$$

or

$$1+\Gamma_h = T_h$$

This is identical to one of the equations for the solution of Problem 1.37 However we do not have another equation from the normal components of the electric field. Therefore we can not solve for Γ_h and T_h using only the normal components of Problem 1.37.

$\boxed{1.39}$ The boundary conditions require continuity of the tangential components of the electric and magnetic fields.

<u>Electric Fields:</u> $(\underline{E}^i+\underline{E}^r)^{tan}_{z=0} = (\underline{E}^t)^{tan}_{z=0}$

$$\cos\theta_i(1+\Gamma_e) = \sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}\;T_e$$

<u>Magnetic Fields:</u> $(\underline{H}^i+\underline{H}^r)^{tan}_{z=0} = (\underline{H}^t)^{tan}_{z=0}$

$$\sqrt{\varepsilon_o}(1-\Gamma_e) = \sqrt{\varepsilon}\;T_e$$

Solving these two equations leads to

$$\Gamma_e = \frac{-\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}{\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

$$T_e = \frac{2\sqrt{\frac{\varepsilon_o}{\varepsilon}}\cos\theta_i}{\cos\theta_i+\sqrt{\frac{\varepsilon_o}{\varepsilon}}\sqrt{1-\frac{\varepsilon_o}{\varepsilon}\sin^2\theta_i}}$$

1.40 The boundary conditions require continuity of the normal components of the electric flux density and magnetic flux density.

<u>Electric Flux Density</u>: $(\underline{D}^i + \underline{D}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

$$\sin\theta_i\, \varepsilon_0(-1 + \Gamma_e) = -\sqrt{\frac{\varepsilon_0}{\varepsilon}}\,\varepsilon \sin\theta_i\, T_e$$

or

$$\sqrt{\varepsilon_0}\,(1 - \Gamma_e) = \sqrt{\varepsilon}\; T_e$$: This equation is identical to one of the equations for the solution of Problem 1.37.

<u>Magnetic Flux Density</u>: $(\underline{B}^i + \underline{B}^r)^{nor}_{z=0} = (\underline{D}^t)^{nor}_{z=0}$

Since there are no normal components, this boundary condition is automatically satisfied. However we only have one equation and two unknowns; therefore we can not solve for Γ_e and T_e using only the normal components of Problem 1.37.

1.41 From Problem 1.16 and at large distances

$$\underline{\mathcal{E}} = \hat{a}_\phi E_0 \sin\theta\, \frac{\cos(\omega t - \beta_0 r)}{r},\quad \underline{\mathcal{E}} = \mathrm{Re}[\underline{E} e^{j\omega t}] \Rightarrow \underline{E} = E_0 \sin\theta\, \frac{e^{-j\beta_0 r}}{r}\hat{a}_\phi$$

$$\underline{\mathcal{H}} = -\hat{a}_\theta \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \sin\theta\, \frac{\cos(\omega t - \beta_0 r)}{r},\quad \underline{\mathcal{H}} = \mathrm{Re}[\underline{H} e^{j\omega t}] \Rightarrow \underline{H} = -\hat{a}_\theta \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \sin\theta\, \frac{e^{-j\beta_0 r}}{r}$$

a. $$\underline{S}_{av} = \underline{S} = \frac{1}{2}\mathrm{Re}(\underline{E}\times\underline{H}^*) = \frac{1}{2}\mathrm{Re}(\hat{a}_\phi E_\phi \times \hat{a}_\theta H^*_\theta) = \frac{1}{2}\mathrm{Re}(\hat{a}_r E_\phi H^*_\theta)$$

$$= \hat{a}_r \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\sin^2\theta}{r^2}$$

b. $$P_{av} = \iint_{S_a} \underline{S}_{av}\cdot d\underline{s} = \iint_{S_a} \underline{S}_{av}\cdot\hat{a}_r r^2 \sin\theta\, d\theta\, d\phi = \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\int_0^{2\pi}\!\!\int_0^{\pi} \sin^3\theta\, d\theta\, d\phi = \frac{|E_0|^2}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}(2\pi)\left(\frac{4}{3}\right)$$

$$= |E_0|^2 \frac{4\pi}{3}\sqrt{\frac{\varepsilon_0}{\mu_0}}$$

1.42 $$\underline{H}^{inc} = \frac{1}{377}(-\hat{a}_x \cos\theta_i + \hat{a}_z \sin\theta_i),\quad \underline{H}^{ref} = \frac{1}{377}(-\hat{a}_x \cos\theta_i - \hat{a}_z \sin\theta_i)$$

$$\underline{H}^{total} = \underline{H}^{inc} + \underline{H}^{ref} = -\hat{a}_x \frac{2}{377}\cos\theta_i = -\hat{a}_x\, 5.31\times10^{-3}\cos\theta_i$$

$$\underline{J}_s = \hat{n}\times\underline{H}^{total} = \hat{a}_y \times(-\hat{a}_x\, 5.31\times10^{-3}\cos\theta_i) = \hat{a}_z\, 5.31\times10^{-3}\cos\theta_i$$

1.43 $$\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

a. $$\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left(-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_z\frac{\partial E_y}{\partial x}\right) = -\hat{a}_x\frac{\beta_z}{\omega\mu_0}E_0\sin\left(\frac{\pi}{a}x\right)e^{-j\beta_z z} - \hat{a}_z\frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)e^{-j\beta_z z}$$

b. $$P_s = -\frac{1}{2}\iiint_V \left(\underline{H}^*\cdot\cancelto{0}{\underline{M}_i} + \underline{E}\cdot\cancelto{0}{\underline{J}_i^*}\right)dv = 0$$

c. $$P_e = \oint\!\!\oint_S \left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot d\underline{s}$$

$\boxed{\text{Cont'd}}$

$\boxed{\text{1.43 cont'd}}$ $\frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times\left(-\hat{a}_x H_x^* - \hat{a}_z H_z^*\right) = \frac{1}{2}\left(\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*\right)$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \hat{a}_z \frac{\beta_z}{2\omega\mu_0}|E_0|^2 \sin^2\left(\frac{\pi}{a}x\right) + \hat{a}_x \frac{|E_0|^2}{j2\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)$$

<u>From the two side walls at $x=0$ and $x=a$:</u>

$$\int_0^1\int_0^b \left(\hat{a}_x S_x\right)\Big|_{x=0}\cdot\left(-\hat{a}_x\,dy\,dz\right) + \int_0^1\int_0^b\left(\hat{a}_x S_x\right)\Big|_{x=a}\cdot\left(\hat{a}_x\,dy\,dz\right) = 0+0=0$$

<u>From the front and back cross sections at $z=0$ and $z=1$:</u>

$$\int_0^b\int_0^a \left(\hat{a}_z S_z\right)\Big|_{z=0}\cdot\left(+\hat{a}_z\,dx\,dy\right) = \frac{\beta_z}{2\omega\mu_0}|E_0|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2$$

$$\int_0^b\int_0^a \left(\hat{a}_z S_z\right)\Big|_{z=1}\cdot\left(-\hat{a}_z\,dx\,dy\right) = -\frac{\beta_z}{2\omega\mu_0}|E_0|^2\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy = -\frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2$$

<u>From the top and bottom walls at $y=0$ and $y=b$:</u>

Since there are no y components of the power density, there is no contribution from the top and bottom walls.

Therefore

$$P_e = 0 + \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2 - \frac{\beta_z}{2\omega\mu_0}\frac{ab}{2}|E_0|^2 = 0$$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $\overline{W}_m = \iiint_V \frac{1}{4}\mu_0|\underline{H}|^2 dv = \frac{\mu_0}{4}|E_0|^2\left\{\int_0^1\int_0^b\int_0^a\left(\frac{\beta_z}{\omega\mu_0}\right)^2\sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\int_0^1\int_0^b\int_0^a\cos^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz\right\}$

$$= \frac{\mu_0}{4}|E_0|^2\left\{\left(\frac{\beta_z}{\omega\mu_0}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\left(\frac{ab}{2}\right)\right\} = \frac{\mu_0}{4}|E_0|^2\left\{\frac{\varepsilon_0}{\mu_0}\frac{ab}{2} - \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2} + \left(\frac{1}{\omega\mu_0}\right)^2\left(\frac{\pi}{a}\right)^2\frac{ab}{2}\right\}$$

$\overline{W}_m = \frac{\varepsilon_0}{8}ab|E_0|^2$

f. $\overline{W}_e = \iiint_V \frac{1}{4}\varepsilon_0|\underline{E}|^2 dv = \frac{\varepsilon_0}{4}|E_0|^2\int_0^1\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx\,dy\,dz = \frac{\varepsilon_0}{4}|E_0|^2\frac{ab}{2} = \frac{\varepsilon_0}{8}ab|E_0|^2$

$\overline{W}_e = \frac{\varepsilon_0}{8}ab|E_0|^2$

Ultimately $\quad P_s = P_e + P_d + j2\omega\left(\overline{W}_m - \overline{W}_e\right)$

$0 = 0 + 0 + j2\omega\left(\frac{\varepsilon_0}{8}ab|E_0|^2 - \frac{\varepsilon_0}{8}ab|E_0|^2\right) = 0+0+0 = 0$

$0 = 0$ QED

$\boxed{1.44}$ $-\nabla\cdot\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)=\frac{1}{2}\underline{H}^*\cdot\cancel{\underline{M}_i}^{\,0}+\frac{1}{2}\underline{E}\cdot\cancel{\underline{J}_i^*}^{\,0}+j2\omega\left(\frac{1}{4}\mu_0|\underline{H}|^2-\frac{1}{4}\varepsilon_0|\underline{E}|^2\right)$

$$\frac{1}{2}(\underline{E}\times\underline{H}^*)=\hat{a}_x\frac{|E_0|^2}{j2\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_0}|E_0|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$=\hat{a}_x\frac{|E_0|^2}{j4\omega\mu_0}\left(\frac{\pi}{a}\right)\sin\left(\frac{2\pi}{a}x\right)+\hat{a}_z\frac{\beta_z}{2\omega\mu_0}|E_0|^2\sin^2\left(\frac{\pi}{a}x\right)$$

$$-\nabla\cdot\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)=-\frac{|E_0|^2}{j4\omega\mu_0}\left(\frac{\pi}{a}\right)\left(\frac{2\pi}{a}\right)\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\frac{1}{4}\mu_0|\underline{H}|^2=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_z^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\cos^2\left(\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_0^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\cos^2\frac{\pi}{a}x-\sin^2\frac{\pi}{a}x\right)\right]$$

$$=\frac{\mu_0}{4}\frac{|E_0|^2}{(\omega\mu_0)^2}\left[\beta_0^2\sin^2\left(\frac{\pi}{a}x\right)+\left(\frac{\pi}{a}\right)^2\left(\frac{1+\cos\left(\frac{2\pi}{a}x\right)}{2}-\frac{1-\cos\left(\frac{2\pi}{a}x\right)}{2}\right)\right]$$

$$\frac{1}{4}\mu_0|\underline{H}|^2=\frac{|E_0|^2}{4}\varepsilon_0\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_0|^2}{4}\frac{1}{\omega^2\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)$$

$$\frac{1}{4}\varepsilon_0|\underline{E}|^2=\frac{|E_0|^2}{4}\varepsilon_0\sin^2\left(\frac{\pi}{a}x\right)$$

Therefore the conservation of energy equation in differential form can be written as

$$j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=0+0+j2\omega\left[\frac{|E_0|^2}{4}\varepsilon_0\sin^2\left(\frac{\pi}{a}x\right)+\frac{|E_0|^2}{4}\frac{1}{\omega^2\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)-\frac{|E_0|^2}{4}\varepsilon_0\sin^2\left(\frac{\pi}{a}x\right)\right]$$

$$j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)=j\frac{|E_0|^2}{2\omega\mu_0}\left(\frac{\pi}{a}\right)^2\cos\left(\frac{2\pi}{a}x\right)\qquad \text{QED}$$

$\boxed{1.45}$ Boundary Conditions on PEC:

(a) $E_x(0\le x\le a,\ y=0,\ 0\le z\le c)=E_x(\text{bottom wall})=0$

$$E_x=\cos(\beta_x x)\sin(\beta_y y)\sin(\beta_z z)\big|_{y=0}=0$$

$E_x(0\le x\le a,\ y=b,\ 0\le z\le c)=E_x(\text{top wall})=0$

$$E_x=\cos(\beta_x x)\sin(\beta_y b)\sin(\beta_z z)=0$$

$$\sin(\beta_y b)=0\Rightarrow\beta_y b=\sin^{-1}(0)=n\pi,\quad n=0,\pm1,\pm2,\ldots$$

$$\boxed{\beta_y=(n\pi/b),\quad n=0,\pm1,\pm2,\ldots}$$

$\boxed{\text{Cont'd}}$

$\boxed{1.45 \text{ (con't)}}$

(b) $E_y(x=0,\ 0\le y\le b,\ 0\le z\le c) = E_y(\text{left wall}) = 0$

$E_y = \sin(0)\cos(\beta_y y)\sin(\beta_z z) = 0$

$E_y(x=a,\ 0\le y\le b,\ 0\le z\le c) = E_y(\text{right wall}) = 0$

$E_y = \sin(\beta_x a)\cos(\beta_y y)\sin(\beta_z z) = 0$

$\sin(\beta_x a) = 0 \Rightarrow \beta_x a = \sin^{-1}(0) = m\pi,\ m = 0, \pm1, \pm2, \ldots$

$$\boxed{\beta_x = (m\pi/a),\quad m = 0, \pm1, \pm2, \ldots}$$

($m = n \neq 0$ simultaneously; $m=n=0$ trivial solution E-fields vanish)

c. $E_y(0\le x\le a,\ 0\le y\le b,\ z=0) = E_y(\text{front wall}) = 0$

$E_y = \sin(\beta_x x)\cos(\beta_y y)\sin(0) = 0$

$E_y(0\le x\le a,\ 0\le y\le b,\ z=c) = E_y(\text{back wall}) = 0$

$E_y = \sin\beta_x x\cos(\beta_y y)\sin(\beta_z c) = 0$

$\sin(\beta_z c) = 0 \Rightarrow \beta_z c = \sin^{-1}(0) = p\pi,\ p = \cancel{0}, \pm1, \pm2, \ldots$

$$\boxed{\beta_z = (p\pi/c) = (p\pi/c),\quad p = \pm1, \pm2, \ldots}$$

$p \neq 0$, because $p = 0$ leads to trivial solution; E-fields vanish

$\boxed{1.46}$ $\underline{E} = \hat{a}_y E_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right),\quad \omega_r = \omega = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}$

a. $\underline{H} = -\frac{1}{j\omega\mu_0}\nabla\times\underline{E} = -\frac{1}{j\omega\mu_0}\left[-\hat{a}_x\frac{\partial E_y}{\partial z} + \hat{a}_z\frac{\partial E_y}{\partial x}\right]$

$= \hat{a}_x\frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{c}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{c}z\right) - \hat{a}_z\frac{E_0}{j\omega\mu_0}\left(\frac{\pi}{a}\right)\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)$

b. $P_s = -\frac{1}{2}\iiint_V\left[\underline{H}^*\cdot\underline{M}_i + \underline{E}\cdot\underline{J}_i^*\right]dv = 0$

(continued)

$\boxed{\text{1.46 cont'd.}}$

c. $P_e = \oiint_{S_a} \left(\frac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot d\underline{s} = \oiint_{S_a} \underline{S}\cdot d\underline{s}$, $\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^*$

$$\underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^* = \frac{1}{2}\hat{a}_y E_y \times\left(\hat{a}_x H_x^* - \hat{a}_z H_z^*\right) = \frac{1}{2}\left(-\hat{a}_z E_y H_x^* - \hat{a}_x E_y H_z^*\right)$$

$$= \frac{1}{2}\left[\hat{a}_x \frac{E_o}{j\omega\mu_o}\left(\frac{\pi}{a}\right)\sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right) + \hat{a}_z \frac{|E_o|^2}{j\omega\mu_o}\left(\frac{\pi}{c}\right)\sin^2\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{c}z\right)\cos\left(\frac{\pi}{c}z\right)\right]$$

Contributions to P_e from the different walls:

Left and right walls:

$$\int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(-\hat{a}_x\,dy\,dz) = 0\;;\quad \int_0^c\int_0^b (\hat{a}_x S_x)\Big|_{x=0}\cdot(\hat{a}_x\,dy\,dz) = 0$$

Back and front walls:

$$\int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(-\hat{a}_z\,dx\,dy) = 0\;;\quad \int_0^b\int_0^a (\hat{a}_z S_z)\Big|_{z=0}\cdot(\hat{a}_z\,dx\,dy) = 0$$

Top and bottom walls:

Since there are no y components of the power density, there are no contributions from the top and bottom walls.

Therefore $P_e = 0+0+0+0+0 = 0$

d. $P_d = \frac{1}{2}\iiint_V \sigma|\underline{E}|^2 dv = 0$

e. $$\bar{W}_m = \frac{\mu_o}{4}\iiint_V |\underline{H}|^2 dv = |E_o|^2\frac{\mu_o}{4}\frac{1}{(\omega\mu_o)^2}\left[\left(\frac{\pi}{c}\right)^2\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\cos^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz + \left(\frac{\pi}{a}\right)^2\int_0^c\int_0^b\int_0^a \cos^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz\right]$$

$$\bar{W}_m = |E_o|^2\frac{\mu_o}{4}\frac{1}{(\omega\mu_o)^2}\left[\left(\frac{\pi}{c}\right)^2\frac{abc}{4} + \left(\frac{\pi}{a}\right)^2\frac{abc}{4}\right] = |E_o|^2\frac{abc}{16}\frac{\varepsilon_o}{\omega^2\mu_o\varepsilon_o}\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{c}\right)^2\right] = |E_o|^2\frac{abc}{16}\varepsilon_o$$

f. $$\bar{W}_e = \frac{\varepsilon_o}{4}\iiint_V |\underline{E}|^2 dv = |E_o|^2\frac{\varepsilon_o}{4}\int_0^c\int_0^b\int_0^a \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{\pi}{c}z\right)dx\,dy\,dz = |E_o|^2\frac{\varepsilon_o}{4}\,\frac{abc}{4} = |E_o|^2\frac{abc}{16}\varepsilon_o$$

Ultimately $P_s = P_e + P_d + j2\omega(\bar{W}_m - \bar{W}_e)$

$$0 = 0 + 0 + j2\omega\left(|E_o|^2\frac{abc}{16}\varepsilon_o - |E_o|^2\frac{abc}{16}\varepsilon_o\right)$$

$0 = 0$ QED

CHAPTER 2

$\boxed{2.1}$ $\underline{P} = \hat{a}_y\, 2.762\times10^{-11}$ C/m², $\underline{E} = \hat{a}_y\, 2$ V/m

a. $q_{sb} = |\underline{P}| = 2.762\times10^{-11}$ C/m²

$q_{sb} = +2.762\times10^{-11}$ C/m² at $y=2$ cm face

$q_{sb} = -2.762\times10^{-11}$ C/m² at $y=0$ cm face

b. $$Q_p = \oiint_S \underline{P}\cdot d\underline{s} = \underbrace{\int_0^{.06}\int_0^{.04} \hat{a}_y P_y \cdot \hat{a}_y\, dx\,dz}_{y=2\text{ cm face}} + \underbrace{\int_0^{.06}\int_0^{.04} \hat{a}_y P_y \cdot \left(-\hat{a}_y\, dx\,dz\right)}_{y=0\text{ cm face}} = \int_0^{.06}\int_0^{.04} P_y\, dx\,dz - \int_0^{.06}\int_0^{.04} P_y\, dx\,dz = 0$$

c. $$\oiint_S \underline{P}\cdot d\underline{s} = \iiint_V \nabla\cdot\underline{P}\, dv = \iiint_V q_{vb}\, dv = Q_b$$

$$q_{vb} = \nabla\cdot\underline{P} = \frac{\partial P_y}{\partial y} = 0$$

d. $$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = \frac{(1+\chi_e)\varepsilon_o}{\varepsilon_o} = 1+\chi_e\,; \qquad \underline{D} = \varepsilon_o \underline{E}_a + \underline{P} = (1+\chi_e)\varepsilon_o \underline{E}_a$$

$$\chi_e = \frac{P}{\varepsilon_o E_a} = \frac{2.762\times10^{-11}}{8.854\times10^{-12}(2)} = \frac{2.762}{1.7708} = 1.5597$$

$$\varepsilon_r = 1 + 1.5597 = 2.5597 \simeq 2.56$$

$\boxed{2.2}$ $\underline{P} = \hat{a}_\rho \frac{2}{\rho}\times10^{-10}$ C/m², $\underline{E} = \hat{a}_\rho \frac{7.53}{\rho}$ V/m; $\quad a \le \rho \le b$

a. $$q_{sb}\Big|_{\rho=a} = -|P| = -\frac{2\times10^{-10}}{2\times10^{-2}} = -10^{-8} \text{ C/m}^2$$

$$q_{sb}\Big|_{\rho=b} = +|P| = \frac{2\times10^{-10}}{6\times10^{-2}} = \frac{1}{3}\times10^{-8} \text{ C/m}^2$$

$q_{sb} = 0$ elsewhere

b. $$Q_p\Big|_{\rho=a} = q_{sb} A_a = -\frac{2}{\rho}\times10^{-10}(2\pi\rho\ell) = -4\pi\ell\times10^{-10} = -1.256\times10^{-10}$$

$$Q_p\Big|_{\rho=b} = q_{sb} A_b = \frac{2}{\rho}\times10^{-10}(2\pi\rho\ell) = 4\pi\ell\times10^{-10} = +1.256\times10^{-10}$$

$$Q_{total} = Q_p\Big|_{\rho=a} + Q_p\Big|_{\rho=b} = 0$$

(continued)

$\boxed{2.2 \text{ cont'd.}}$

c. $$Q_p = \oint\!\!\oint_S \underline{P}\cdot d\underline{s} = \iiint_V \nabla\cdot\underline{P}\, dv = \iiint_V q_{vb}\, dv$$

$$q_{vb} = \nabla\cdot\underline{P} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho P_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\,\frac{2}{\rho}\times 10^{-10}\right) = 0$$

d. $$\underline{D} = \varepsilon\underline{E}_a = \varepsilon_0\underline{E}_a + \underline{P} = \varepsilon_0\underline{E}_a + \chi_e\varepsilon_0\underline{E}_a = \varepsilon_0(1+\chi_e)\underline{E}_a$$

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1+\chi_e)$$

$$\chi_e = \frac{P}{\varepsilon_0 E_a} = \frac{\frac{2}{\rho}\times 10^{-10}}{\varepsilon_0\,\frac{7.53}{\rho}} = \frac{2\times 10^{-10}}{7.53\,\varepsilon_0} = 2.9998$$

$$\varepsilon_r = 1+\chi_e = 1+2.9998 = 3.9998 \simeq 4$$

$\boxed{2.3}$ $\underline{P} = \hat{a}_r\,\frac{31.87}{r^2}\times 10^{-12}$ C/m^2, $\underline{E} = \hat{a}_r\,\frac{0.45}{r^2}\, V/m$; $a \le r \le b$

a. $$q_{sb}\big|_{r=a} = -|\underline{P}| = -\frac{31.87\times 10^{-12}}{(2\times 10^{-2})^2} = -\frac{31.87\times 10^{-12}}{4\times 10^{-4}} = -7.9675\times 10^{-8}\ C/m^2$$

$$q_{sb}\big|_{r=b} = +|\underline{P}| = \frac{31.87\times 10^{-12}}{(4\times 10^{-2})^2} = \frac{31.87\times 10^{-12}}{16\times 10^{-4}} = 1.9919\times 10^{-8}\ C/m^2$$

b. $$Q_p\big|_{r=a} = q_{sb}A_a = -\frac{31.87\times 10^{-12}}{r^2}(4\pi r^2) = -31.87\,(4\pi\times 10^{-12}) = -4.005\times 10^{-10}$$

$$Q_p\big|_{r=b} = q_{sb}A_b = \frac{31.87\times 10^{-12}}{r^2}(4\pi r^2) = 31.87\,(4\pi\times 10^{-12}) = 4.005\times 10^{-10}$$

$$Q_{total} = Q_p\big|_{r=a} + Q_p\big|_{r=b} = 0$$

c. $$Q_p = \oint\!\!\oint_S \underline{P}\cdot d\underline{s} = \iiint_V \nabla\cdot\underline{P}\, dv = \iiint_V q_{vb}\, dv$$

$$q_{vb} = \nabla\cdot\underline{P} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{31.87}{r^2}\times 10^{-12}\right) = 0$$

d. $$\chi_e = \frac{P}{\varepsilon_0 E_a} = \frac{\frac{31.87\times 10^{-12}}{r^2}}{\varepsilon_0\,\frac{0.45}{r^2}} = \frac{31.87\times 10^{-12}}{0.45\,\varepsilon_0} = 7.9989$$

$$\varepsilon_r = 1+\chi_e = 1+7.9989 = 8.9989 \simeq 9$$

$\boxed{2.4}$

a. $E_o(0.25\times10^{-3}) + E_1(1\times10^{-3}) = 100$ Volts

Also $\varepsilon_o E_o = \varepsilon_1 E_1 \Rightarrow E_1 = \frac{\varepsilon_o}{\varepsilon_1}E_o$

Thus $E_o(0.25\times10^{-3}) + E_o\left(\frac{10^{-3}}{5}\right) = E_o(0.25+0.2)\times10^{-3} = 0.45\times10^{-3}E_o = 100$

$$E_o = \frac{100\times10^{3}}{0.45} = 2.222\times10^{5}\ \text{V/m}$$

$$E_1 = \frac{\varepsilon_o}{\varepsilon_1}E_o = \frac{2.222\times10^{5}}{5} = 0.4444\times10^{5}\ \text{V/m}$$

b. $D_o = \varepsilon_o E_o = 8.854\times10^{-12}(2.222\times10^{5}) = 19.6755\times10^{-7} = 1.96755\times10^{-6}\ \text{C/m}^2$

$D_1 = \varepsilon_1 E_1 = 5(8.854\times10^{-12})(0.4444\times10^{5}) = 1.96755\times10^{-6} = D_o \quad \text{C/m}^2$

c. $q_s = 1.96755\times10^{-6}\ \text{C/m}^2$ in upper plate where voltage is positive

$q_s = -1.96755\times10^{-6}\ \text{C/m}^2$ in lower plate where voltage is negative

d. $Q = q_s A = 1.96755\times10^{-6}(2\times10^{-2}) = 3.9351\times10^{-8}$ C in upper plate

$Q = q_s A = -1.96755\times10^{-6}(2\times10^{-2}) = -3.9351\times10^{-8}$ C in lower plate

e. $C_o = \frac{Q}{V_o}$, $V_o = E_o(0.25\times10^{-3}) = 2.222\times10^{5}(0.25\times10^{-3}) = 55.5556$ Volts

$$C_o = \frac{3.9351\times10^{-8}}{55.5556} = 7.0832\times10^{-10}$$

$C_1 = \frac{Q}{V_1}$, $V_1 = E_1(1\times10^{-3}) = 44.4444$ Volts

$$C_1 = \frac{3.9351\times10^{-8}}{44.4444} = 8.854\times10^{-10}$$

$$C_t = \frac{Q}{V} = \frac{3.9351\times10^{-8}}{100} = 3.9351\times10^{-10} = \frac{C_oC_1}{C_o+C_1} = \frac{7.0775(8.8628)}{7.0775+8.8628}\times10^{-10}$$

f. $W_{eo} = \frac{1}{2}C_oV_o^2 = \frac{1}{2}(7.0832\times10^{-10})(55.5556)^2 = 1.0931\times10^{-6}$ Joules

$W_{e1} = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(8.854\times10^{-10})(44.4444)^2 = 0.87445\times10^{-6}$ Joules

$W_{et} = W_{eo} + W_{e1} = 1.96755\times10^{-6}$ Joules

Also $W_{et} = \frac{1}{2}C_tV^2 = \frac{1}{2}(3.9351\times10^{-10})(100)^2 = 1.96755\times10^{-6}$ Joules

$\boxed{2.5}$ Before the insertion of the slab the electric flux density between the plates was equal to

$$D = \varepsilon_o E_o = 8.854\times10^{-12}\left(\frac{100}{1.25\times10^{-3}}\right) = 7.0832\times10^{-7}\ C/m^2$$

Thus the surface charge density was equal to $|\rho_s| = D = 7.0832\times10^{-7}\ C/m^2$

a,b. After the removal of the battery and the insertion of the dielectric sheet, the charge density on the plates remains the same as before the insertion of the dielectric sheet. Thus

$q_s = +7.0832\times10^{-7}\ C/m^2$ on the upper plate where the voltage is positive.

$$Q = q_s A = 7.0832\times10^{-7}(2\times10^{-2}) = 14.1664\times10^{-9}\ C$$

$q_s = -7.0832\times10^{-7}\ C/m^2$ on the lower plate where the voltage is negative.

$$Q = q_s A = -7.0832\times10^{-7}(2\times10^{-2}) = -14.1664\times10^{-9}\ C$$

c. Using the boundary conditions on the normal components of the electric flux density

$$\hat{n}\cdot(\underline{D}_2 - \underline{D}_1) = q_s \quad \text{or} \quad D_{2n} - \cancel{D_{1n}}^{\,0} = D_{2n} = q_s$$

Thus the electric flux density in the dielectric slab and free space is equal to

$$D_o = D_1 = 7.0832\times10^{-7}\ C/m^2$$

d. $$E_o = \frac{D_o}{\varepsilon_o} = \frac{7.0832\times10^{-7}}{8.854\times10^{-12}} = 80\times10^3\ V/m$$

$$E_1 = \frac{D_1}{\varepsilon_1} = \frac{7.0832\times10^{-7}}{5\,\varepsilon_o} = 16\times10^3\ V/m$$

e. $$V_o = E_o(0.25\times10^{-3}) = 20\ \text{Volts}$$

$$V_1 = E_1(1\times10^{-3}) = 16\ \text{Volts}$$

$$V_t = V_o + V_1 = 20+16 = 36\ \text{Volts}$$

f. $$C_o = \frac{Q}{V_o} = \frac{14.1664\times10^{-9}}{20} = 0.70832\times10^{-9} = 7.0832\times10^{-10}$$

$$C_1 = \frac{Q}{V_1} = \frac{14.1664\times10^{-9}}{16} = 0.8854\times10^{-9} = 8.854\times10^{-10}$$

$$C_t = \frac{Q}{V_t} = \frac{14.1664\times10^{-9}}{36} = 0.39351\times10^{-9} = 3.9351\times10^{-10} = \frac{C_o C_1}{C_o + C_1} = \frac{7.0832(8.854)\times10^{-10}}{7.0832+8.854}$$

g. $$W_{eo} = \tfrac{1}{2}C_o V_o^2 = \tfrac{1}{2}(7.0832\times10^{-10})(20)^2 = 0.14166\times10^{-6}\ \text{Joules}$$

$$W_{e1} = \tfrac{1}{2}C_1 V_1^2 = \tfrac{1}{2}(8.854\times10^{-10})(16)^2 = 0.11333\times10^{-6}\ \text{Joules}$$

$$W_{et} = \tfrac{1}{2}C_t V_t^2 = \tfrac{1}{2}(3.9351\times10^{-10})(36)^2 = 0.25499\times10^{-6} = W_{eo} + W_{e1}\ \text{Joules}$$

$\boxed{2.6}$ Before the insertion of the slab

$$E = \frac{V}{4\times10^{-2}} = \frac{8}{4\times10^{-2}} = 200 \text{ V/m}$$

$$D = \varepsilon_o E = 8.854\times10^{-12}(200) = 17.708\times10^{-10} = 1.7708\times10^{-9} \text{ C/m}^2$$

a. $q_s = \pm D = \pm 1.7708\times10^{-9}$ C/m²

$$Q = \pm q_s A = \pm 1.7708\times10^{-9}(64\times10^{-4}) = \pm 113.3312\times10^{-13} = \pm 1.133312\times10^{-11} \text{ C}$$

b. $E = 200$ V/m

c. $D = 1.7708\times10^{-9}$ C/m²

d. $C = \varepsilon_o \dfrac{A}{d} = 8.854\times10^{-12}\dfrac{64\times10^{-4}}{4\times10^{-2}} = 141.664\times10^{-14} = 1.41664\times10^{-12}$

e. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(1.41664\times10^{-12})(8)^2 = 45.3326\times10^{-12}$ Joules

After the insertion of the slab the electric field intensity remained the same; that is $E_o = E_1 = 200$ Volts/m

$$D_o = \varepsilon_o E_o = 8.854\times10^{-12}(200) = 1.7708\times10^{-9} \Rightarrow q_{so} = 1.7708\times10^{-9}$$

$$D_1 = \varepsilon_1 E_1 = 2.56 D_o = 2.56(1.7708\times10^{-9}) = 4.5332\times10^{-9} \Rightarrow q_{s1} = 4.5332\times10^{-9}$$

f. $Q_o = \pm q_{so} A = \pm 1.7708\times10^{-9}(32\times10^{-4}) = \pm 5.66656\times10^{-12}$ C (in free space)

$Q_1 = \pm q_{s1} A = \pm 4.5332\times10^{-9}(32\times10^{-4}) = \pm 14.50639\times10^{-12}$ C (in dielectric)

g. $E_o = E_1 = 200$ V/m

h. $D_o = 1.7708\times10^{-9}$ C/m² (in free space)

$D_1 = 4.5332\times10^{-9}$ C/m² (in dielectric)

i. $C_o = \dfrac{Q_o}{V} = \dfrac{5.66656\times10^{-12}}{8} = 0.70832\times10^{-12}$ (in free space)

$C_1 = \dfrac{Q_1}{V} = \dfrac{14.50639\times10^{-12}}{8} = 1.813299\times10^{-12}$ (in dielectric)

j. $C_t = \dfrac{Q_t}{V} = \dfrac{(5.66656+14.50639)\times10^{-12}}{8} = \dfrac{20.17295\times10^{-12}}{8} = 2.521619\times10^{-12} = C_o + C_1$

k. $W_{eo} = \frac{1}{2}C_oV^2 = \frac{1}{2}(0.70832\times10^{-12})(8)^2 = 22.66624\times10^{-12}$ (in free space)

$W_{e1} = \frac{1}{2}C_1V^2 = \frac{1}{2}(1.813299\times10^{-12})(8)^2 = 58.02528\times10^{-12}$ (in dielectric)

l. $W_{et} = \frac{1}{2}C_tV^2 = \frac{1}{2}(2.521619\times10^{-12})(8)^2 = 80.6918\times10^{-12}$ Joules

$W_{et} = W_{eo} + W_{e1}$

$\boxed{2.7}$ Parts (a)-(e) are the same as in the solution of Problem 2.6.

f. After the removal of the voltage source and the insertion of the slab, the total charge on the plates stays the same as before the removal of the source. Therefore from part a of Problem 2.6

$$Q_0 + Q_1 = Q_t = 1.133312\times10^{-11}$$

Also along the interface between the free space and the dielectric slab the tangential components of the electric field must be continuous. Therefore

$$E_0 = E_1 \Rightarrow \frac{D_0}{\varepsilon_0} = \frac{D_1}{\varepsilon_1} \Rightarrow \frac{q_{s0}}{\varepsilon_0} = \frac{q_{s1}}{\varepsilon_1} \Rightarrow \frac{Q_0/A_0}{\varepsilon_0} = \frac{Q_1/A_1}{\varepsilon_1} \Rightarrow Q_1 = \frac{\varepsilon_1 A_1}{\varepsilon_0 A_0} Q_0$$

Since the two areas are the same $A_0 = A_1$. Thus

$$Q_1 = \frac{\varepsilon_1}{\varepsilon_0} Q_0 = \varepsilon_r Q_0 \quad \text{where } \varepsilon_r = \frac{\varepsilon_1}{\varepsilon_0}$$

Using these two equations

$$Q_0 + Q_1 = Q_t$$

$$Q_1 = \varepsilon_r Q_0$$

we can write that

$$Q_0 = \frac{Q_t}{1+\varepsilon_r} = \frac{1.133312\times10^{-11}}{1+2.56} = 0.31835\times10^{-11} = 3.1835\times10^{-12}\ \text{C} \quad \text{(in free space)}$$

$$Q_1 = \varepsilon_r Q_0 = 2.56(3.1835\times10^{-1}) = 8.14966\times10^{-12}\ \text{C} \quad \text{(in dielectric)}$$

g. $$q_{s0} = \frac{Q_0}{A_0} = \frac{3.1835\times10^{-12}}{32\times10^{-4}} = 0.9948\times10^{-9} = D_0 \Rightarrow E_0 = \frac{0.9948\times10^{-9}}{8.854\times10^{-12}} = 112.36\ \text{V/m} \quad \text{(in free space)}$$

$$q_{s1} = \frac{Q_1}{A_1} = \frac{8.14966\times10^{-12}}{32\times10^{-4}} = 2.5468\times10^{-9} = D_1 \Rightarrow E_1 = \frac{2.5468\times10^{-9}}{2.56(8.854\times10^{-12})} = 112.36\ \text{V/m} \quad \text{(in dielectric)}$$

h. $D_0 = 0.9948\times10^{-9}\ \text{C/m}^2$ (in free space); $D_1 = 2.5468\times10^{-9}\ \text{C/m}^2$ (in dielectric)

i. $$C_0 = \frac{Q_0}{V_0} = \frac{3.1835\times10^{-12}}{112.36(4\times10^{-2})} = 0.708326\times10^{-12}\ \text{farads} = \varepsilon_0 \frac{A_0}{d}$$

$$C_1 = \frac{Q_1}{V_1} = \frac{8.14966\times10^{-12}}{112.36(4\times10^{-2})} = 1.81329\times10^{-12}\ \text{farads} = \varepsilon_1 \frac{A_1}{d}$$

j. $C_t = C_0 + C_1 = 2.521616\times10^{-12}$ farads

k. $$W_0 = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(0.708326\times10^{-12})[112.36(4\times10^{-2})]^2 = 7.15396\times10^{-12}\ \text{Joules}$$

$$W_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(1.81329\times10^{-12})[112.36(4\times10^{-2})]^2 = 18.31389\ \text{Joules}$$

l. $$W_t = W_0 + W_1 = 25.46785\times10^{-12}\ \text{Joules}$$

$\boxed{2.8}$ $V = 100$ Volts, $A = 2cm^2 = 2\,cm^2\left(\frac{m}{100\,cm}\right)^2 = 2\times10^{-4}$

(a) Before the insertion of the slab, the electric flux density between the plates is equal to

$$D = \varepsilon_0 E_0 = 8.854\times10^{-12}\left(\frac{100}{1.25\times10^{-3}}\right) = 7.0832\times10^{-7}\ C/m^2$$

Thus the surface charge density is equal to $|q_s| = D = 7.0832\times10^{-7}\ C/m^2$

After removal of the battery and the insertion of the dielectric slab, the charge density remains the same. Thus,

$$Q_u = Q(\text{upper}) = q_s(\text{upper})A = +7.0832\times10^{-7}(2\times10^{-4}) = +14.1664\times10^{-11}\ C$$
$$Q_l = Q(\text{lower}) = q_s(\text{lower})A = -7.0832\times10^{-7}(2\times10^{-4}) = -14.1664\times10^{-11}\ C$$

(b) $q_s(\text{upper}) = +7.0832\times10^{-7}\ C/m^2$

$q_s(\text{lower}) = -7.032\times10^{-7}\ C/m^2$

(c) The electric flux density in the dielectric slab and free space is

$$\hat{n}\cdot(D_2 - D_1) = q_s \Rightarrow D_{2n} - \cancel{D_{1n}}^{\,0} = D_{2n} = q_s$$
$$D_d = 7.0832\times10^{-7}\ C/m^2$$
$$D_0 = 7.0832\times10^{-7}\ C/m^2$$

(d) $$\varepsilon_d E_d = D_d \Rightarrow E_d = \frac{D_d}{\varepsilon_d} = \frac{7.0832\times10^{-7}}{5(8.854\times10^{-12})} = \frac{0.8\times10^5}{5} = 16\times10^{+3}\ V/m$$
$$\varepsilon_0 E_0 = D_0 \Rightarrow E_0 = \frac{D_0}{\varepsilon_0} = \frac{7.0832\times10^{-7}}{8.854\times10^{-12}} = 0.8\times10^5 = 80\times10^3\ V/m$$

(e) $$V_d = E_d d_d = 16\times10^{+3}(1\times10^{-3}) = 16\ \text{Volts}$$
$$V_0 = E_0 d_0 = 80\times10^{+3}(0.25\times10^{-3}) = 20\ \text{Volts}$$
$$V_t = V_0 + V_d = 20 + 16 = 36\ \text{Volts}$$

(f) $$C_d = \frac{Q}{V_d} = \frac{14.1664\times10^{-11}}{16} = 0.8854\times10^{-11} = 8.854\times10^{-12}\ \text{farads}$$
$$C_0 = \frac{Q}{20} = \frac{14.1664\times10^{-11}}{20} = 0.70832\times10^{-11} = 7.0832\times10^{-12}\ \text{farads}$$

$$C_t = \frac{Q}{V_t} = \frac{14.1664\times10^{-11}}{36} = 3.9351\times10^{-12} = \frac{C_0 C_1}{C_0 + C_1} = \frac{7.0832(8.854)\times10^{-24}}{(7.0832+8.854)\times10^{-12}}$$

$\boxed{2.9}$ $A = 1\ cm^2,\ V = 10\ volt,\ \varepsilon_{r1} = 2,\ \varepsilon_{r2} = 6,\ d_1 = d_2 = 1\ cm$

(a) $E_1 d_1 + E_2 d_2 = 10 \Rightarrow E_1 + E_2 = \frac{10}{10^{-2}} = 10^3\ V/m$

$D_1 = D_2 \Rightarrow \varepsilon_1 E_1 = \varepsilon_2 E_2 \Rightarrow E_1 = \frac{E_2}{\varepsilon_1}(\varepsilon_2) = \frac{\varepsilon_2}{\varepsilon_1} E_2 = \frac{6}{2} E_2 = 3E_2$

$E_1 + E_2 = 3E_2 + E_2 = 4E_2 = 10^3 \Rightarrow E_2 = 0.25\times10^3\ V/m,\ E_1 = 3E_2 = 0.75\times10^3\ V/m$

$E_1 = 0.75\times10^3\ V/m,\ E_2 = 0.25\times10^3\ V/m$

(b) $D_1 = \varepsilon_1 E_1 = 2\varepsilon_0(0.75\times10^3) = 1.5\varepsilon_0\times10^3\ C/m^2$

$D_2 = \varepsilon_2 E_2 = 6\varepsilon_0(0.25\times10^3) = 1.5\varepsilon_0\times10^3\ C/m^2$

(c) $Q_{upper} = A_1 q_{s1} = 10^{-4}\varepsilon_0(1.5\times10^3) = 0.15\varepsilon_0\ C$

$Q_{lower} = A_2 q_{s2} = 10^{-4}\varepsilon_0(1.5\times10^3) = 0.15\varepsilon_0\ C$

(d) $C_1 = \frac{Q}{V_1} = \frac{Q_1}{E_1 d_1} = \frac{0.15\varepsilon_0}{(0.75\times10^3)(10^{-2})} = 0.02\varepsilon_0\ F$

$C_2 = \frac{Q_2}{V_2} = \frac{0.15\varepsilon_0}{E_2 d_2} = \frac{0.15\varepsilon_0}{(0.25\times10^3)(10^{-2})} = 0.06\varepsilon_0\ F$

$C_T = \frac{Q}{V} = \frac{0.15\varepsilon_0}{10} = 0.015\varepsilon_0\ F$

(e)

$C_1 = \varepsilon_1 \frac{A_1}{d_1} = 2\varepsilon_0 \frac{10^{-4}}{10^{-2}} = 0.02\varepsilon_0$

$C_2 = \varepsilon_2 \frac{A_2}{d_2} = 6\varepsilon_0 \frac{10^{-4}}{10^{-2}} = 0.06\varepsilon_0$

$C_T = \frac{C_1 C_2}{C_1 + C_2} = \frac{(0.02\varepsilon_0)(0.06\varepsilon_0)}{(0.02\varepsilon_0 + 0.06\varepsilon_0)} = \frac{0.02(0.06)}{0.08}\varepsilon_0$

$C_T = \frac{2(6)}{8}\times10^{-2}\varepsilon_0 = 0.015\varepsilon_0$

(f) They are the same, as they should be.

$\boxed{2.10}$

(a) $E_1 = \frac{10}{1\text{ cm}}\left(\frac{100\text{ cm}}{1\text{ m}}\right) = \boxed{10^{+3}\text{ V/m}}$

$E_2 = \frac{10}{1\text{ cm}}\left(\frac{100\text{ cm}}{1\text{ m}}\right) = \boxed{10^{+3}\text{ V/m}}$

(b) $D_1 = E_1\varepsilon_1 = 10^{+3}\left(2\times 8.854\times 10^{-12}\right) = \boxed{17.708\times 10^{-9}\text{ C/m}^2}$

$D_2 = E_2\varepsilon_2 = 10^{+3}\left(6\times 8.854\times 10^{-12}\right) = \boxed{53.124\times 10^{-9}\text{ C/m}^2}$

(c) $q_1 = \boxed{17.708\times 10^{-9}\text{ C/m}^2}$

$q_2 = \boxed{53.124\times 10^{-9}\text{ C/m}^2}$

(d) $Q_1 = q_1 A = 17.708\times 10^{-9}\frac{\text{C}}{\text{m}^2}\left(1\times 10^{-4}\text{ m}^2\right) = 17.708\times 10^{-13} = \boxed{1.7708\times 10^{-12}\text{ C}}$

$Q_2 = q_2 A = 53.124\times 10^{-9}\frac{\text{C}}{\text{m}^2}\left(1\times 10^{-4}\text{ m}^2\right) = 53.124\times 10^{-13} = \boxed{5.3124\times 10^{-12}\text{ C}}$

(e) $C_1 = \frac{Q_1}{V_1} = \frac{1.7708\times 10^{-12}}{10} = \boxed{1.7708\times 10^{-13}\text{ farads}}$

$C_2 = \frac{Q_2}{V_2} = \frac{5.3124\times 10^{-12}}{10} = \boxed{5.3124\times 10^{-13}\text{ farads}}$

(f) $C_T = C_1 + C_2 = (1.7708+5.3124)\times 10^{-13} = 7.0832\times 10^{-13} = \boxed{0.70832\times 10^{-12}\text{ f}}$

(g) $C_1 = \varepsilon_1\frac{A_1}{S_1} = 2\left(8.854\times 10^{-12}\right)\frac{10^{-4}}{10^{-2}} = 17.708\times 10^{-14} = \boxed{1.7708\times 10^{-13}\text{ farads}}$

$C_2 = \varepsilon_2\frac{A_2}{S_2} = 6\left(8.854\times 10^{-12}\right)\frac{10^{-4}}{10^{-2}} = 53.124\times 10^{-14} = \boxed{5.3124\times 10^{-13}\text{ farad}}$

(h) $C_T = C_1 + C_2 = (1.7708+5.3124)\times 10^{-13} = 7.0824\times 10^{-13} = \boxed{0.70824\times 10^{-12}\text{ farads}}$

(i) They are the same as they should. Both are derived based on basic and related principles.

2.11

Before the insertion of the slabs and before the power supply is disconnected

$$E_1 = E_2 = \frac{10}{1\times10^{-2}} = 1{,}000\ V/m = 10^3\ V/m$$

$$D_1 = D_2 = \varepsilon_0 E_1 = \varepsilon_0 E_2 = 8.854\times10^{-12}(10^3) = 8.854\times10^{-9}\ C/m^2$$

$$q_{s1} = q_{s2} = D_1 = D_2 = 8.854\times10^{-9}\ C/m^2$$

$$Q_1 = Q_2 = q_{s1}A_1 = q_{s2}A_2 = 8.854\times10^{-9}(10^{-4}) = 8.854\times10^{-13}$$

After the removal of the power supply, the charge density (C/m^2) in each plate remains in the presence of the slabs is the same before the insertion of the slabs.

(a) $q_{s1} = -q_{s2} = 8.854\times10^{-9}\ C/m^2$

(b) $D_1 = D_2 = |q_{s1}| = |q_{s2}| = 8.854\times10^{-9}\ C/m^2$

(c) $Q_1 = -Q_2 = q_{s1}A_1 = q_{s2}A_2 = 8.854\times10^{-13}\ C$

(d) $D_1 = \varepsilon_1 E_1 \Rightarrow E_1 = \dfrac{D_1}{\varepsilon_1} = \dfrac{8.854\times10^{-9}}{2(8.854\times10^{-12})} = \dfrac{10^3}{2} = 500\ V/m$

$D_2 = \varepsilon_2 E_2 \Rightarrow E_2 = \dfrac{D_2}{\varepsilon_2} = \dfrac{8.854\times10^{-9}}{6(8.854\times10^{-12})} = \dfrac{10^3}{6} = 166.67\ V/m$

(e) $V_1 = E_1 d_1 = 500(10^{-2}) = 5$ Volts

$V_2 = E_2 d_2 = 166.67(10^{-2}) = 1.6667$ Volts

(f) $C_1 = \varepsilon_1 \dfrac{A_1}{d_1} = 2(8.854\times10^{-12})\dfrac{10^{-4}}{10^{-2}} = 17.708\times10^{-14}\ f$

$C_2 = \varepsilon_2 \dfrac{A_2}{d_2} = 6(8.854\times10^{-12})\dfrac{10^{-4}}{10^{-2}} = 53.124\times10^{-14}\ f$

(g) $C_1 = \dfrac{Q_1}{V_1} = \dfrac{8.854\times10^{-13}}{5} = 1.7708\times10^{-13} = 17.708\times10^{-14}\ f$

$C_2 = \dfrac{Q_2}{V_2} = \dfrac{8.854\times10^{-13}}{1.6667} = 5.3124\times10^{-13} = 53.708\times10^{-14}\ f$

(h) They should be the same. Same definition

2.12 Using Coulomb's Law

a. $$\oiint_S \underline{D}\cdot d\underline{s} = \int_0^{\ell}\int_0^{2\pi}(\hat{a}_\rho D_\rho)\cdot\hat{a}_\rho \rho\, d\phi\, dz = \int_0^{\ell}\int_0^{2\pi}\varepsilon_0 E_\rho \rho\, d\phi\, dz = 2\pi\varepsilon_0 \ell \rho E_\rho = Q$$

or

$$E_\rho = \frac{Q}{2\pi\varepsilon_0 \ell}\frac{1}{\rho} = \frac{Q}{\rho}(3\times10^{11}) = 3\times10^{11}\frac{Q}{\rho}$$

Since the voltage is 10 Volts, then

$$V = \int_a^b \underline{E}\cdot d\underline{\ell} = \int_a^b (\hat{a}_\rho E_\rho)\cdot\hat{a}_\rho d\rho = 3\times10^{11} Q\int_a^b \frac{d\rho}{\rho} = 3\times10^{11} Q \ln(\rho)\Big|_a^b$$

$$V = 10 = 3\times10^{11} Q \ln\left(\frac{b}{a}\right) = 3\times10^{11} Q \ln(2) = 3\times10^{11} Q(0.69315)$$

$$Q = \frac{10}{3\times10^{11}(0.69315)} = 48.0898\times10^{-12} \text{ C}$$

$$E_\rho = 3\times10^{11}\frac{Q}{\rho} = 3\times10^{11}\frac{48.0898\times10^{-12}}{\rho} = \frac{14.42695}{\rho}$$

b. $Q = 48.0898\times10^{-12}$ C

c. $$q_{sa} = -\frac{Q}{A_a} = -\frac{Q}{2\pi a\ell} = -\frac{48.0898\times10^{-12}}{2\pi(2\times10^{-2})(6\times10^{-2})} = -6.378\times10^{-9} \text{ C/m}^2$$

$$q_{sb} = \frac{Q}{A_b} = \frac{Q}{2\pi b\ell} = \frac{48.0898\times10^{-12}}{2\pi(4\times10^{-2})(6\times10^{-2})} = 3.189\times10^{-9} \text{ C/m}^2$$

d. $$D = \varepsilon_0 E_\rho = 8.854\times10^{-12}\left(\frac{14.42695}{\rho}\right) = \frac{1.27736}{\rho}\times10^{-10} \text{ C/m}^2$$

e. $$C = \frac{Q}{V} = \frac{48.0898\times10^{-12}}{10} = 4.80898\times10^{-12} \text{ farads}$$

f. $$W_e = \frac{1}{2}CV^2 = \frac{1}{2}(4.80898\times10^{-12})(100) = 2.4045\times10^{-10} \text{ Joules}$$

2.13 After the removal of the voltage source, the total charge stays the same. Using this we can answer the questions.

a. Same as in part b of Problem 2.12

$Q = 48.0898\times10^{-12}$ C

b. Same as in part c of Problem 2.12

$q_{sa} = -6.378\times10^{-9}$ C/m²

$q_{sb} = 3.189\times10^{-9}$ C/m² (continued)

$\boxed{2.13 \text{ cont'd}}$ Same as in part d of Problem 2.12 (in free space and dielectric)

c. $D_{\rho o} = D_{\rho 1} = \dfrac{1.27736\times10^{-10}}{\rho}$

d.
$$E_{\rho o} = \frac{D_{\rho o}}{\varepsilon_o} = \frac{1.27736\times10^{-10}}{8.854\times10^{-12}\,\rho} = \frac{14.4269}{\rho}$$
$$E_{\rho 1} = \frac{D_{\rho 1}}{\varepsilon_1} = \frac{1.27736\times10^{-10}}{2.56(8.854\times10^{-12})\,\rho} = \frac{5.6355}{\rho}$$

e.
$$V = \int_a^c \underline{E}_1\cdot d\underline{\ell} + \int_c^b \underline{E}_o\cdot d\underline{\ell} = 5.6355\int_a^c \frac{d\rho}{\rho} + 14.4269\int_c^b \frac{d\rho}{\rho} = 5.6355\,\ln\left(\frac{c}{a}\right) + 14.4269\,\ln\left(\frac{b}{c}\right)$$
$$= 5.6355\,\ln\left(\frac{3}{2}\right) + 14.4269\,\ln\left(\frac{4}{3}\right) = 5.6355(0.405465) + 14.4269(0.28768)$$
$$V = 2.285 + 4.1504 = 6.4354 \text{ Volts}$$

f. $C = \dfrac{Q}{V} = \dfrac{48.0898\times10^{-12}}{6.4354} = 7.472698\times10^{-12}$ farads

g. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(7.472698)(6.4354)^2\times10^{-12} = 1.54738\times10^{-10}$ Joules

$\boxed{2.14}$ Since the voltage source is connected at all times, the ~~total~~ voltage is maintained at 10 Volts. Also the normal components of the electric flux density is continuous along the interface.

a.
$$V = \int_a^c \underline{E}_1\cdot d\underline{\ell} + \int_c^b \underline{E}_o\cdot d\underline{\ell} = 10 \text{ Volts} = \int_a^c \hat{a}_\rho E_{\rho 1}\cdot\hat{a}_\rho\, d\rho + \int_c^b \hat{a}_\rho E_{\rho o}\cdot\hat{a}_\rho\, d\rho$$
$$V = \int_a^c E_{\rho 1}\,d\rho + \int_c^b E_{\rho o}\,d\rho = \int_a^c \frac{D_{\rho 1}}{\varepsilon_1}\,d\rho + \int_c^b \frac{D_{\rho o}}{\varepsilon_o}\,d\rho = \frac{1}{\varepsilon_1}\int_a^c D_{\rho 1}\,d\rho + \int_c^b D_{\rho o}\,d\rho$$

However $D_{\rho 1} = D_{\rho o} = \dfrac{Q}{2\pi\ell}\dfrac{1}{\rho}$

Thus
$$V = \frac{1}{\varepsilon_1}\frac{Q}{2\pi\ell}\ln\left(\frac{c}{a}\right) + \frac{1}{\varepsilon_o}\frac{Q}{2\pi\ell}\ln\left(\frac{b}{c}\right) = \frac{Q}{\varepsilon_o(2\pi\ell)}\left[\frac{\ln(c/a)}{\varepsilon_r} + \ln(b/c)\right] = 10$$
$$Q = \frac{10(2\pi\varepsilon_o\ell)}{\frac{\ln(3/2)}{2.56} + \ln(4/3)} = \frac{10^{-10}}{3\left[\frac{0.405465}{2.56} + 0.28768\right]} = 7.47276\times10^{-9}\ C$$
$$E_{\rho o} = \frac{Q}{2\pi\varepsilon_o\ell}\frac{1}{\rho} = \frac{7.47276\times10^{-9}}{2\pi\varepsilon_o(6\times10^{-2})\,\rho} = \frac{2{,}241.827}{\rho} \quad \text{(in free space)}$$

(continued)

$\boxed{\text{2.14 cont'd.}}$ $E_{\rho 1} = \frac{Q}{2\pi\varepsilon_1 \ell}\frac{1}{\rho} = \frac{E_{\rho o}}{\varepsilon_r} = \frac{875.7136}{\rho}$ (in dielectric)

b. $D_{\rho o} = \varepsilon_o E_{\rho o} = \frac{19.849\times10^{-9}}{\rho}$ C/m²

$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \frac{19.849\times10^{-9}}{\rho}$ C/m²

c. $q_{sa} = D_{\rho 1}\Big|_{\rho=2\times10^{-2}} = \frac{19.849\times10^{-9}}{2\times10^{-2}} = 0.99246\times10^{-6}$ C/m²

$q_{sb} = D_{\rho o}\Big|_{\rho=4\times10^{-3}} = \frac{q_{sa}}{2} = 0.49623\times10^{-6}$ C/m²

d. $Q_a = q_{sa}(2\pi a\ell) = 0.99246\times10^{-6}(2\pi\times2\times10^{-2}\times6\times10^{-2}) = 74.8295\times10^{-10}$ C

$Q_b = q_{sb}(2\pi b\ell) = 0.49623\times10^{-6}(2\pi\times4\times10^{-2}\times6\times10^{-2}) = 74.8295\times10^{-10}$ C

e. $C = \frac{Q}{V} = \frac{74.8295\times10^{-10}}{10} = 7.48295\times10^{-10} = 0.748295\times10^{-9}$ farads

f. $W_e = \frac{1}{2}CV^2 = \frac{1}{2}(0.748295\times10^{-9})(100) = 37.41475\times10^{-9}$ Joules

$\boxed{2.15}$ The voltage source is connected across at all times.

a. $E_o = \frac{100}{4\times10^{-2}} = 2{,}500$ V/m $= E_1$

b. $D_o = \varepsilon_o E_o = 22.135\times10^{-9}$ C/m² (in free space)

$D_1 = \varepsilon_1 E_1 = 2.56 D_o = 56.6656\times10^{-9}$ C/m² (in dielectric)

c. $Q_o = 22.135\times10^{-9}(2\times10^{-2}) = 0.4427\times10^{-9}$ C (in freespace)

$Q_1 = 56.6656\times10^{-9}(2\times10^{-2}) = 1.13331\times10^{-9}$ C (in dielectric)

d. $W_{eo} = \frac{1}{2}C_o V_o^2 = \frac{1}{2}\left(\frac{Q_o}{V_o}\right)V_o^2 = \frac{1}{2}Q_o V_o = \frac{1}{2}(0.4427\times10^{-9})(100) = 22.135\times10^{-9}$ J (free space)

$W_{e1} = \frac{1}{2}C_1 V_1^2 = \frac{1}{2}\left(\frac{Q_1}{V_1}\right)V_1^2 = \frac{1}{2}Q_1 V_1 = \frac{1}{2}(1.13331\times10^{-9})(100) = 56.6655\times10^{-9}$ J (dielectric)

e. $Q_t = Q_o + Q_1 = 1.57601\times10^{-9}$

f. $C_t = \frac{Q_t}{V} = \frac{1.57601\times10^{-9}}{100} = 1.57601\times10^{-11} = 15.7601\times10^{-12} = C_o + C_1$ farads

g. $W_{et} = \frac{1}{2}C_t V^2 = \frac{1}{2}(15.7601\times10^{-12})(100)^2 = 78.8005\times10^{-9}$ J $= W_{eo} + W_{e1}$

$\boxed{2.16}$

(a) $\oiint_S \underline{D}\cdot d\underline{s} = \int_0^{\ell}\int_0^{\pi} \hat{a}_\rho D_{\rho 0}\cdot\hat{a}_\rho\, dz\rho d\phi + \int_0^{\ell}\int_{\pi}^{2\pi} \hat{a}_\rho D_{\rho 1}\cdot\hat{a}_\rho\, dz\rho d\phi$

$\oiint_S \underline{D}\cdot d\underline{s} = \pi\ell(D_{\rho 0}+D_{\rho 1})\rho = Q_t \Rightarrow D_{\rho 0}+D_{\rho 1} = \frac{Q_t}{2\pi\ell}\left(\frac{1}{\rho}\right)$

Also, because the tangential electric field components ar continuous along the radial interfaces between the dielectric & free space

$$E_{\rho 0} = E_{\rho 1} \Rightarrow \frac{D_{\rho 0}}{\varepsilon_0} = \frac{D_{\rho 1}}{\varepsilon_1} \Rightarrow D_{\rho 1} = \frac{\varepsilon_1}{\varepsilon_0}D_{\rho 0} = \varepsilon_r D_{\rho 0},\ \varepsilon_r = \frac{\varepsilon_1}{\varepsilon_0}$$

Thus

$$D_{\rho 0}(1+\varepsilon_r) = \frac{Q_t}{2\pi\ell}\frac{1}{\rho} \Rightarrow D_{\rho 0} = \frac{Q_t}{2\pi\ell(1+\varepsilon_r)}\frac{1}{\rho},\ D_{\rho 1} = \frac{Q_t}{2\pi\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho}$$

$$E_{\rho 0} = \frac{D_{\rho 0}}{\varepsilon_0} = \frac{Q_t}{\pi\varepsilon_0\ell(1+\varepsilon_r)}\frac{1}{\rho},\quad \text{in free space}$$

$$E_{\rho 1} = \frac{D_{\rho 1}}{\varepsilon_1} = \frac{Q_t}{\pi\varepsilon_1\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho},\ \text{in dielectric}$$

(b) $\int_a^b \underline{E}\cdot d\underline{\ell} = \int_a^b \hat{a}_\rho E_{\rho 0}\cdot\hat{a}_\rho d\rho = \int_a^b E_{\rho 0} d\rho = \frac{Q_t \ln(b/a)}{2\pi\varepsilon_0\ell(1+\varepsilon_r)} = V = 10$

$$Q_t = \frac{10(\pi\varepsilon_0\ell)(1+\varepsilon_r)}{\ln(2)} = \frac{(1+2.56)\times10^{-10}}{6\ln(2)} = 0.856\times10^{-10}\ C$$

(c,d) $D_{\rho 0} = \varepsilon_0 E_{\rho 0} = \frac{Q_t}{\pi\ell(1+\varepsilon_r)}\frac{1}{\rho} = \frac{0.856\times10^{-10}}{2\pi(6\times10^{-2})(1+2.56)}\frac{1}{\rho} = \frac{1.2756\times10^{-10}}{\rho}\ C/m^2$

$$D_{\rho 0}\big|_{\rho=a} = \frac{1.2756\times10^{-10}}{2\times10^{-2}} = 0.6378\times10^{-8}\ C/m^2 \Rightarrow q_{s0}\big|_{\rho=a} = 0.6378\times10^{-8}\ C/m^2$$

$$D_{\rho 0}\big|_{\rho=b} = \frac{1.2756\times10^{-10}}{4\times10^{-2}} = 0.3189\times10^{-8}\ C/m^2 \Rightarrow q_{s0}\big|_{\rho=b} = 0.3189\times10^{-8}\ C/m^2$$

$$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \frac{Q_t}{\pi\ell}\frac{\varepsilon_r}{1+\varepsilon_r}\frac{1}{\rho} = \varepsilon_r D_{\rho 0} = \frac{3.2655\times10^{-10}}{\rho}\ C/m^2$$

$$D_{\rho 1}\big|_{\rho=a} = \frac{3.2655\times10^{-10}}{2\times10^{-2}} = 1.63277\times10^{-8}\ C/m^2 \Rightarrow q_{s1}\big|_{\rho=a} = 1.63277\times10^{-8}\ C/m^2$$

$$D_{\rho 1}\big|_{\rho=b} = \frac{3.2655\times10^{-10}}{4\times10^{-2}} = 0.81638\times10^{-8}\ C/m^2 \Rightarrow q_{s1}\big|_{\rho=b} = 0.81638\times10^{-8}\ C/m^2$$

(e)

$$C_0 = \frac{Q_0}{V} = \frac{0.6378\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2})}{10} = 2.40445\times10^{-12}\ \text{farads (in free space)}$$

$$C_1 = \frac{Q_1}{V} = \frac{1.63277\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2})}{10} = 6.155398\times10^{-12}\ \text{farads (in dielectric)}$$

$$C_t = C_0 + C_1 = 8.559847\times10^{-12}\ \text{farads}$$

(f) $W_{e0} = \frac{1}{2}C_0V^2 = \frac{1}{2}(2.40445\times10^{-12})(100) = 1.2022\times10^{-10}$ Joules (in free space)

$W_{e1} = \frac{1}{2}C_1V^2 = \frac{1}{2}(6.155398\times10^{-12})(100) = 3.077699\times10^{-10}$ Joules (in dielectric)

$W_t = W_{e0} + W_{e1} = 4.279949\times10^{-10}$ Joules (total)

$\boxed{2.17}$ From Problem 2.16 $E_\rho = \dfrac{Q}{2\pi\varepsilon l\rho}$ where Q is the total charge.

a. Since the charge stays the same at all times and is the same as that of Problem 2.16, or $Q_t = 85.5985\times10^{-12}$ C. Also to satisfy the boundary conditions along the interface between the two media

$$E_{\rho 0} = E_{\rho 1}$$

b. $D_{\rho 0} = \varepsilon_0 E_{\rho 0} = \dfrac{1.2756\times10^{-10}}{\rho}$ C/m²

$D_{\rho 1} = \varepsilon_1 E_{\rho 1} = \varepsilon_r E_{\rho 0} = \dfrac{3.2655\times10^{-10}}{\rho}$ C/m²

c. $q_{s0}|_{\rho=a} = -D_{\rho 0}|_{\rho=a} = -\dfrac{1.2756\times10^{-10}}{2\times10^{-2}} = -0.6378\times10^{-8}$ C/m²

$q_{s0}|_{\rho=b} = D_{\rho 0}|_{\rho=b} = \dfrac{1.2756\times10^{-10}}{4\times10^{-2}} = 0.3189\times10^{-8}$ C/m²

$q_{s1}|_{\rho=a} = -D_{\rho 1}|_{\rho=a} = -\dfrac{3.2655\times10^{-10}}{2\times10^{-2}} = -1.63277\times10^{-8}$ C/m²

$q_{s1}|_{\rho=b} = D_{\rho 1}|_{\rho=b} = \dfrac{3.2756\times10^{-10}}{4\times10^{-2}} = 0.81638\times10^{-8}$ C/m²

d. $Q_0|_{\rho=a} = q_{s0}|_{\rho=a}(\pi a l) = -0.6378\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2}) = -24.0445\times10^{-12}$ C

$Q_0|_{\rho=b} = q_{s0}|_{\rho=b}(\pi b l) = 0.3189\times10^{-8}(\pi\times4\times10^{-2}\times6\times10^{-2}) = 24.0445\times10^{-12}$ C

$Q_1|_{\rho=a} = q_{s1}|_{\rho=a}(\pi a l) = -1.63277\times10^{-8}(\pi\times2\times10^{-2}\times6\times10^{-2}) = -61.5540\times10^{-12}$ C

$Q_1|_{\rho=b} = q_{s1}|_{\rho=b}(\pi b l) = 1.63277\times10^{-8}(\pi\times4\times10^{-2}\times6\times10^{-2}) = 61.5540\times10^{-12}$ C

e. $C_0 = \dfrac{Q_0}{V_0} = \dfrac{24.0445\times10^{-12}}{10} = 2.40445\times10^{-12}$ farads in free space

$C_1 = \dfrac{Q_1}{V_1} = \dfrac{61.5540\times10^{-12}}{10} = 6.15540\times10^{-12}$ farads in polystyrene

$C_t = C_0 + C_1 = (2.40445 + 6.15540)\times10^{-12} = 8.55985\times10^{-12}$

f. $W_{e0} = \frac{1}{2}C_0V_0^2 = \frac{1}{2}(2.40445\times10^{-12})(100) = 1.2022\times10^{-10}$ Joules (in free space)

$W_{e1} = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(6.15540\times10^{-12})(100) = 3.0777\times10^{-10}$ Joules (in polystyrene)

$W_{et} = \frac{1}{2}C_tV_0^2 = \frac{1}{2}C_tV_1^2 = \frac{1}{2}(8.55985\times10^{-12})(100) = 4.2799\times10^{-10}$ Joules

$= W_{e0} + W_{e1}$

$\boxed{2.18}$

$\underline{\mathcal{E}} = \hat{a}_z 10^{-3} \sin(2\pi\times10^7 t)$, $\varepsilon_{sr} = 2.56$

(a) $1+\chi_e = \dfrac{\varepsilon_s}{\varepsilon_0} = \varepsilon_{sr} \Rightarrow \chi_e = \varepsilon_{sr} - 1 = 2.56 - 1 = 1.56$

(b) $\underline{\mathcal{D}} = \varepsilon_s \underline{\mathcal{E}} = \hat{a}_z 2.56(\varepsilon_0) 10^{-3} \sin(2\pi\times10^7 t) = \hat{a}_z 2.56\times10^{-3}\varepsilon_0 \sin(2\pi\times10^7 t)$

$\underline{\mathcal{D}} = \hat{a}_z 2.56\times10^{-3}\varepsilon_0 \sin(2\pi\times10^7 t)$

(c) $\underline{\mathcal{P}} = \varepsilon_0 \chi_e \underline{\mathcal{E}} = \hat{a}_z \varepsilon_0 (1.56\times10^{-3}\varepsilon_0)\sin(2\pi\times10^7 t) = \hat{a}_z 1.56\times10^{-3}\varepsilon_0 \sin(2\pi\times10^7 t)$

$\underline{\mathcal{P}} = \hat{a}_z 1.56\times10^{-3}\varepsilon_0 \sin(2\pi\times10^7 t)$

(d) $\underline{\mathcal{J}}_d = \dfrac{\partial \underline{\mathcal{D}}}{\partial t} = \hat{a}_z 2.56\times10^{-3}\varepsilon_0 \dfrac{\partial}{\partial t}\sin(2\pi\times10^7 t)$

$= \hat{a}_z (2.56\times10^{-3}\times2\pi\times10^7)\varepsilon_0 \cos(2\pi\times10^7 t)$

$\underline{\mathcal{J}}_d = \hat{a}_z 16.085\times10^4 \varepsilon_0 \cos(2\pi\times10^7 t)$

(e) $\underline{\mathcal{J}}_p = \dfrac{\partial \underline{\mathcal{P}}}{\partial t} = \hat{a}_z \dfrac{\partial}{\partial t}\left[1.56\times10^{-3}\varepsilon_0 \sin(2\pi\times10^7 t)\right]$

$= \hat{a}_z 1.56\times10^{-3}(2\pi\times10^7)\varepsilon_0 \cos(2\pi\times10^7 t)$

$\underline{\mathcal{J}}_p = \hat{a}_z 9.802\times10^4 \varepsilon_0 \cos(2\pi\times10^7 t)$

$\boxed{2.19}$ $\underline{M} = \hat{a}_z 1.245\times10^6$ A/m , $\underline{H} = \hat{a}_z 5\times10^3$ A/m

a. $\underline{J}_{ms} = \underline{M}\times\hat{n}$

$y=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_y) = \hat{a}_x M_z = \hat{a}_x 1.245\times10^6$

$x=4$ cm: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_x = \hat{a}_y M_z = \hat{a}_y 1.245\times10^6$

$y=6$ cm: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_y = -\hat{a}_x M_z = -\hat{a}_x 1.245\times10^6$

$x=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_x) = -\hat{a}_y M_z = -\hat{a}_y 1.245\times10^6$

$z=0$: $\underline{J}_{ms} = \hat{a}_z M_z \times(-\hat{a}_z) = 0$

$z=1$ cm: $\underline{J}_{ms} = \hat{a}_z M_z \times \hat{a}_z = 0$

b. $\underline{J}_m = \nabla\times\underline{M} = 0$

c. $I_m = \oiint_S \underline{J}_m \cdot d\underline{s} = \iiint_V (\nabla\cdot\underline{J}_m)\, dv = 0$

d. $\chi_m = \dfrac{M}{H} = \dfrac{1.245\times10^6}{5\times10^3} = 249$

$\mu_r = 1+\chi_m = 1+249 = 250$

$\boxed{2.20}$ $\underline{H} = \hat{a}_\phi \dfrac{0.3183}{\rho}$ A/m , $\underline{M} = \hat{a}_\phi \dfrac{190.67}{\rho}$ A/m

a. $\underline{J}_{ms} = \underline{M}\times\hat{n}$

$\rho=a$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times(-\hat{a}_\rho)\Big|_{\rho=a} = \hat{a}_z M_\phi\Big|_{\rho=a} = \hat{a}_z \dfrac{190.67}{1\times10^{-2}} = \hat{a}_z 19.067\times10^3$

$\rho=b$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times \hat{a}_\rho\Big|_{\rho=b} = -\hat{a}_z M_\phi\Big|_{\rho=b} = -\hat{a}_z \dfrac{190.67}{3\times10^{-2}} = -\hat{a}_z 6.3557\times10^3$

b. $\underline{J}_m = \nabla\times\underline{M} = \hat{a}_z \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho M_\phi) = \hat{a}_z \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(190.67) = 0$

c. $I_m = \oiint_S \underline{J}_m \cdot d\underline{s} = \iiint_V (\nabla\cdot\underline{J}_m)\, dv = 0$

d. $\chi_m = \dfrac{M}{H} = \dfrac{190.67/\rho}{0.3183/\rho} = \dfrac{190.67}{0.3183} = 599.03 \simeq 599$

$\mu_r = 1+\chi_m = 1+599 = 600$

$z=0$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times \hat{a}_z = \hat{a}_\rho M_\phi$

$\underline{J}_{ms} = \hat{a}_\rho \dfrac{160.67}{\rho}$

$z=-l$: $\underline{J}_{ms} = \hat{a}_\phi M_\phi \times(-\hat{a}_z)$

$= -\hat{a}_\rho M_\phi$

$\underline{J}_{ms} = -\hat{a}_\rho \dfrac{160.67}{\rho}$

$\boxed{2.21}$ $\underline{M} = \hat{a}_\phi 10 \quad \frac{A}{m}$

a. $\underline{J}_{ms} = \underline{M} \times \hat{n}\Big|_{\rho=1} = (\hat{a}_\phi M_\phi \times \hat{a}_\rho)\Big|_{\rho=1} = -\hat{a}_z M_\phi\Big|_{\rho=1} = -\hat{a}_z 10 \quad A/m$

b. $\underline{J}_m = \nabla \times \underline{M} = \hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho M_\phi) = \hat{a}_z \frac{1}{\rho}\frac{\partial}{\partial\rho}(10\rho) = \hat{a}_z \frac{10}{\rho} \quad A/m^2$

c. $I_m = \iint_S \underline{J}_m \cdot d\underline{s} = \int_0^{2\pi}\int_0^1 \hat{a}_z J_{mz} \cdot \hat{a}_z \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^1 10\, d\rho\, d\phi = 2\pi(10) = 62.83185\ A$

$\boxed{2.22}$ $\underline{J} \simeq \hat{a}_z J_o e^{-10^2[(a-|x|)+(a-|y|)]} \quad A/m^2$

$$I = \iint_{S_o} \underline{J} \cdot d\underline{s} = 4\int_0^a\int_0^a \hat{a}_z J_z \cdot \hat{a}_z\, dx\, dy = 4J_o\left[\int_0^a e^{-100(a-x)}dx \int_0^a e^{-100(a-y)}dy\right]$$

$$= 4J_o\left[\int_0^a e^{-100(a-x)}dx\right]^2 = 4J_o\left\{\left[\frac{1}{100}e^{-100(a-x)}\right]_0^a\right\}^2 = 4J_o\left\{\frac{1}{100}\left[1-e^{-100a}\right]\right\}^2$$

$$I = 4J_o\left\{\frac{1}{100}\left[1-e^{-1}\right]\right\}^2 = 4J_o\left[\frac{1}{100}(1-0.36788)\right]^2 = 2(7.9915\times10^{-5})J_o \quad A$$

$$= 15.983\times10^{-5} J_o \quad A$$

$\boxed{2.23}$ $\underline{J} = \hat{a}_z J_o e^{-10^4(a-\rho)} \quad A/m^2$

a. $I = \iint_{S_o} \underline{J} \cdot \hat{a}_z d\underline{s} = \int_0^{2\pi}\int_0^a \hat{a}_z J_z \cdot \hat{a}_z \rho\, d\rho\, d\phi = \int_0^{2\pi}\int_0^a J_o e^{-10^4(a-\rho)}\rho\, d\rho\, d\phi$

$$= 2\pi J_o \int_0^a e^{-10^4(a-\rho)}\rho\, d\rho = 2\pi J_o \int_0^a e^{-10^4 a} e^{10^4\rho}\rho\, d\rho$$

Let $u = 10^4\rho \Rightarrow du = 10^4 d\rho \Rightarrow d\rho = 10^{-4}du,\ \rho = 10^{-4}u$

$$I = 2\pi\times10^{-8} J_o \int_0^{10^4 a} e^{-10^4 a} e^{u} u\, du = 2\pi\times10^{-8} J_o e^{-10^4 a}\int_0^{10^4 a} e^{u} u\, du$$

Using the integral $\int_0^c e^u u\, du = e^u(u-1)\Big|_0^c$ we can write that

$$I = 2\pi J_o\times10^{-8} e^{-10^4 a}\left[e^{10^4 a}(10^4 a-1) - e^0(0-1)\right]_{a=0.01} = 0.622\times10^{-5} J_o = 10\ A$$

$$I = 10 \simeq 0.622\times10^{-5} J_o \Rightarrow J_o = 10/0.622\times10^{-5} = 1.6077\times10^6\ A/m^2$$

b. $J = J_o e^{-10^4(a-\rho)} = e^{-1} J_o = 0.368 J_o \Rightarrow 10^4(a-\rho) = 1$

$$a-\rho = 10^{-4} \Rightarrow \rho = a - 10^{-4}$$

Distance from surface $= 10^{-4}$ m (very small; almost on the surface)

$\boxed{2.24}$ $\sigma = 5.76\times10^{7}$ S/m, $\varepsilon = 8.854\times10^{-12}$ f/m

$$T_r = \frac{\varepsilon}{\sigma} = \frac{8.854\times10^{-12}}{5.76\times10^{7}} = 1.5372\times10^{-19}\ \text{secs.}$$

In the microwave region (f= 1-10 GHz) the period is

$$f=1\ \text{GHz:}\quad T=\frac{1}{f}=\frac{1}{10^9}=10^{-9} >> T_r = 1.5372\times10^{-19}$$

$$f=2\ \text{GHz:}\quad T=\frac{1}{f}=\frac{1}{10^{10}}=10^{-10} >> T_r = 1.5372\times10^{-19}$$

For x-rays $[\lambda=(1-10)\times10^{-8}\,\text{cm}]$ the period is

$$\lambda = 1\times10^{-8}\,\text{cm:}\quad f=\frac{v}{\lambda}=\frac{3\times10^{10}}{10^{-8}}=3\times10^{+18}$$

$$T=\frac{1}{f}=\frac{1}{3\times10^{18}}=0.333\times10^{-18}=3.33\times10^{-19}$$

$$\lambda = 10\times10^{-8}\,\text{cm:}\quad f=\frac{v}{\lambda}=\frac{3\times10^{10}}{10\times10^{-8}}=3\times10^{+17}$$

$$T=\frac{1}{f}=\frac{1}{3\times10^{17}}=0.333\times10^{-17}=3.33\times10^{-18}$$

which are comparable to the relaxation time constant.

$\boxed{2.25}$ $\sigma = 3.96\times10^{7}$ S/m, $\mu_e = 2.2\times10^{-3}\ m^2/(V\text{-}s)$, $\underline{E} = \hat{a}_x 2$ V/m

a. $\sigma = -\mu_e q_{ve} \Rightarrow q_{ve} = -\dfrac{\sigma}{\mu_e} = -\dfrac{3.96\times10^{7}}{2.2\times10^{-3}} = -1.8\times10^{10}\ \ C/m^3$

b. $\underline{v}_e = -\mu_e \underline{E} = -2.2\times10^{-3}(\hat{a}_x E_x) = -\hat{a}_x(2.2\times2\times10^{-3}) = -\hat{a}_x\, 4.4\times10^{-3}$ m/sec.

c. $\underline{J}_c = \sigma\underline{E} = 3.96\times10^{+7}(\hat{a}_x 2) = \hat{a}_x\, 7.92\times10^{7}$ A/m²

d. $I = \iint_{S_o} \underline{J}_c\cdot d\underline{s} = \iint_{S_o} \hat{a}_x J_{cx}\cdot\hat{a}_x\, ds = \iint_{S_o} J_{cx}\, ds = 7.92\times10^{7}(S_o) = 7.92\times10^{7}(10^{-3})$

$I = 7.92\times10^{4}$ A

e. $N_e = \dfrac{q_{ve}}{q_e} = \dfrac{-1.8\times10^{10}}{-1.602\times10^{-19}} = 1.1236\times10^{29}\ \dfrac{\text{electrons}}{m^3}$

Chapter 3

$\boxed{3.1}$ $\nabla\times\underline{E} = -\underline{M}_i - j\omega\mu\underline{H}$, $\nabla\times\underline{H} = \underline{J}_i + \sigma\underline{E} + j\omega\varepsilon\underline{E}$

$$\nabla\times\nabla\times\underline{E} = -\nabla\times\underline{M}_i - j\omega\mu\nabla\times\underline{H}$$

$$\nabla\times\nabla\times\underline{H} = \nabla\times\underline{J}_i + \sigma\nabla\times\underline{E} + j\omega\varepsilon\nabla\times\underline{E}$$

Using Maxwell's equations from above and the vector identify of

$$\nabla\times\nabla\times\underline{F} = \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}$$

we can write

$$\nabla(\nabla\cdot\underline{E}) - \nabla^2\underline{E} = -\nabla\times\underline{M}_i - j\omega\mu\left[\underline{J}_i + \sigma\underline{E} + j\omega\varepsilon\underline{E}\right]$$

$$= -\nabla\times\underline{M}_i - j\omega\mu\underline{J}_i - j\omega\mu\sigma\underline{E} + \omega^2\mu\varepsilon\underline{E}$$

Since $\nabla\cdot\underline{D} = \nabla\cdot(\varepsilon\underline{E}) = \varepsilon\nabla\cdot\underline{E} = q_{ve} \Rightarrow \nabla\cdot\underline{E} = \frac{q_{ve}}{\varepsilon}$

Now we can write that

$$\nabla^2\underline{E} = \nabla\times\underline{M}_i + j\omega\mu\underline{J}_i + \frac{1}{\varepsilon}\nabla q_{ve} + j\omega\mu\sigma\underline{E} - \omega^2\mu\varepsilon\underline{E}$$

which is an uncoupled second-order differential equation.

Using the equation for the magnetic field from above along with Maxwell's equations and the vector identity, we can write

$$\nabla(\nabla\cdot\underline{H}) - \nabla^2\underline{H} = \nabla\times\underline{J}_i + (\sigma + j\omega\varepsilon)\nabla\times\underline{E} = \nabla\times\underline{J}_i + (\sigma + j\omega\varepsilon)(-\underline{M}_i - j\omega\mu\underline{H})$$

Since $\nabla\cdot\underline{B} = \nabla\cdot(\mu\underline{H}) = \mu\nabla\cdot\underline{H} = q_{vm} \Rightarrow \nabla\cdot\underline{H} = \frac{1}{\mu}q_{vm}$

then

$$\nabla^2\underline{H} = -\nabla\times\underline{J}_i + \sigma\underline{M}_i + \frac{1}{\mu}\nabla q_{vm} + j\omega\varepsilon\underline{M}_i + j\omega\mu\sigma\underline{H} - \omega^2\mu\varepsilon\underline{H}$$

which also is an uncoupled second order differential equation.

$\boxed{3.2}$ $\frac{d^2f}{dx^2} = -\beta_x^2 f$, $f = f_1 = A_1 e^{-j\beta_x x} + B_1 e^{+j\beta_x x}$

Using $f = f_1 = A_1 e^{-j\beta_x x}$, then

$$(-j\beta_x)^2 A_1 e^{-j\beta_x x} = -\beta_x^2 A_1 e^{-j\beta_x x} = -\beta_x A_1 e^{-j\beta_x x} \qquad \text{Q.E.D.}$$

(continued)

$\boxed{3.2 \text{ cont'd.}}$ The same can be shown by letting $f = f_1 = B_1 e^{+j\beta_x x}$

Now let us try the cosinusoidal solutions.

Let $f = f_2 = C_1 \cos(\beta_x x)$

Substituting this into the differential equation leads to

$$-\beta_x^2 C_1 \cos(\beta_x) = -\beta_x^2 C_1 \cos(\beta_x x) \qquad \text{Q.E.D.}$$

The same can be shown by letting $f = f_2 = D_1 \sin(\beta_x x)$

$\boxed{3.3}$ $\mathcal{E}_x^-(x,y,z;t) = [C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)][C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)] B_3 \cos(\omega t + \beta_z z)$

To follow a point z_p for different values of t we must maintain constant the amplitude of the cosine term. This is accomplished by letting

$$\omega t + \beta_z z = C_1 = \text{constant}$$

Taking a derivative of both sides with respect to time, we can write

$$\omega(1) + \beta_z \frac{dz}{dt} = 0 \Rightarrow \omega + \beta_z v_p = 0 \Rightarrow v_p = -\frac{\omega}{\beta_z}$$

which indicates that the wave is moving in the $-z$ direction.

$\boxed{3.4}$ $\nabla^2 E_x - \gamma^2 E_x = 0 = \dfrac{d^2 E_x}{dx^2} + \dfrac{d^2 E_x}{dy^2} + \dfrac{d^2 E_x}{dz^2} - \gamma^2 E_x$

Letting $E_x(x,y,z) = f(x)\, g(y)\, h(z)$ and substituting above leads to

$$gh \frac{d^2 f}{dx^2} + fh \frac{d^2 g}{dy^2} + fg \frac{d^2 h}{dz^2} - \gamma^2 fgh = 0$$

Diving both sides by fgh, we can write

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = \gamma^2$$

By letting each term on the left equal to a constant leads to

$$\frac{1}{f}\frac{d^2 f}{dx^2} = \gamma_x^2 \Rightarrow f = f_1 = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x}; \; f = f_2 = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x)$$

$$\frac{1}{g}\frac{d^2 g}{dy^2} = \gamma_y^2 \Rightarrow g = g_1 = A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y}; \; g = g_2 = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y)$$

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\gamma_z^2 \Rightarrow h = h_1 = A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z}; \; h = h_2 = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z)$$

provided that $\qquad \gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2$

$\boxed{3.5}$ $\nabla(\nabla\cdot\underline{E}) - \nabla\times\nabla\times\underline{E} = -\beta^2\underline{E}$, $\underline{E} = \hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z$

Using cylindrical coordinates

$$\nabla\cdot\underline{E} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

$$\nabla(\nabla\cdot\underline{E}) = \nabla\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]$$

$$= \hat{a}_\rho\left\{\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\} + \hat{a}_\phi\frac{1}{\rho}\left\{\frac{\partial}{\partial\phi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\}$$

$$+ \hat{a}_z\left\{\frac{\partial}{\partial z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\rho) + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}\right]\right\}$$

$$\nabla(\nabla\cdot\underline{E}) = \hat{a}_\rho\left[\frac{\partial^2 E_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial\rho} - \frac{E_\rho}{\rho^2} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\rho\partial\phi} - \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial^2 E_z}{\partial\rho\partial z}\right]$$

$$+ \hat{a}_\phi\left[\frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial\rho\partial\phi} + \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial\phi} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\partial z}\right] + \hat{a}_z\left[\frac{\partial^2 E_\rho}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\phi\partial z} + \frac{\partial^2 E_z}{\partial z^2}\right]$$

$$\nabla\times\underline{E} = \hat{a}_\rho\left[\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right] + \hat{a}_\phi\left[\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial\rho}\right] + \hat{a}_z\left[\frac{\partial E_\phi}{\partial\rho} + \frac{E_\phi}{\rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial\phi}\right]$$

$$\nabla\times\nabla\times\underline{E} = \hat{a}_\rho\left\{\frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\rho\partial\phi} + \frac{1}{\rho^2}\frac{\partial E_\phi}{\partial\phi} - \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} - \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial^2 E_z}{\partial\rho\partial z}\right\}$$

$$+ \hat{a}_\phi\left\{\frac{1}{\rho}\frac{\partial^2 E_z}{\partial\phi\partial z} - \frac{\partial^2 E_\phi}{\partial z^2} - \frac{\partial^2 E_\phi}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial E_\phi}{\partial\rho} + \frac{E_\phi}{\rho^2} + \frac{1}{\rho}\frac{\partial^2 E_\rho}{\partial\rho\partial\phi} - \frac{1}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right\}$$

$$+ \hat{a}_z\left\{\frac{\partial^2 E_\rho}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial z} - \frac{\partial^2 E_z}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} - \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial^2 E_\phi}{\partial\phi\partial z}\right\}$$

Substituting all these into the wave equation and equating identical components from the left and right sides, it leads to

<u>ρ component:</u>

$$\left[\frac{\partial^2 E_\rho}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\rho}{\partial\phi^2} + \frac{\partial^2 E_\rho}{\partial z^2}\right] - \frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi} = -\beta^2 E_\rho$$

or $\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial\phi}\right) = -\beta^2 E_\rho$

<u>ϕ component:</u>

$$\left[\frac{\partial^2 E_\phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial\phi^2} + \frac{\partial^2 E_\phi}{\partial z^2}\right] - \frac{1}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi} = -\beta^2 E_\phi$$

or $\nabla^2 E_\phi + \left[-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial\phi}\right] = -\beta^2 E_\phi$

<u>z component:</u>

$$\left[\frac{\partial^2 E_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right] = -\beta^2 E_z$$

or $\nabla^2 E_z = -\beta^2 E_z$

$\boxed{3.6}$ $\quad \underline{E} = \hat{a}_\rho E_\rho(\rho,\phi,z) + \hat{a}_\phi E_\phi(\rho,\phi,z) + \hat{a}_z E_z$

Using the rectangular to cylindrical coordinate transformation, we can write

$$\hat{a}_\rho = \hat{a}_x \cos\phi + \hat{a}_y \sin\phi$$

$$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$$

Therefore

$$\frac{\partial \hat{a}_\rho}{\partial \rho} = 0 \;;\quad \frac{\partial \hat{a}_\phi}{\partial \rho} = 0 \;;\quad \frac{\partial \hat{a}_z}{\partial \rho} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial \phi} = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi = \hat{a}_\phi \;;\quad \frac{\partial^2 \hat{a}_\rho}{\partial \phi^2} = -\hat{a}_x \cos\phi - \hat{a}_y \sin\phi = -\hat{a}_\rho$$

$$\frac{\partial \hat{a}_\phi}{\partial \phi} = -\hat{a}_x \cos\phi - \hat{a}_y \sin\phi = -\hat{a}_\rho \;;\quad \frac{\partial^2 \hat{a}_\phi}{\partial \phi^2} = \hat{a}_x \sin\phi - \hat{a}_y \cos\phi = -\hat{a}_\phi$$

$$\frac{\partial \hat{a}_z}{\partial \phi} = 0$$

$$\frac{\partial \hat{a}_\rho}{\partial z} = 0 \;;\quad \frac{\partial \hat{a}_\phi}{\partial z} = 0 \;;\quad \frac{\partial \hat{a}_z}{\partial z} = 0$$

$$\nabla^2 \underline{E} = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \nabla^2(\hat{a}_z E_z) = \nabla^2(\hat{a}_\rho E_\rho) + \nabla^2(\hat{a}_\phi E_\phi) + \hat{a}_z \nabla^2 E_z$$

$$\nabla^2(\hat{a}_\rho E_\rho) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial}{\partial \rho}(\hat{a}_\rho E_\rho)\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(\hat{a}_\rho E_\rho) + \frac{\partial^2(\hat{a}_\rho E_\rho)}{\partial z^2}$$

$$\frac{\partial^2}{\partial \phi^2}(\hat{a}_\rho E_\rho) = \frac{\partial}{\partial \phi}\left[\frac{\partial}{\partial \phi}(\hat{a}_\rho E_\rho)\right] = \frac{\partial}{\partial \phi}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\rho}{\partial \phi}\right] = \frac{\partial}{\partial \phi}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi E_\rho\right]$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \frac{\partial E_\rho}{\partial \phi}\frac{\partial \hat{a}_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + E_\rho \frac{\partial \hat{a}_\phi}{\partial \phi}$$

$$= \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} + \hat{a}_\phi \frac{\partial E_\rho}{\partial \phi} - \hat{a}_\rho E_\rho = -\hat{a}_\rho E_\rho + \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial \phi^2} + \hat{a}_\phi 2 \frac{\partial E_\rho}{\partial \phi}$$

$$\frac{\partial^2}{\partial z^2}(\hat{a}_\rho E_\rho) = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}(\hat{a}_\rho E_\rho)\right] = \frac{\partial}{\partial z}\left[\hat{a}_\rho \frac{\partial E_\rho}{\partial z} + E_\rho \cancelto{0}{\frac{\partial \hat{a}_\rho}{\partial z}}\right] = \hat{a}_\rho \frac{\partial^2 E_\rho}{\partial z^2} + \frac{\partial E_\rho}{\partial z}\cancelto{0}{\frac{\partial \hat{a}_\rho}{\partial z}}$$

Therefore

$$\nabla^2(\hat{a}_\rho E_\rho) = \hat{a}_\rho\left[\frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2}\left(\frac{\partial^2 E_\rho}{\partial \phi^2} - E_\rho\right) + \frac{\partial^2 E_\rho}{\partial z^2}\right] + \hat{a}_\phi \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi}$$

Similarly

$$\nabla^2(\hat{a}_\phi E_\phi) = \frac{\partial^2}{\partial \rho^2}(\hat{a}_\phi E_\phi) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\hat{a}_\phi E_\phi) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}(\hat{a}_\phi E_\phi) + \hat{a}_\phi \frac{\partial^2 E_\phi}{\partial z^2}$$

$$= -\hat{a}_\rho \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial \phi} + \hat{a}_\phi\left[\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_\phi}{\partial \phi^2} - \frac{E_\phi}{\rho^2} + \frac{\partial^2 E_\phi}{\partial z^2}\right]$$

$$\nabla^2(\hat{a}_z E_z) = \hat{a}_z \nabla^2 E_z = \hat{a}_z\left[\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2}\right]$$

(continued)