INSTRUCTOR'S SOLUTIONS MANUAL

BARBARA BOSCHMANS
Northern Arizona University

BRIAN BEAUDRIE
Northern Arizona University

A PROBLEM SOLVING APPROACH TO MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

TWELFTH EDITION

Rick Billstein
University of Montana

Shlomo Libeskind
University of Oregon

Johnny W. Lott
University of Montana

PEARSON

Boston Columbus Indianapolis New York San Francisco
Amsterdam Cape Town Dubai London Madrid Milan Munich Paris Montreal Toronto
Delhi Mexico City São Paulo Sydney Hong Kong Seoul Singapore Taipei Tokyo

Reproduced by Pearson from electronic files supplied by the author.

Publishing as Pearson, 501 Boylston Street, Boston, MA 02116.

ISBN-13: 978-0-321-99057-0
ISBN-10: 0-321-99057-9

www.pearsonhighered.com

PEARSON

Contents

CHAPTER 1

AN INTRODUCTION TO PROBLEM SOLVING

Assessment 1-1A: Mathematics and Problem Solving

1. **(a)** List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 2 & + & \cdots & + & 98 & + & 99 \\ 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\ \hline 100 & + & 100 & + & \cdots & + & 100 & + & 100 \end{array}$$

There are 99 sums of 100. Thus the total can be found by computing $\frac{99\cdot 100}{2} = \mathbf{4950}$.

(Another way of looking at this problem is to realize there are $\frac{99}{2} = 49.5$ pairs of sums, each of 100; thus $49.5 \cdot 100 = 4950$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1001-1}{2} + 1 = 501$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 3 & + & \cdots & + & 999 & + & 1001 \\ 1001 & + & 999 & + & \cdots & + & 3 & + & 1 \\ \hline 1002 & + & 1002 & + & \cdots & + & 1002 & + & 1002 \end{array}$$

There are 501 sums of 1002. Thus the total can be found by computing $\frac{501\cdot 1002}{2} = \mathbf{251{,}001}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{300-3}{3} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 3 & + & 6 & + & \cdots & + & 297 & + & 300 \\ 300 & + & 297 & + & \cdots & + & 6 & + & 3 \\ \hline 303 & + & 303 & + & \cdots & + & 303 & + & 303 \end{array}$$

There are 100 sums of 303. Thus the total can be found by computing $\frac{100\cdot 303}{2} = 15{,}150$.

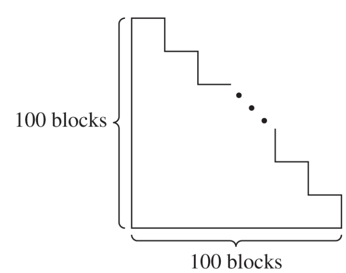
(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{400-4}{4} + 1 = 100$ terms.

List the numbers:

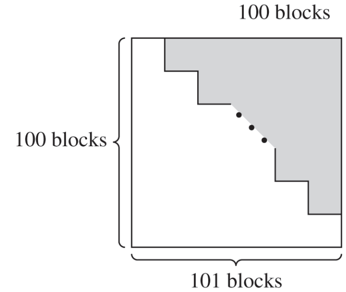
$$\begin{array}{rcrcccrcr} 4 & + & 8 & + & \cdots & + & 396 & + & 400 \\ 400 & + & 396 & + & \cdots & + & 8 & + & 4 \\ \hline 404 & + & 404 & + & \cdots & + & 404 & + & 404 \end{array}$$

There are 100 sums of 404. Thus the total can be found by computing $\frac{100\cdot 404}{2} = 20{,}200$.

2. **(a)**



(b)



When the stack in (a) and a stack of the same size is placed differently next to the original stack in (a), a rectangle containing 100 (101) blocks is created. Since each block is represented twice, the desired sum is $100(101)/2 = \mathbf{5050}$.

While the above represents a specific example, the same thinking can be used for any natural number n to arrive at a formula $n(n+1)/2$.

3. There are $\frac{147-36}{1} + 1 = 112$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 36 & + & 37 & + & \cdots & + & 146 & + & 147 \\ 147 & + & 146 & + & \cdots & + & 37 & + & 36 \\ \hline 183 & + & 183 & + & \cdots & + & 183 & + & 183 \end{array}$$

There are 112 sums of 183. Thus the total can be found by computing $\frac{112 \cdot 183}{2} = \mathbf{10,248}$.

4. **(a)** Make a table as follows; there are 9 rows so there are 9 different ways.

6-cookie packages	2-cookie packages	single-cookie packages
1	2	0
1	1	2
1	0	4
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

(b) Make a table as follows; there are 12 rows so there are **12 different ways**.

6-cookie packages	2-cookie packages	single-cookie packages
2	0	0
1	3	0
1	2	2
1	1	4
1	0	6
0	6	0
0	5	2
0	4	4
0	3	6
0	2	8
0	1	10
0	0	12

5. If each layer of boxes has 7 more than the previous layer we can add powers of 7:

$7^0 = 1$ (red box)

$7^1 = 7$ (blue boxes)

$7^2 = 49$ (black boxes)

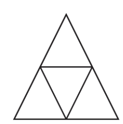
$7^3 = 343$ (yellow boxes)

$7^4 = 2401$ (gold boxes)

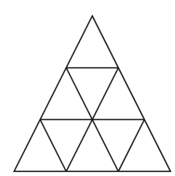
$1 + 7 + 49 + 343 + 2401 = 2801$ boxes altogether.

6. Using strategies from Poyla's problem solving list identify subgoals (solve simpler problems) and make diagrams to solve the original problem.

1 triangle; name this the "unit" triangle.

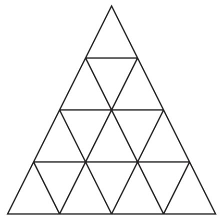


This triangle is made of 4 unit triangles. Counting the large triangle there are 5 triangles



Unit triangles	4 unit triangles	9 unit triangles
9	3	1

13 total triangles



Unit triangles	4 unit triangles	9 unit triangles	16 unit triangles
16	7	3	1

There are 27 triangles in the original figure.

7. Observe that $E = (1+1) + (3+1) + \cdots + (97+1) = O + 49$. Thus, E is 49 more than O..

Alternative strategy:

$$O + E = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 97 + 98$$

$$= \frac{98(99)}{2} = 49(99)$$

$$E = 2(1 + 2 + 3 + 4 + \cdots + 49) = 2\left(\frac{49(50)}{2}\right) = 49(50)$$

$$O = O + E - E = 49(99) - 49(50) = 49(49).$$

So O is 49 less than E.

8. Bubba is last; Cory must be between Alababa and Dandy; Dandy is faster than Cory. Listing from fastest to slowest, the finishing order is then Dandy, Cory, Alababa, and Bubba.

9. Make a table.

\$20 bills	\$10 bills	\$5 bills
2	1	0
2	0	2
1	3	0
1	2	2
1	1	4
1	0	6
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

There are twelve rows so there are twelve different ways. 12

10. The diagonal from the left, top corner to the right, bottom corner sums to $17 + 22 + 27 = 66$.

The first row sums to $17 + a + 7 = 24 + a$. So $a = 66 - 24 = \mathbf{42}$. The last column sums to $7 + b + 27 = 34 + b$. So $b = 66 - 34 = \mathbf{32}$. The first column sums to $17 + 12 + c = 29 + c$. So $c = 66 - 29 = \mathbf{37}$. The second column sums to $42 + 22 + d = 64 + d$. So $d = 66 - 64 = \mathbf{2}$.

11. Debbie and Amy began reading on the same day, since 72 pages for Debbie ÷ 9 pages per day = 8 days. Thus Amy is on 6 pages per day × 8 days = **page 4**8.

12. The last three digits must sum to 20, so the second to last digit must be $20 - (7 + 4) = 9$. Since the sum of the 11^{th} , 12^{th}, and 13^{th} digits is also 20, the 11^{th} digit is $20 - (7 + 9) = 4$.

A		7								4	7	9	4

We can continue in this fashion until we find that A is 9, or we can observe the repeating pattern from back to front, 4, 9, 7, 4, 9, 7, … and discover that A is 9.

13. Choose the box labeled Oranges and Apples (Box B). Retrieve a fruit from Box B. Since Box B is mislabeled, Box B should be labeled as having the fruit you retrieved. For example, if you retrieved an apple, then Box B should be labeled Apples. Since Box A is mislabeled, the Oranges and Apples label should be placed on Box A. These leave only one possibility for Box C; it should be labeled Oranges. If an orange was retrieved from Box B, then Box C would be labeled Oranges and Apples and Box A should be labeled Apples.

14. The electrician made \$1315 for 4 days at \$50 per hour. She spent \$15 per day on gasoline so 4 • \$15 = \$60 on gasoline. The total is then \$1315 + \$60 = \$ 1375. At \$50 per hour, she worked $\frac{1375}{50} = 27.5$ hours.

15. Working backward: Top – 6 rungs – 7 rungs + 5 rungs – 3 rungs = top – 11 rungs, which is located at the middle.

From the middle rung travel up 11 to the top or down 11 to the bottom. Along with the starting rung, then, there are 11 + 11 + 1 = **23 rungs.**

Assessment 1-1B

1. **(a)** List the numbers:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & \cdots & + & 48 & + & 49 \\ 49 & + & 48 & + & \cdots & + & 2 & + & 1 \\ \hline 50 & + & 50 & + & \cdots & + & 50 & + & 50 \end{array}$$

There are 49 sums of 50. Thus the total can be found by computing $\frac{49 \cdot 50}{2} = \mathbf{1225}$. (Another way of looking at this problem is to realize there are $\frac{49}{2} = 24.5$ pairs of sums, each of 50; thus $24.5 \cdot 50 = 1225$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{2009-1}{2} + 1 = 1005$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 1 & + & 3 & + & \cdots & + & 2007 & + & 2009 \\ 2009 & + & 2007 & + & \cdots & + & 3 & + & 1 \\ \hline 2010 & + & 2010 & + & \cdots & + & 2010 & + & 2010 \end{array}$$

There are 1005 sums of 2010. Thus the total can be found by computing $\frac{1005 \cdot 2010}{2} = \mathbf{1{,}010{,}025}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{600-6}{6} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 6 & + & 12 & + & \cdots & + & 594 & + & 600 \\ 600 & + & 594 & + & \cdots & + & 12 & + & 6 \\ \hline 606 & + & 606 & + & \cdots & + & 606 & + & 606 \end{array}$$

There are 100 sums of 606. Thus the total can be found by computing $\frac{606 \cdot 100}{2} = 30{,}300$.

(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last (or the last from the first if the first is greater than the last), dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1000-5}{5} + 1 = 200$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 1000 & + & 995 & + & \cdots & + & 10 & + & 5 \\ 5 & + & 10 & + & \cdots & + & 995 & + & 1000 \\ \hline 1005 & + & 1005 & + & \cdots & + & 1005 & + & 1005 \end{array}$$

There are 200 sums of 1005. Thus the total can be found by computing $\frac{1005 \cdot 200}{2} = 100{,}500$.

2. **(a)** The diagram illustrates how the numbers can be paired to form 50 sums of 101. The sum of the first 100 natural numbers is $50(101) = \mathbf{5050}$.

(b) A diagram similar to the one in 2a would illustrate how the numbers can be paired to form 100 sums of 202. Since the last term is odd, the middle term, 101, is left unpaired. So, the sum of the first 201 natural numbers is 100 • 202 + 101 = 20,301.

3. There are $\frac{203-58}{1} + 1 = 146$ terms. terms

List the numbers:

$$\begin{array}{ccccccccc} 58 & + & 59 & + & \cdots & + & 202 & + & 203 \\ 203 & + & 202 & + & \cdots & + & 59 & + & 58 \\ \hline 261 & + & 261 & + & \cdots & + & 261 & + & 261 \end{array}$$

There are 146 sums of 261. Thus the total can be found by computing $\frac{146 \cdot 261}{2} = \mathbf{19{,}053}$.

4. There are many answers to this problem. A systematic list is a good approach. Using only two numbers and addition, 5 rows give 5 different ways.

One number	Eleven minus the number
1	10
2	9
3	8
4	7
5	6

We can view 6 + 5 as a different way than 5 + 6 and continue in this manner to find 10 different ways. Or, we can find 7 more ways using three numbers as follows.

1	1	9
2	1	8
3	1	7
4	1	6
5	1	5
1	2	8

5. After two socks are drawn, the two socks match or they don't. If they match, we are done. If they don't match, the next sock drawn will match one of the two socks already drawn.

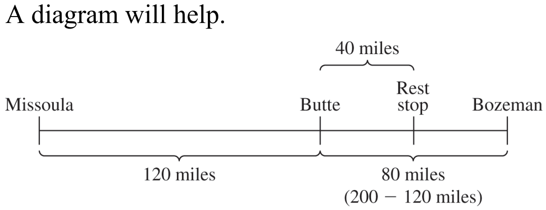
6. There are 13 squares of one unit each; 4 squares of four units each; and one square 9 units; for a total of 18 squares.

7.

$$\begin{aligned} P &= 1 + 3 + 5 + 7 + \ldots + 99 \\ Q &= \qquad\quad 5 + 7 + \ldots + 99 + 101 \\ Q - P &= (5 + \ldots + 99 + 101) - \\ &\quad (1 + 3 + 5 + \ldots + 99) \\ &= (101) - (1 + 3) \\ &= 97 \end{aligned}$$

Q is larger than P by 97.

8. A diagram will help.



The next step is 120 miles + 40 miles = **160 miles** from Missoula.

9. **(a)** Marc must have five pennies to make an even \$1.00. The minimum number of coins would have as many quarters as possible, or three quarters. The remaining 20¢ must consist of at least one dime and one nickel; the only possibility is one dime and two nickels. The minimum is 5 pennies, 2 nickels, 1 dime, and 3 quarters, or 11 coins.

(b) The maximum number of coins is achieved by having as many pennies as possible. It is a requirement to have one quarter, one dime, and one nickel = 40¢, so there may then be 60 pennies for a total of 63 coins.

10. Adding the numbers gives 99. This means that each row, diagonal, and column must add to 99 ÷ 3 = 33. Write 33 as a sum of the numbers in all possible ways:

$19 + 11 + 3$
$19 + 9 + 5$
$17 + 13 + 3$
$17 + 11 + 5$
$17 + 9 + 7$
$15 + 13 + 5$
$15 + 11 + 7$
$13 + 11 + 9$

Summarizing the pattern:

Number	Nr. sums with number
3	2
5	3
7	2
9	3
11	4
13	3
15	2
17	3
19	2

Thus 11 must be in the center of the square and 5, 9, 13, and 17 must be in the corners. One solution would be:

17	7	9
3	11	19
13	15	5

11. Answers may vary; two solutions might be to:

(a) Put four marbles on each tray of the balance scale. Take the heavier four and weigh two on each tray. Take the heavier two and weigh one on each tray; the heavier marble will be evident on this third weighing.

(b) This alternative shows the heavier marble can be found more efficiently, two steps rather than three. Put three marbles in each tray of the balance scale.

(***i***) If the two trays are the same weight, the heavier marble is one of the remaining two. Weigh them to find the heavier.

(***ii***) If one side is heavier, take two of the three marbles and weigh them. If they are the same weight, the remaining marble is the heavier. If not, the heavier will be evident on this second weighing.

12. **(a)** There are:

1 partridge $\times$ 12 days $=$ 12 gifts;
2 doves $\times$ 11 days $=$ 22 gifts;
3 hens $\times$ 10 days $=$ 30 gifts;
4 birds $\times$ 9 days $=$ 36 gifts;
5 rings $\times$ 8 days $=$ 40 gifts;
6 geese $\times$ 7 days $=$ 42 gifts;
7 swans $\times$ 6 days $=$ 42 gifts;
8 maids $\times$ 5 days $=$ 40 gifts;

9 ladies $\times$ 4 days $=$ 36 gifts;
10 lords $\times$ 3 days $=$ 30 gifts;
11 pipers $\times$ 2 days $=$ 22 gifts; and
12 drummers $\times$ 1 day $=$ 12 gifts.

So the gifts given the most by your true love was 42 geese and 42 swans.

(b) $12 + 22 + \cdots + 22 + 12 =$ **364** gifts total.

13. **(a)** There must be 1 or 3 quarters for an amount ending in 5. Then dimes can add to \$1.15 plus 4 pennies to realize \$1.19. Thus:

Quarters	Dimes	Pennies	Total
3	4	4	\$1.19
1	9	4	\$1.19

and in neither case can change for \$1.00 be made.

(b) Two or zero quarters would allow an amount ending in 0. Then more combinations of dimes or pennies could add to \$1.00.

14. If the price of 15 sandwiches equals the price of 20 salads, each sandwich will buy $\frac{20}{15} = \frac{4}{3}$ salads. Thus 3 sandwiches $= 3\left(\frac{4}{3}\right) =$ **4 salads**.

15. Use a variable and a table

12 AM	5 AM	9 AM	12 PM
T	T − 15	2(T − 15)	2(T − 15) + 10

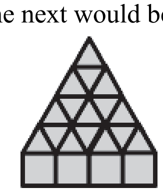
So,
$$\begin{aligned} 2(T - 15) + 10 &= 32 \\ 2T - 30 + 10 &= 32 \\ 2T - 20 &= 32 \\ 2T &= 52 \\ T &= \textbf{26 degrees}. \end{aligned}$$

Assessment 1-2A: Explorations with Patterns

1. **(a)** Each figure in the sequence adds one box each to the top and bottom rows. The next would be:

(b) Each figure in the sequence adds one upright and one inverted triangle. The next would be:

(c) Each figure in the sequence adds one box to the base and one row to the overall triangle. The next would be:



2. **(a)** Terms that continue a pattern are 17, 21, 25, , … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are 220, 270, 320, … . This is arithmetic because each successive term is obtained from the previous term by addition of 50.

(c) Terms that continue a pattern are 27, 81, 243, … . This is geometric because each successive term is obtained from the previous term by multiplying by 3.

(d) Terms that continue a pattern are $10^9, 10^{11}, 10^{13}, \ldots$. This is geometric because each successive term is obtained from the previous term by multiplying by 10^2.

(e) Terms that continue a pattern are $193+10\cdot 2^{30}, 193+11\cdot 2^{30}, 193+12\cdot 2^{30}, \ldots$. This is arithmetic because each successive term is obtained from the previous term by addition of 230.

3. In these problems, let a_n represent the nth term in a sequence, a_1 represent the first term, d represent the common difference between terms in an arithmetic sequence, and r represent the common ratio between terms in a geometric sequence. In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 1$ and $d = 4$:

(*i*) $a_{100} = 1 + (100-1)\cdot 4$
$= 1 + 99 \cdot 4 = 397.$

(*ii*) $a_n = 1 + (n-1)\cdot 4$
$= 1 + 4n - 4 = 4n - 3.$

(b) Arithmetic sequence: $a_1 = 70$ and $d = 50$:

(*i*) $a_{100} = 70 + (100-1)\cdot 50$
$= 70 + 99 \cdot 50 = 5020.$

(*ii*) $a_n = 70 + (n-1)\cdot 50$
$= 70 + 50n - 50 \text{ or } 50n + 20.$

(c) Geometric sequence: $a_1 = 1$ and $r = 3$:

(*i*) $a_{100} = 1 \cdot 3^{100-1} = 3^{99}.$

(*ii*) $a_n = 1 \cdot 3^{n-1} = 3^{n-1}.$

(d) Geometric sequence: $a_1 = 10$ and $r = 10^2$:

(*i*) $a_{100} = 10 \cdot (10^2)^{(100-1)} = 10 \cdot (10^2)^{99}$
$= 10 \cdot 10^{198} = 10^{199}.$

(*ii*) $a_n = 10 \cdot (10^2)^{(n-1)}$
$= 10 \cdot 10^{(2n-2)} = 10^{2n-1}.$

(e) Arithmetic sequence:
$a_1 = 193 + 7 \cdot 2^{30}$ and $d = 2^{30}$:

(*i*) $a_{100} = 193 + 7 \cdot 2^{30} + (100-1)\cdot 2^{30}$
$= 193 + 7 \cdot 2^{30} + 99 \cdot 2^{30}$
$= 193 + 106 \cdot 2^{30}.$

(*ii*) $a_n = 193 + 7 \cdot 2^{30} + (n-1)\cdot 2^{30}$
$= 193 + (n+6)\cdot 2^{30}.$

4. 2, 7, 12, … . Each term is the 5th number on a clock face (clockwise) from, the preceding term.

5. (a) Make a table.

Number of term	Term
1	$1 \cdot 1 \cdot 1 = 1$
2	$2 \cdot 2 \cdot 2 = 8$
3	$3 \cdot 3 \cdot 3 = 27$
4	$4 \cdot 4 \cdot 4 = 64$
5	$5 \cdot 5 \cdot 5 = 125$
6	$6 \cdot 6 \cdot 6 = 216$
7	$7 \cdot 7 \cdot 7 = 343$
8	$8 \cdot 8 \cdot 8 = 512$
9	$9 \cdot 9 \cdot 9 = 729$
10	$10 \cdot 10 \cdot 10 = 1000$
11	$11 \cdot 11 \cdot 11 = 1331$

The 11th term 1331 is the least 4-digit number greater than 1000.

(b) The 9th term 729 is the greatest 3-digit number in this pattern.

(c) The greatest number less than 10^4 is $21 \cdot 21 \cdot 21 = 9261$.

(d) The cell A14 corresponds to the 14th term, which is $14 \cdot 14 \cdot 14 = 2744$.

6. (a) The number of matchstick squares in each windmill form an arithmetic sequence with $a_1 = 5$ and $d = 4$. The number of matchstick squares required to build the 10th windmill is thus $5 + (10 - 1) \cdot 4 =$ $5 + 9 \cdot 4 = \mathbf{41}$ **squares**.

(b) The ***n***th windmill would require $5 + (n - 1) \cdot 4 = 5 + 4n - 4 = \mathbf{4n + 1}$ squares.

(c) There are 16 matchsticks in the original windmill. Each additional windmill adds 12 matchsticks.

This is an arithmetic sequence with $a_1 = 16$ and $d = 12$, so $a_n = 16 +$ $(n - 1) \cdot 12 = \mathbf{12n + 4}$ matchsticks.

7. (a) Each cube adds four squares to the preceding figure; or 6, 10, 14, … . This is an arithmetic sequence with $a_1 = 6$ and $d = 4$. Thus $a_{15} = 6 + (15 - 1) \cdot 4 = 62\,squares$ to be painted in the 10th figure.

(b) This is an arithmetic sequence with $a_1 = 6$ and $d = 6$. The ***n***th term is thus: $a_n = 6 + (n - 1) \cdot 4 = \mathbf{4n + 2}$.

8. Since the first year begins with 700 students, after the first year there would be 760, after the second there would be 820, … , and after the twelfth year the number of students would be the 13th term in the sequence.

This then is an arithmetic sequence with $a_1 = 700$ and $d = 60$, so the 13th term (current enrollment + twelve more years) is:

$700 + (13 - 1) \cdot 60 = 1420\,students.$

9. Using the general expression for the *n*th term of an arithmetic sequence with $a_1 = 24{,}000$ and $a_9 = 31{,}680$ yields:

$$31680 = 24000 + (9 - 1)d$$
$$31680 = 24000 + 8d \Rightarrow d = 960,$$

the amount by which Joe's income increased each year.

To find the year in which his income was \$45120:

$$45120 = 24000 + (n - 1) \cdot 960$$
$$45120 = 23040 + 960n$$
$$\Rightarrow n = 23.$$

Joe's income was \$45,120 in his **23rd year**.

10. (a) If the first difference of the sequence increases by 2 for each term, then the five first differences between the first six terms of the original sequence are 2, 4, 6, 8, 10. If the first term of the original sequence is 3, then the first six terms are 3, 5, 9, 15, 23, 33.

(b) If the first term is a, then $a + (a + 2) =$ $10 \Rightarrow a = 4$. Thus the first six terms of the original sequence are 4, 6, 10, 16, 24, 34.

(c) If the fifth term is 35, then:

Sixth term $35 + 10 = 45$

Fourth term $35 - 8 = 27$

Third term $27 - 6 = 21$

Second term $21 - 4 = 17$

First term $17 - 2 = 15$.

Thus the sequence is 15, 17, 21, 27, 35, 45.

11. (a) Look for the differences:

```
5     6     14     32     64     115     191
   1     8     18     32     51     76
      7     10     14     19     25
         3      4      5      6
            1      1      1
```

The third difference row is an arithmetic sequence with fixed difference of 1. Thus the 6th term in the second difference row $25 + 7 = 32$; the 7th term in the first difference row is $76 + 32 = 108$; and the 8th term in the original sequence is $191 + 108 = 299$. Using the same reasoning, the next three terms in the original sequence will be 299, 447, 644.

(b) Look for the differences:

0		2		6		12		20		30		42
	2		4		6		8		10		12	
		2		2		2		2		2		2

The first difference row is an arithmetic sequence with fixed difference 2. Thus the 7th term in the first difference row is $12 + 2 = 14$; the 8th term in the original sequence is $42 + 14 = 56$. Using the same reasoning, the next three terms in the original sequence are 56, 72, 90.

12. **(a)** Using the general expression for the nth term of an arithmetic sequence with $a_1 = 51, a_n = 251$, and $d = 1$ yields:

$$251 = 51 + (n-1) \cdot 1 \Rightarrow$$
$$251 = 50 + n \Rightarrow n = 201.$$

There are 201 terms in the sequence.

(b) Using the general expression for the ***n***th term of a geometric sequence with $a_1 = 1$, $a_n = 2^{60}$, and $r = 2$ yields:

$$2^{60} = 1(2)^{n-1} = 2^{n-1}$$
$$\Rightarrow n - 1 = 60 \Rightarrow n = 61.$$

There are 61 terms in the sequence.

(c) Using the general expression for the ***n***th term of an arithmetic sequence with $a_1 = 10$, $a_n = 2000$, and $d = 10$ yields:

$$2000 = 10 + (n-1) \cdot 10 \Rightarrow$$
$$2000 = 10n \Rightarrow n = 200.$$

There are 200 terms in the sequence.

(d) Using the general expression for the n^{th} term of a geometric sequence with $a_1 = 1$, $a_n = 1024$, and $r = 2$ yields:

$$1024 = 1(2)^{n-1} \Rightarrow$$
$$2^{10} = 2^{n-1} \Rightarrow n - 1 = 10$$
$$\Rightarrow n = 11.$$

There are 11 terms in the sequence.

13. **(a)**
First term: $(1)^2 + 2 = \mathbf{3}$;
Second term: $(2)^2 + 2 = \mathbf{6}$;
Third term: $(3)^2 + 2 = \mathbf{11}$;
Fourth term: $(4)^2 + 2 = \mathbf{18}$; and
Fifth term: $(5)^2 + 2 = \mathbf{27}$.

(b)
First term: $5(1) + 1 = 6$;
Second term: $5(2) + 1 = 11$;
Third term: $5(3) + 1 = 16$;
Fourth term: $5(4) + 1 = 21$; and
Fifth term: $5(5) + 1 = 26$.

(c)
First term: $10^{(1)} - 1 = \mathbf{9}$;
Second term: $10^{(2)} - 1 = \mathbf{99}$;
Third term: $10^{(3)} - 1 = \mathbf{999}$;
Fourth term: $10^{(4)} - 1 = \mathbf{9999}$; and
Fifth term: $10^{(5)} - 1 = \mathbf{99999}$.

(d)
First term: $3(1) - 2 = 1$**;**
Second term: $3(2) - 2 = 4$**;**
Third term: $3(3) - 2 = 7$**;**
Fourth term: $3(4) - 2 = 10$**;** and
Fifth term: $3(5) - 2 = 13$**.**

14. Answers may vary; examples are:

(a) If $n = 5$, then $\frac{5+5}{5} = 2 \neq 5 + 1 = 6$.

(b) If $n = 2$, then $(2 + 4)^2 = 36 \neq 2^2 + 16 = 20$.

15. **(a)** There are 1, 5, 11, 19, 29 tiles in the five figures. Each figure adds 2n tiles to the preceding figure, thus a_6 the 6th term has $29 + 12 =$ **41 tiles**.

(b) $n^2 = 1, 4, 9, 16, 25, \ldots$. Adding $(n-1)$ to n^2 yields $1, 5, 11, 19, 29, \ldots$, which is the proper sequence. Thus the n*th* term has $n^2 + (n-1)$.

(c) If $n^2 + (n - 1) = 1259$;

Then $n^2 + n - 1260 = 0$. This implies $(n - 35)(n + 36) = 0$, so $n = 35$.

There are 1259 tiles in the 35th figure.

16. The nth term of the arithmetic sequence is $200 + n(200)$. The sequence can also be generated by adding 200 to the previous term. The n^{th} term of the geometric sequence is 2^n. The sequence can also be generated by multiplying the previous term by 2. Make a table.

Number of the term	Arithmetic term	Geometric term
7	1600	128
8	1800	256
9	2000	512
10	2200	1024
11	2400	2048
12	2600	4096

With the 12th term, the geometric sequence is greater.

17. **(a)** Start with one piece of paper. Cutting it into five pieces gives us 5. Taking each of the pieces and cutting it into five pieces again gives $5 \cdot 5 = 25$ pieces. Continuing this process gives a geometric sequence: 1, 5, 25, 125, … . After the 5^{th} cut there are $5^5 = 3125$ pieces of paper.

(b) The number of pieces after the nth cut would be 5^n.

18. **(a)** For an arithmetic sequence there is a common difference between the terms. Between 39 and 69 there are three differences so we can find the common difference by subtracting 39 from 69 and dividing the answer by three: $69 - 39 = 30$ and $30 \div 3 = 10$. The common difference is 10 and we can find the missing terms: 39 – 10 = 29 and 39 + 10 = 49 and 49 + 10 = 59.

(b) For an arithmetic sequence there is a common difference between the terms. Between 200 and 800 there are three differences so we can find the common difference by subtracting 200 from 800 and dividing the answer by three: $800 - 200 = 600$ and $600 \div 3 = 200$. The common difference is 200 and we can find the missing terms: 200 – 200 = 0 and 200 + 200 = 400 and 400 + 200 = 600.

(c) For a geometric sequence there is a common ration between the terms. Between 5^4 and 5^{10} there are three common ratios used so we can find the common ratio by dividing 5^{10} by 5^4 and then taking the cube root:

$5^{10} \div 5^4 = 5^6$ and $(5^6)^{(\frac{1}{3})} = 5^2$. The common ratio is 52 and we can find the missing terms: $5^4 \div 5^2 = 5^2, 5^4 \cdot 5^2 = 5^6, 5^6 \cdot 5^2 = 5^8$.

19. **(a)** Let's call the missing terms a, b, c, d, and f, then the sequence becomes:

$a, b, 1, 1, c, d, e, f$

$b + 1 = 1 \to b = 0$

$a + b = 1 \to a + 0 = 1 \to a = 1$

$1 + 1 = c \to c = 2$

$1 + c = d \to 1 + 2 = d \to d = 3$

$c + d = e \to 2 + 3 = e \to e = 5$

$d + e = f \to 3 + 5 = f \to= 8.$

The missing terms are 1, 0, 2, 3, 5, and 8.

(b) Let's call the missing terms a, b, c, and d, then the sequence becomes:

$a, b, c, 10, 13, d, 36, 59$

$c + 10 = 13 \to c = 3$

$b + c = 10 \to b + 3 = 10 \to b = 7$

$a + b = c \to a + 7 = 3 \to= {}^{-}4$

$10 + 13 = d \to d = 23$

The missing terms are -4, 7, 3, and 23.

(c) If a Fibonacci-type sequence is a sequence in which the first two terms are arbitrary and in which every term starting from the third is the sum of the previous two terms, then we can add 0 and 2 to get the third term and continue the pattern:

$0 + 2 = 2$

$2 + 2 = 4$

$2 + 4 = 6$

$4 + 6 = 10$

$6 + 10 = 16$

$10 + 16 = 26$

The missing terms are 2, 4, 6, 10, 16, and 26.

20. Starting with 1 and 1 the Fibonacci sequence would be 1,1,2,3,5,8,13,21,34,55, … .

(a) 1 + 1 + 2 = 4 which is one less than the fifth term.

(b) 1 + 1 + 2 + 3 = 7 which is one less than the sixth term.

(c) 1 + 1 + 2 + 3 + 5 = 12 which is one less than the seventh term.

(d) The sum of the first n terms is one less than the ($\boldsymbol{n}$ + 2) term. OR $\boldsymbol{F}1 + \boldsymbol{F}2 + \boldsymbol{F}3 + \ldots + \boldsymbol{F_n} = \boldsymbol{F_n}+2-1$.

(e) The sum of the first n-2 terms of the Fibonacci sequence is one less than the nth term.

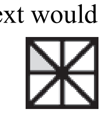
21. **(a)**

Year 1	$80+.05(80)=84$
Year 2	$84+.05(84)=88.2$
Year 3	$88.2+.05(88.2)=92.61$
Year 4	$92.61+.05(92.61)=97.2405$
Year 5	$97.2405+.05(97.2405)=102.102525$
	$\approx \$102.10.$

(b) This is a geometric sequence with $\boldsymbol{a}1$ = 80 and $\boldsymbol{r}$ = 1.05, so the price after $\boldsymbol{n}$ years is $80 \cdot 1.05^{\boldsymbol{n}}$.

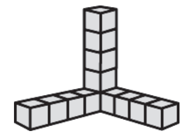
Assessment 1-2B

1. **(a)** In a clockwise direction, the shaded area moves to a new position separated from the original by one open space, then two open spaces, then by three, etc. The separation in each successive step increases by one unit; next would be:



(b) Each figure in the sequence adds one row of boxes to the base. Next would be:

(c) Each figure in the sequence adds one box to the top and each leg of the figure. Next would be:



2. **(a)** Terms that continue a pattern are 18, 22, 26, … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are 39, 52, 65, … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 13.

(c) Terms that continue a pattern are 256, 1024, 4096, … . This is a geometric sequence because each successive term is obtained from the previous term by multiplying by 4.

(d) Terms that continue a pattern are $2^{14}, 2^{18}, 2^{22}$, … . This is a geometric sequence because each successive term is obtained from the previous term by multiplying by 24.

(e) Terms that continue a pattern are $100+4\cdot 2^{50}, 100+6\cdot 2^{50}, 100+8\cdot 2^{50}, \ldots.$ This is a arithmetic sequence because each successive term is obtained from the previous term by adding by $2 \cdot 2^{50}$.

3. In these problems, a_n represents the nth term in a sequence, a_1 represents the first term, d represent the common difference between terms in an arithmetic sequence, and r represents the common ratio between terms in a geometric sequence.

In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence, $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 2$ and $d = 4$.

(*i*) $a_{100} = 2 + (100-1)\cdot 4 = 398.$

(*ii*) $a_n = 2 + (n-1)\cdot 4$
$= 2 + 4n - 4 = 4n - 2.$

(b) Arithmetic sequence: $a_1 = 0$ **and** $d = 13$.

(*i*) $a_{100} = 0 + (100 - 1) \cdot 13 = 1287$.

(*ii*) $a_n = 0 + (n - 1) \cdot 13$
$= 13n - 13$.

(c) Geometric sequence: $a_1 = 4$ and $r = 4$.

(*i*) $a_{100} = 4 \cdot 4^{99} = 4^{100}$.

(*ii*) $a_n = 4 \cdot 4^{n-1} = 4^n$.

(d) Geometric sequence: $a_1 = 2^2$ and $r = 2^4$.

(*i*) $a_{100} = 2^2 \cdot (2^4)^{99} = 2^2 \cdot 2^{396} = 2^{398}$.

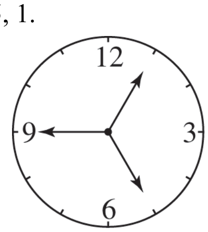
(*ii*) $a_n = 2^2 \cdot (2^4)^{(n-1)}$
$= 2^2 \cdot 2^{4n-4} = 2^{4n-2}$.

(e) Arithmetic sequence:
$a_1 = 100 + 4 \cdot 2^{50}$ and $d = 2^{51}$:

(*i*) $a_{100} = 100 + 4 \cdot 2^{50} + (100 - 1) \cdot 2^{51}$
$= 100 + 2 \cdot 2^{51} + 99 \cdot 2^{51}$
$= 100 + 101 \cdot 2^{51}$.

(*ii*) $a_n = 100 + 4 \cdot 2^{50} + (n - 1) \cdot 2^{51}$
$= 100 + 2 \cdot 2^{51} + (n - 1) \cdot 2^{51}$
$= 100 + (n + 1) \cdot 2^{51}$.

4. The hands must move 8 hours to move from 1 to 9 on the clock face. To move from 9 to 5, the hand must move 8 hours also. To move from 5 to 1, the hand must move another 8 hours. If we add 8 hours to 1 o'clock, we will land on the 9. This pattern will continue, so the next three terms are 9, 5, 1.



5. (a) Answers may vary:

(*i*) **The sum of the first** *n* **odd numbers is** $\boldsymbol{n^2}$**;** e.g., $1 + 3 + 5 + 7 = 4^2$.

(*ii*) **Square the average of the first and last terms;** e.g.,
$1 + 3 + 5 + 7 = \left(\frac{1+7}{2}\right)^2 = 4^2$.

(b) There are $\frac{35-1}{2} + 1 = 18$ terms in this sequence.

(*i*) $1 + 3 + 5 + 7 + \cdots + 35 = 18^2$
$= \mathbf{324}$.

(*ii*) $\left(\frac{1+35}{2}\right)^2 = 18^2 = \mathbf{324}$.

6. (a) Note that 5 toothpicks are added to form each succeeding hexagon. This is an arithmetic sequence $a_1 = 6$ and $d = 5$, so $a_{10} = 6 + (10 - 1) \cdot 5 = 6 + 9 \cdot 5 = \mathbf{51}$ toothpicks.

(b) *n* hexagons would require $6 + (n - 1) \cdot 5 = 6 + 5n - 5 = \mathbf{5n + 1}$ toothpicks.

7. (a) Looking at the third figure, there are $5 + 3 + 1 = 9$ triangles. The fourth figure would than have $7 + 5 + 3 + 1 = 16$ triangles. An alternative to simply adding 7, 5, 3, and 1 together is to note that $7 + 1 = 8$ and $5 + 3 = 8$. There are $\frac{4}{2} = 2$ of these sums, and $2 \cdot 8 = 16$. Then the 100th figure would have $100 + 99 = 199$ triangles in the base, $99 + 98 = 197$ triangles in the second row, and so on until the 100th row where there would be 1 triangle. $199 + 1 = 200$; $197 + 3 = 200$; etc. and so the sum of each pair is 200 and there are $\frac{100}{2} = 50$ of these pairs. $50 \cdot 200 = 10{,}000$, or 10,000 triangles in the 100th figure.

(b) The number of triangles in the ***n***th figure is $\frac{n}{2}$(number of triangles in base + 1). The number of triangles in the base is $n + (n - 1)$, or $2n - 1$. $(2n - 1) + 1 = 2n$. Then $\frac{n}{2}(2n) = n^2$, or ***n*2** triangles in the ***n***th figure.

8. This is a geometric sequence with $a_1 = \frac{15,360}{2}$ and $r = \frac{1}{2}$. The nth term of a geometric sequence is $a_n = a_1 r^{n-1}$; thus the 10th term would be $15360\left(\frac{1}{2}\right)^{10} =$ **15 liters**.

Note the progression of terms in the following table:

After Day	Amount of Water Remaining
1	$15,360 \cdot \frac{1}{2} = 7680$ liters
2	$7680 \cdot \frac{1}{2} = 3840$ liters
⋮	⋮
9	$60 \cdot \frac{1}{2} = 30$ liters
10	$30 \cdot \frac{1}{2} = 15$ liters

9. This is an arithmetic sequence with $a_1 = 8\frac{1}{6}$ (i.e., 8 a.m. plus 10 minutes, or $\frac{10}{60}$ of an hour) and $d = \frac{5}{6}$ (or $\frac{50}{60}$ of an hour). Thus

$a_8 = 8\frac{1}{6} + (8 - 1) \cdot \frac{5}{6} = 14$, or **2:00 p.m**.

(14 is 2:00 p.m. on a 24-hour clock.)

10. **(a)** If the first difference of the sequence increases by 3 for each term, then five first differences between the first six terms of the original sequence are 3, 6, 9, 12, 15. If the first term of the original sequence is 3, then the first six terms are 3, 6, 12, 21, 33, 48.

(b) If the first term is a, then $a + (a + 3) = 7 \Rightarrow a = 2$. Thus the first six terms of the original sequence are 2, 5, 11, 20, 32, 47.

(c) If the fifth term (a_5) is 34, then:

$a_6 = 34 + 15 = 49$
$a_4 = 34 - 12 = 22$
$a_3 = 22 - 9 = 13$
$a_2 = 13 - 6 = 7$
$a_1 = 7 - 3 = 4$.

Thus the sequence is 4, 7, 13, 22, 34, 49.

11. **(a)** Look for the differences:

3 8 15 24 35 48
5 7 9 11 13

The first difference row is an arithmetic sequence with fixed difference of 2. Thus the 6th term in the first difference row is $13 + 2 = 15$; the 7th term in the original sequence is $48 + 15 = 63$. Using the same reasoning, the next three terms in the original sequence are 63, 80, 99.

(b) Look for the differences:

1 7 18 37 67 111
6 11 19 30 44
5 8 11 14

The second difference row is an arithmetic sequence with fixed difference of 3. Thus the 5th term in the second difference row is $14 + 3 = 17$; the 6th term in the original sequence is $111 + 61 = 172$. Using the same reasoning, the next three terms in the original sequence are 172, 253, 357.

12. **(a)** The ***n^th^*** term for this geometric sequence is 3^{n-1}. Thus $3^{99} = 3^{n-1}$.

So $99 = n - 1$, and $n = 100$.

There are 100 terms in the sequence.

(b) The ***n^th^*** term for this arithmetic sequence is $9 + (n - 1) \cdot 4$. Thus $353 = 9 + (n - 1) \cdot 4$. Solving for ***n***, $n = 87$. There are 87 terms in the sequence.

(c) The ***n^th^*** term for this arithmetic sequence is $38 + (n - 1) \cdot 1$. Thus $238 = 38 + (n - 1) \cdot 1$. Solving for ***n***, $n = 201$. There are 201 terms in the sequence.

13. **(a)** First term: $5(1) - 1 = 4$
Second term: $5(2) - 1 = 9$
Third term: $5(3) - 1 = 14$
Fourth term: $5(4) - 1 = 19$
Fifth term: $5(5) - 1 = 24$

(b) First term: $6(1) - 2 = \mathbf{4}$
Second term: $6(2) - 2 = \mathbf{10}$
Third term: $6(3) - 2 = \mathbf{16}$
Fourth term: $6(4) - 2 = \mathbf{22}$
Fifth term: $6(5) - 2 = \mathbf{28}$

(c) First term: $5 \cdot 1 + 1 = \mathbf{6}$
Second term: $5 \cdot 2 + 1 = \mathbf{11}$
Third term: $5 \cdot 3 + 1 = \mathbf{16}$
Fourth term: $5 \cdot 4 + 1 = \mathbf{21}$
Fifth term: $5 \cdot 5 + 1 = \mathbf{26}$

(d) First term: $1^2 - 1 = \mathbf{0}$
Second term: $2^2 - 1 = \mathbf{3}$
Third term: $3^2 - 1 = \mathbf{8}$
Fourth term: $4^2 - 1 = \mathbf{15}$
Fifth term: $5^2 - 1 = \mathbf{24}$

14. Answers may vary; examples are:

(a) If $n = 6$, then $\frac{3+6}{3} = 3 \neq 6$.

(b) If $n = 4$, then $(4 - 2)^2 = 4 \neq 4^2 - 2^2 = 12$.

15. **(a)** The first figure has 2 tiles, the second has 5 tiles, the third has 8 tiles, … . This is an arithmetic sequence where the ***n****th* term is $2 + (n - 1) \cdot 3$.

Thus the 7^{th} term has $2 + (7 - 1) \cdot 3 = 20$ tiles.

(b) The *n*th term is $2 + (n - 1) \cdot 3 = 2 + 3n - 3 = \mathbf{3n - 1}$.

(c) The question can be written as: Is there an *n* such that $3n - 1 = 449$. Since $3n - 1 = 449 \Rightarrow 3n = 450 \Rightarrow n = 150$, the answer is yes, the 150th figure.

16. The n^{th} term of the arithmetic sequence is $-100 + n(300)$. The sequence can also be generated by adding 300 to the previous term. The n^{th} term of the geometric sequence is 3^{n-1}. The sequence can also be generated by multiplying the previous term by 3. Make a table

Number of the term	Arithmetic term	Geometric term
7	2000	729
8	2300	2187
9	2600	6561

With the **9th term**, the geometric sequence is greater.

17. Use a table of Fibonacci numbers to find the pattern, F_n is the nth Fibonacci number:

Generation	Male	Female	Number in Generation	Total
1	1	0	1	1
2	0	1	1	2
3	1	1	2	4
4	1	2	3	7
5	2	3	5	12
6	3	5	8	20
⋮	⋮	⋮	⋮	⋮
n	F_{n-2}	F_{n-1}	F_n	$F_{n+2}-1$

The sum of the first *n* Fibonacci numbers is $F_{n+2} - 1$. $F_{12} = 144$, so there are **143 bees** in all 10 generations.

18. **(a)** For an arithmetic sequence there is a common difference between the terms. Between 49 and 64 there are three differences so we can find the common difference by subtracting 49 from 64 and dividing the answer by three: $64 - 49 = 15 \text{ and } 15 \div 3 = 5$. The common difference is 5 and we can find the missing terms: 49 – 5 = 44 and 49 + 5 = 54 and 54 + 5 = 59.

(b) For a geometric sequence there is a common ratio between the terms. Between 1 and 625 there are four common ratios used so we can find the common ratio by dividing 625 by 1 and then taking the fourth root:

$625 \div 1 = 625$ and $625^{(\frac{1}{4})} = 5$. The common ratio is 5 and we can find the missing terms: $1 \cdot 5 = 5, 5 \cdot 5 = 25, 25 \cdot 5 = 125$.

(c) For a geometric sequence there is a common ratio between the terms. Between 310 and 319 there are three common ratios used so we can find the common ratio by dividing 319 by 310 and then taking the cube root:

$3^{19} \div 3^{10} = 3^9$ and $(3^9)^{(\frac{1}{3})} = 3^3$. The common ratio is 33 and we can find the missing terms: $3^{10} \div 3^3 = 3^7, 3^{10} \cdot 3^3 = 3^{13}, 3^{13} \cdot 3^3 = 3^{16}$.

(d) For an arithmetic sequence there is a common difference between the terms. Between ***a*** and 5***a*** there are four differences so we can find the common difference by subtracting ***a*** from 5***a*** and dividing the answer by four:
$5a - a = 4a$ and $4a \div 4 = a$. The common difference is ***a*** and we can find the missing terms: ***a*** + ***a*** = 2***a***, : 2***a*** + ***a*** = 3***a***, : 3***a*** + ***a*** = 4***a***.

19. (a) Let's call the missing terms *x* and *y,* then the sequence becomes 1, *x*, *y*, 7, 11 and if it is a Fibonacci-type sequence then:

$1 + x = y$

$x + y = 7$

$y + 7 = 11 \to y = 11 - 7 = 4$

and $x + y = 7 \to x + 4 = 7 \to x = 3$.

The missing terms are 3 and 4.

(b) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, 2, *y*, 4, *z* and if it is a Fibonacci-type sequence then:

$x + 2 = y$

$2 + y = 4 \to y = 2$

$y + 4 = z \to 2 + 4 = z \to z = 6$

and $x + 2 = 2 \to x = 0$.

The missing terms are 0, 2, and 6.

(c) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, *y*, 3, 4, *z* and if it is a Fibonacci-type sequence then:

$x + y = 3$

$y + 3 = 4 \to y = 4 - 3 = 1$

$3 + 4 = 7$

and $x + y = 3 \to x + 1 = 3 \to x = 2$.

The missing terms are 2, 1, and 7.

20. Starting with 1 and 1 the Fibonacci sequence would be 1,1,2,3,5,8,13,21,34,55, … .

(a) $F_1 + F_3 = F_4 \to 1 + 2 = 3$

(b) $F_1 + F_3 + F_5 = F_6 \to 1 + 2 + 5 = 8$.

(c) $F_1 + F_3 + F_5 + F_7 = F_8 \to 1 + 2 + 5 + 13 = 21$.

(d) $F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k}$.

Mathematical Connections 1-2: Review Problems

17. Order the teams from 1 to 10, and consider a simpler problem of counting how many games are played if each team plays each other once. The first team plays nine teams. The second team also plays nine teams, but one of these games has already been counted. The third team also plays 9 teams, but two of these games were counted in the previous two summands. Continuing in this manner, the total is 10 + 9 + 8 + … + 3 + 2 + 1 = 9(10) / 2=45 games. Double this amount to obtain 90 games must be played for each team to play each other twice.

18. 7 ways. Make a table:

Quarters	Dimes	Nickels
1	1	1
1	0	3
0	4	0
0	3	2
0	2	4
0	1	6
0	0	8

19. If the problem is interpreted to stated that at least one 12-person tent is used, then there are 10 ways. This can be seen by the table below, which illustrates the ways 2,3,5, and -6 person tents can be combined accommodate 14 people.

6-Person	5-Person	3-Person	2-Person
2	0	0	1
1	1	1	0
1	0	2	1
1	0	0	4
0	2	0	2
0	1	3	0
0	1	1	3
0	0	4	1
0	0	2	4
0	0	0	7

Chapter 1 Review

1. Make a plan. Every 7 days (every week) the day will change from Sunday to Sunday. 365 days per year $\div 7$ days per week ≈ 52 weeks per year $+\frac{1}{7}$ weeks per year. Thus the day of the week will change from Sunday to Sunday 52 times and then change from Sunday to Monday. July 4 will be a Monday.

2. $\$5.90 \div 2 = \$2.95$ more on one of the items. That is $\$20 + \$2.95 = \$22.95$ for the more expensive item and $\$20 - \$2.95 = \$17.05$ for the less expensive item. Check that both items add up to \$40: $\$22.95 + \$17.05 = \$40$.

3. The information in the rhyme is that the scholar is a "ten o'clock scholar" who "used to come at ten o'clock." The question is "what makes you come sooner." If we assume that ten o'clock means 10 AM, then the rhyme makes no sense because the scholar came later than usual. If we assume that ten o'clock means 10 PM, the question makes sense but is not answered.

4. **(a)** 15, 21, 28 Neither. The successive differences of terms increases by one; e.g., 10 + 5, 15 + 6, … .

(b) 32, 27, 22. Arithmetic Subtract 5 from each term to obtain the subsequent term.

(c) 400, 200, 100. Geometric Each term is half the previous term.

(d) 21, 34, 55. Neither Each term is the sum of the previous two terms—this is the Fibonacci sequence.

(e) 17, 20, 23. Arithmetic Add 3 to each term to obtain the subsequent term.

(f) 256, 1024, 4096. Geometric Multiply each term by 4 to obtain the subsequent term.

(g) 16, 20, 24. Arithmetic Add 4 to each term to obtain the subsequent term.

(h) 125, 216, 343. Neither Each term is the 3rd power of the counting numbers = 13, 23, 33, … .

5. **(a)** The successive differences are 3. Each term is 3 more than the previous term. This suggests that it is an arithmetic sequence of the form $3n + ?$. Since the first term is 5, $3(1) + ? = 5$. The n^{th} term would be $\mathbf{3n + 2}$.

(b) Each term given is 3 times the previous term. This suggests that the sequence is geometric. $\boldsymbol{n^{th}}$ term will be $3^{\boldsymbol{n}}$.

(c) We are reminded of the sequence 1; 8; 27; 64; … , which is given by n^3. Since the terms in the original sequence are one less than the terms given by n^3, $\mathbf{n^3 - 1}$ is a possible n^{th} term.

6. **(a)** $3(1) - 2 = \mathbf{1}$;
$3(2) - 2 = \mathbf{4}$;
$3(3) - 2 = \mathbf{7}$;
$3(4) - 2 = \mathbf{10}$; and
$3(5) - 2 = \mathbf{13}$.

(b) $1^2 + 1 = \mathbf{2}$;
$2^2 + 2 = \mathbf{6}$;
$3^2 + 3 = \mathbf{12}$;
$4^2 + 4 = \mathbf{20}$; and
$5^2 + 5 = \mathbf{30}$.

(c) $4(1) - 1 = \mathbf{3}$;
$4(2) - 1 = \mathbf{7}$;
$4(3) - 1 = \mathbf{11}$;
$4(4) - 1 = \mathbf{15}$; and
$4(5) - 1 = \mathbf{19}$.

7. **(a)** $a_1 = 2, d = 2, a_n = 200.$

So $200 = 2 + (n-1) \cdot 2 \Rightarrow n = 100.$

Sum is $\frac{100(2+200)}{2} = \mathbf{10,100}$.

(b) $a_1 = 51,\ d = 1, a_n = 151.$

So $151 = 51 + (n-1) \cdot 1 \Rightarrow n = 101.$

Sum is $\frac{101 \cdot (51+151)}{2} = \mathbf{10,201}$.

8. **(a)** $5 + 3 = 8$, which is not odd.

(b) 15 is odd; and it does not end in a 1 or a 3.

(c) The sum of any two even numbers is always even. An even number is one divisible by 2, so any even number can be represented by $2 + 2 + 2 + \cdots$. Regardless of how many twos are added, the result is always a multiple of 2, or an even number.

9. All rows, columns, and diagonals must add to 34; i.e., the sum of the digits in row 1. Complete rows or columns with one number missing, then two, etc. to work through the square:

16	3	2	13
5	10	**11**	**8**
9	**6**	7	12
4	**15**	14	**1**

10. The ten middle tables will hold two each and the two end tables will hold three each, totaling 26 people.

11. **(a)** $\square + 2^{60} = 2^{61}$

$\square = 2^{61} - 2^{60}$

$\square = 2(2^{60}) - 2^{60}$

$\square = 2^{60}(2-1)$

$\square = 2^{60}$

(b) $\square^2 = 625$

$\sqrt{\square^2} = \sqrt{625}$

$\square = 25$

12. $100 \div 5 = 20$ plus $1 = \mathbf{21}$ posts. 1 must be added because both end posts must be counted.

13. 1 mile $= 5280$ feet.

$5280 \div 6$ feet $= 880$ turns per mile.

880×50000 miles $=$ **44,000,000 turns**.

14. There are 9 students between 7 and 17 (8 through 16). There must be 9 between them in both directions, since they are direct opposites. $9 + 9 + 2 =$ **20 students**.

15. Let l be a large box, m be a medium box, and s be a small box:

$3l + (3l \times 2m \text{ each}) + [(3 \times 2)m \times 5s \text{ each}]$

$3l + 6m + 30s =$ **39 total boxes**.

16. Look for the differences:

5	15	37	77	141
10	22	40	64	
12	18	24		

The second difference row is an arithmetic sequence with $d = 6$. Thus the 4th term in the second difference row is $24 + 6 = 30$; the5th term in the first difference row is $64 + 30 = 94$; and the 6th term in the original sequence is $141 + 94 = \mathbf{235}$.

17. Extend the pattern of doubling the number of ants each day. This is a geometric sequence with $a_1 = 1500,\ a_n = 100{,}000$, and $r = 2$.

$$100{,}000 = 1500 \cdot 2^{n-1} \Rightarrow$$
$$66\tfrac{2}{3} = 2^{n-1}.$$

Since $2^{7-1} < 66\frac{2}{3}$ and $2^{8-1} > 66\frac{2}{3}$, the ant farm will fill sometime between **the 7th and 8th day**.

18. The best strategy would be one of guessing and checking:

(***i***) Ten 3's + two 5's = 40… close but too low.

(***ii***) Nine 3's + three 5's = 42… still too low.

(***iii***) Eight 3's + Four 5's = 44.

They must have answered four 5-point questions.

19. **Yes**. Let ℓ = length of the longest piece,

m = length of the middle-sized piece, and

s = lenth of the shortest piece.

Then $\ell = 3m$ and $s = m - 10$.

So $\ell + m + s = 90 \Rightarrow$

$3m + m + (m - 10) = 90 \Rightarrow$

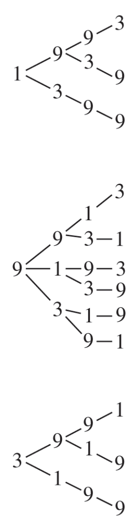
$5m = 100$.

Thus $m = \mathbf{20\,cm}$;

$\ell = 3m = \mathbf{60\,cm}$; and

$s = m - 10 = 10$ **cm**.

20. Make a diagram that demonstrates all the ways four-digit numbers can be formed from left (thousands place) to right (ones place). **12 four-digit** numbers can be formed.



21. If n is 1 larger than m (i.e., n-m = 1, or equivalently, $n = m+1$), there would be 2 terms: m, and n (since $n = m + 1$). The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 2$.

If n is 2 larger than m (i.e., n-m = 2, or equivalently, $n = m+2$) there would be 3 terms: m, $m + 1$, and n, since $n = m + 2$). The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 3$.

If n is 3 larger than m there would be 4 terms: m, $m + 1$, $m + 2$, and n, since $n = m + 3$. The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 4$.

Therefore, given that n is larger than m there would be $n - m + 1$ terms.

22. Answer may vary. Fill the 4-cup container with water and pour the water into the 7-cup container. Fill the 4-cup container again and pour water into the 7-cup container until it is full. Four minus three (1) cups of water will remain in the 4-cup container. Empty the 7-cup container and pour the contents of the 4-cup container into the 7- cup container. The 7-cup container now holds 1 cup of water. Refill the 4-cup container and pour it into the 7-cup container. The 7-cup container now contains exactly 5 cups of water.

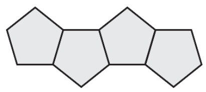
23. A possible pattern is to increase each rectangle by one row of dots and one column of dots to obtain the next term in the sequence. Make a table.

Number of the term	Row of dots	Column of dots	Term (row × column)
1	1	2	2
2	2	3	6
3	3	4	12
4	4	5	20
5	5	6	30
6	6	7	42
7	7	8	56
⋮			
100	100	101	10100
⋮			
n	n	$n + 1$	$n(n + 1)$

We also observe that the number of the term corresponds to the number of rows in the arrays and that the number of columns in the array is the number of the term plus one. Thus, the next three terms are **30, 42, and 56**. The 100th term is 10100 and the n^{th} term is $n(n + 1)$.

24. A possible pattern is that each successive figure is constructed by adjoining another pentagon to the previous figure

(a)



(b) Observe that the perimeter of the first figure is 5 and that when a new pentagon is adjoined 4 new sides are added and one side (where the new pentagon is adjoined) is lost. Make a table.

Number of terms	1-unit sides (perimeter)
1	5
2	$5 - 1 + 4 = 8$
3	$8 - 1 + 4 = 11$
4	$11 - 1 + 4 = 14$

(c) and **(d)** Looking at the terms in the sequence and noting that the difference of terms is three, we suspect that the sequence is arithmetic and conjecture that the n^{th} term is $3n+2$. However, we need to be sure. Looking at the 4th term (part a) we observe that the pentagons on the end Contribute 4 sides to the perimeter and the "middle" pentagons contribute 3 sides. Thus, in the n^{th} figure there will be 2 "end" pentagons that contribute 4 1-unit sides and $n - 2$ "middle" pentagons that contribute $3(n - 2)\ 1$ unit sides. The total will be $\mathbf{3(n - 2) + 2(4)} = 3n + 2$ units. Thus the 100th term is $3(100) + 2 = \mathbf{30}2$.

25. (a) The circled terms will constitute an arithmetic sequence because the common difference will be twice the difference in the original series.

(b) The new sequence will be a geometric sequence because the ratio will be the square of the ratio of the original series.

26. If the sequence is $a_1, a_2, a_3, a_4, \ldots$ to find the first term we substitute $n = 1$ and get $1^2 - 1$ or 0, so $a_1=0$. For $n = 2$, we get 2^2-2 or 2. Thus $a_1 + a_2=2$; hence $a_2 = 2$. For $n = 3$, $a_1 + a_2 + a_3 = 3^2 - 3 = 6$. Substituting for a_1 and a_2, we get $a_3 = 6 - 2 = 4$. For $n = 4$, $a_1 + a_2 + a_3 + a_4 = 4^2 - 4 = 12$. Substituting for a_1, a_2, and a_3, we get $0 + 2 + 4 + a_4 = 12$. Hence $a_4 = 6$.

27. (a) Let's call the missing terms a, and b then the sequence becomes:

$$13, a, b, 27$$
$$13 + a = b \to a = b - 13$$
$$a + b = 27 \to b - 13 + b = 27$$
$$\to 2b = 40$$
$$\to b = 20$$
$$a = b - 13 \to a = 7$$

So 7 and 20 are the missing terms.

(b) Let's call the missing terms a, and b then the sequence becomes:

$$137, a, b, 163$$
$$137 + a = b \to a = b - 137$$
$$a + b = 163 \to b - 137 + b = 163$$
$$\to 2b = 300$$
$$\to b = 150$$
$$a = b - 137 \to a = 13$$

So 13 and 150 are the missing terms.

(c) Let's call the missing terms x, and y, then the sequence becomes:

$$b, x, y, a$$
$$b + x = y \to x = y - b$$
$$x + y = a \to y - b + y = a$$
$$\to 2y = a + b$$
$$\to y = \frac{a+b}{2}$$
$$x = \frac{a+b}{2} - b \to x = \frac{a+b}{2} - \frac{2b}{2}$$
$$\to x = \frac{a-b}{2}$$

So the missing terms are $\frac{a-b}{2}$ and $\frac{a+b}{2}$.

CHAPTER 2

INNTRODUCTION TO LOGIC AND SETS

Assessment 2-1A: Reasoning and Logic: An Introduction

1. **(a)** **False statement**. A statement is a sentence that is either true or false, but not both.

   (b) **False statement.**

   (c) **Not a statement.**

   (d) **True statement.**

2. **(a)** **There exists at least one** natural number n such that $n + 8 = 11.$

   (b) **There exists at least one** natural number n such that $n^2 = 4.$

   (c) For **all** natural numbers $n,\ n + 3 = 3 + n.$

   (d) For **all** natural numbers $n,\ 5n + 4n = 9n.$

3. **(a)** For **all** natural numbers $n,\ n + 8 = 11.$

   (b) For **all** natural numbers $n,\ n^2 = 4.$

   (c) There is **no** natural number x such that $x + 3 = 3 + x.$

   (d) There is **no** natural number x such that $5x + 4x = 9x.$

4. **(a)** The book **does not have** 500 pages.

   (b) $3 \cdot 5 \neq 15.$

   (c) **Some** dogs **do not have** four legs.

   (d) **No** rectangles are squares.

   (e) **All** rectangles are squares.

   (f) **Some** dogs have fleas.

5. **(a)** If $n = 4,$ or $n = 5,$ then $n < 6$ *and* $n > 3,$ so the statement is **true**, since it can be shown to work for some natural numbers $n.$.

   (b) If $n = 10,$ then $n > 0;$ *or* if $n = 1,$ then $n < 5,$ so the statement is **true**.

6. **(a)**

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

   (b)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

   (c) **Yes**. The truth table entries are the same.

   (d) **No**. The truth table entries are not the same.

7. **(a)**

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	
T	T	T	F	T	
T	F	F	F	F	
F	T	T	T	T	
F	F	T	T	T	

   (b) Answers will vary. Here are two possible examples:1. Let p be "the Bobcats win" and q be "the Bobcats make the playoffs". Then Column 3 would read "If the Bobcats win, then the Bobcats make the playoffs". Column 5 would read "The Bobcats lose or the Bobcats make the playoffs."2. Let p be "it is summer vacation" and q be "I am at home". Then Column 3 would read "If it is summer vacation, then I am at home". Column 5 would read "It is not summer vacation or I am at home."

   (c) In this problem, p is the statement "2 + 3 = 5" and q is the statement "4 + 6 = 10". That would make the statement in this problem be in the form of $\sim\ p \vee q$. So, it will be logically equivalent to a statement in the form of $p \to q$ or "if 2 + 3 = 5, then 4 + 6 = 10."

8. **(a)** $q \wedge r$. Both q and r are true.

(b) $r \vee \sim q$. r is true or q is not true.

(c) $\sim(q \wedge r)$. q and r are not both true.

(d) $\sim q$. q is not true.

9. **(a)** **False**. The statement is a conjunction. The two parts could be stated as such: p is the statement $2 + 3 = 5$ and q is the statement $4 + 7 = 10$. In this situation, p is true, but q is false. In order for a conjunction to be true, both p and q must be true; otherwise, the conjunction is false.

(b) **True**. This statement is false, since Barack Obama was president in 2013.

(c) **False**. The United States Supreme Court currently has nine justices.

(d) **True**. The only triangles that have three sides of the same length are equilateral triangles. In every case, an equilateral triangle will have two sides the same length as well.

(e) **False**. Isosceles triangles have two sides equal in length, but the third side is not equal to the other two.

10. **(a)** By DeMorgan's Laws, the negation of $p \wedge q$ is $\sim p \vee \sim q$. Therefore, the answer is $2 + 3 \neq 5$ or $4 + 7 \neq 10$.

(b) The president of the United States in 2013 was Barack Obama.

(c) With every seat filled, the Supreme Court of the United States does not have 12 justices.

(d) The triangle has three sides of the same length and the triangle does not have two sides of the same length.

(e) The triangle has two sides of the same length or the triangle does not have three sides of the same length.

In both (d) and (e) above, the negation of a conditional statement $p \rightarrow q$ is $p \wedge \sim q$.

11. **(a)**

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \vee q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	T

Since the truth values for $\sim p \vee \sim q$ are not the same as for $\sim(p \vee q)$, the statements are **not logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are not the same as for $\sim p \wedge \sim q$, the statements are **not logically equivalent**.

12. Dr. No is a male spy. He is not poor, and he is not tall.

13. If p = "it is raining" and q = "the grass is wet":

(a) $\boldsymbol{p \rightarrow q}$. **(b)** $\boldsymbol{\sim p \rightarrow q}$.

(c) $\boldsymbol{p \rightarrow \sim q}$.

(d) $\boldsymbol{p \rightarrow q}$. The hypothesis is "it is raining;" the conclusion is "the grass is wet."

(e) $\boldsymbol{\sim q \rightarrow \sim p}$. **(f)** $\boldsymbol{p \leftrightarrow q}$.

14. **(a)** **Converse**: If a triangle has no two sides of the same length, then the triangle is scalene. **Inverse**: If a triangle is not scalene, then the triangle has (at least) two sides of the same length. **Contrapositive**:If a triangle does not have two sides of the same length, then the triangle is not scalene. Note that this statement is a biconditional.

(b) **Converse:** If an angle is a right angle, then it is not acute. **Inverse**: If an angle is acute, then it is not a right angle. **Contrapositive:** If an angle is not a right angle, then it is acute. Note that the original statement and the contrapositive are not true, while the converse and inverse are true.

(c) **Converse**: If Mary is not a citizen of Cuba, then she is a U.S. citizen. **Inverse**: If Mary is not a U.S. citizen, then she is a citizen of Cuba. **Contrapositive**: If Mary is a citizen of Cuba, then she is not a U.S. citizen. Note that the original statement and the contrapositive are true, while the converse and inverse are not true

(d) **Converse**: If a number is not a natural number, then it is a whole number. **Inverse**: If a number is not a whole number, then it is a natural number. **Contrapositive**: If a number is a natural number, then it is not a whole number.

15. The statements are negations of each other.

p	q	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$	$p \to q$	$\sim (p \to q)$
T	T	F	F	T	T	F
T	F	T	T	F	F	T
F	T	T	F	T	T	F
F	F	F	F	T	T	F

16. The contrapositive is logically equivalent: "If a number is not a multiple of 4 then it is not a multiple of 8."

17. **(a)** **Valid.** This is valid by the transitivity property. "All squares are quadrilaterals" is $p \to q$ "all quadrilaterals are polygons" is $q \to r$; and "all squares are polygons" is $p \to r$.

(b) **Invalid.** We do not know what will happen to students who are not freshman. There is no statement "sophomores, juniors, and seniors do not take mathematics."

18.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

19. In the set of teachers, some have IQ's of 145 or more, and some are in Mensa. Notice that those two subsets intersect; their intersection represents the group of teachers who belong in both categories. Therefore, the argument is valid: some teachers with IQ's of 145 or more are in Mensa.

20. **(a)** Since all students in Integrated Mathematics I make A's, and some of those students are in Beta Club, then **some Beta Club students make A's**.

(b) Let $p =$ I study for the final, $q =$ I pass the final, $r =$ I pass the course, $s =$ I look for a teaching job. Then $p \to q$, if I study for the final, then I will pass the final. $q \to r$, if I pass the final, then I will pass the course. $r \to s$, if I pass the course, I will look for a teaching job. So $p \to s$, **if I study for the final I will look for a teaching job**.

(c) The first statement could be rephrased as "If a triangle is equilateral, then it is isosceles." Let $p =$ equilateral triangle and $q =$ isosceles triangle. So the first statement is $p \to q$, . The second statement is simply p; then the conclusion should be q or **there exist triangles that are isosceles**.

21. **(a)** If a figure is a square, then it is a rectangle.

(b) If a number is an integer, then it is a rational number.

(c) If a polygon has exactly three sides, then it is a triangle.

22. **(a)** If $\sim p \vee \sim q \equiv \sim(p \wedge q)$, then
$\mathbf{3 \cdot 2 \neq 6}$ ***or*** $\mathbf{1 + 1 = 3}$.

(b) If $\sim p \wedge \sim q \equiv \sim(p \vee q)$, then you cannot pay me now and you cannot pay me later.

Assessment 2-1B

1. **(a)** **Not a statement**. A statement is a sentence that is either true or false.

(b) **Not a statement**. A statement must be either true of false; this could be either.

(c) True statement.

(d) **False statement**. $2 + 3 \neq 8$.

(e) Not a statement.

2. **(a)** For **all** natural numbers n, $n + 0 = n$.

(b) There exists **no** natural number n such that $n + 1 = n + 2$.

(c) **There exists at least one** natural number n such that $3 \cdot (n + 2) = 12$.

(d) **There exists at least one** natural number n such that $n^3 = 8$.

3. **(a)** There is **no** natural number n such that $n + 0 = n$.

(b) **There exists at least one** natural number n such that $n + 1 = n + 2$.

(c) For **all** natural numbers n, $3 \cdot (n + 2) \neq 12$.

(d) For **all** natural numbers n, $n^3 \neq 8$.

4. **(a)** Six is **greater than** or equal to 8. Another way to express this would be to say 6 **is not less than** 8.

(b) **All** cats **have** nine lives. Another way to express this would be to say that **no** cats **do not have** nine lives.

(c) There exists **a** square that is **not** a rectangle. Another way to express this would be to say **some** squares **are not** rectangles.

(d) **All** numbers **are** positive.

(e) **No** people **have** blond hair.

5. **(a)** If $n = 10$, then $n > 5$ and $n > 2$, so the statement is **true**.

(b) x could equal 5, so the statement is **false**.

6. **(a)** $p \vee q$ is false only if both p and q are false, so if p is true the statement is **true** regardless of the truth value of q.

(b) An implication is false only when p is true and q is false, so if p is false then the statement is **true** regardless of the truth value of q.

7. **(a)** $\boldsymbol{q \wedge r}$.

(b) $\boldsymbol{q \wedge \sim r}$.

(c) $\boldsymbol{\sim r \vee \sim q}$.

(d) $\boldsymbol{\sim(q \wedge r)}$.

8. **(a)** **True**. This statement is a disjunction. The two parts could be stated as such: p is the statement "4 + 6 = 10", while q is the statement "2 + 3 = 5". In this situation p is true, and q is true. In order for a disjunction to be true, either p or q (or both) have to be true; the only way a disjunction can be false is if both p and q are false. So therefore, this statement is true.

(b) **False**. If a team has more than 11 players on the field, it is a penalty and the play will not count.

(c) True.

(d) **False**. To see, sketch a drawing where three sides are the same length, but with the two angles where the sides intersect being different measures (in fact, make one a right angle, the other an obtuse angle). You should easily make a quadrilateral with a side length different from the other three.

(e) **True**. If a rectangle has four sides of the same length, then by default it must have three sides the same length. Of course, a rectangle with four equal sides is a square!.

9. (a) By DeMorgan's Laws, the negation of the disjunction $p \vee q$ is $\sim p \wedge \sim q$. So, the statement would be $4 + 6 \neq 10$ and $2 + 3 \neq 5$

(b) A National Football League team cannot have more than 11 players on the field while a game is in progress.

(c) The first president of the United States was not George Washington.

(d) A quadrilateral has three sides of the same length and the quadrilateral does not have four sides of the same length.

(e) A rectangle has four sides of the same length and that rectangle does not have three sides of the same length.

In both (d) and (e), the negation of the conditional statement $\boldsymbol{p \to q}$ is $\boldsymbol{p \wedge \sim q}$.

10. (a)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \vee q)$ are the same as for $\sim p \wedge \sim q$, the statements **are logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are the same as for $\sim p \vee \sim q$, the statements are **logically equivalent**.

11. Ms. Makeover is not single. She has straight blond hair.

12. (a) $\boldsymbol{p \to q}$.

(b) $\boldsymbol{\sim p \to q}$.

(c) $\boldsymbol{p \to \sim q}$.

(d) q if p, or $\boldsymbol{p \to q}$.

(e) $\boldsymbol{\sim p \to \sim q}$.

(f) $\sim q \to \sim p$.

13. (a) Converse: **If $\boldsymbol{x^2 = 9}$, then $\boldsymbol{x = 3}$**.

Inverse: **If $\boldsymbol{x \neq 3}$, then $\boldsymbol{x^2 \neq 9}$**.

Contrapositive: **If $\boldsymbol{x^2 \neq 9}$, then $\boldsymbol{x \neq 3}$**.

(b) Converse: If classes are canceled, then it snowed.

Inverse: If it does not snow, then classes are not canceled.

Contrapositive: If classes are not canceled, then it did not snow.

14. **No**. This is the inverse; i.e., if it does not rain then lris can either go to the movies or not without making her statement false.

15. (a) **Valid**. Use modus ponens: Hypatia was a woman → all women are mortal → Hypatia was mortal.

(b) **Valid**. Since *Dirty Harry* was not written by J.K. Rowling, and she wrote all the *Harry Potter* books, then *Dirty Harry* cannot be a *Harry Potter* book.

(c) **Not valid**. There exist some whole numbers that are not natural numbers. It might be easier to understand if the word seven is replaced by a variable "x". So, it reads: Some whole numbers are not natural numbers. "x" is a whole number. Conclusion: "x" is a natural number.

16. **(a)** Since all students in Integrated Mathematics I are in Kappa Mu Epsilon, and Helen is in Integrated Mathematics I, then the conclusion is that **Helen is in Kappa Mu Epsilon**.

(b) Let p = all engineers need mathematics and q = Ron needs mathematics.

Then $p \to q$, or if all engineers need mathematics then Ron needs mathematics.

p is true, but q is false, Ron does not need mathematics.

So Ron is not an engineer.

(c) Since all bicycles have tires and all tires use rubber, then the conclusion is **all bicycles use rubber.**

17. **(a)** **If** a number is a natural number, **then** it is a real number.

(b) **If** a figure is a circle, **then** it is a closed figure.

18. DeMorgan's Laws are that:

$\sim(p \wedge q)$ is the logical equivalent of $\sim p \vee \sim q$.

$\sim(p \vee q)$ is the logical equivalent of $\sim p \wedge \sim q$.

Thus:

(a) The negation is $\mathbf{3 + 5 = 9}$ **or** $\mathbf{3 \cdot 5 \neq 15}$.

(b) The negation is I am not going and she is not going.

19. Therefore, a square is a parallelogram.

Assessment 2-2A: Describing Sets

1. **(a)** Either a list or set-builder notation may be used: $\{\mathbf{a,s,e,m,n,t}\}$ or $\{\,x\,|\,x$ **is a letter in the word** ***assessment***$\}$.

(b) $\{\mathbf{21,22,23,24,\ldots}\}$ or $\{\,x\,|\,x$ **is a natural number and** $\boldsymbol{x > 20}\}$ or $\{x\,|\,x \in N$ **and** $\boldsymbol{x > 20}\}$.

2. **(a)** $\boldsymbol{P} = \{p,q,r,s\}$.

(b) $\{\mathbf{1,2}\} \subset \{\mathbf{1,2,3}\}$. The symbol $\subset$ refers to a proper subset.

(c) $\{\mathbf{0,1}\} \not\subseteq \{\mathbf{1,2,3}\}$. The symbol $\subseteq$ refers to a subset.

3. **(a)** **Yes**. $\{1,2,3,4,5\} \sim \{m,n,o,p,q\}$ because both sets have the same number of elements and thus exhibit a one-to-one correspondence.

(b) **Yes**. $\{a,b,c,d,e,f,\ldots,m\} \sim \{1,2,3,\ldots,13\}$ because both sets have the same number of elements.

(c) **No**. $\{\boldsymbol{x}\,|\,\boldsymbol{x}$ is a letter in the word *mathematics*$\}$ $\not\sim \{1,2,3,4,\ldots,11\}$; there are only eight unduplicated letters in the word *mathematics*.

4. **(a)** The first element of the first set can be paired with any of the six in the second set, leaving five possible pairings for the second element, four for the third, three for the fourth, two for the fifth, and one for the sixth. Thus there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{720}$ one-to-one correspondences.

(b) There are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$ possible one-to-one correspondences. The first element of the first set can be paired with any of the n elements of the second set; for each of those n ways to make the first pairing, there are $n - 1$ ways the second element of the first set can be paired with any element of the second set; which means there are $n - 2$ ways the third element of the first set can be paired with any element of the third set; and so on. The Fundamental Counting Principle states that the choices can be multiplied to find the total number of correspondences.

5. **(a)** If x must correspond to 5, then y may correspond to any of the four remaining elements of $\{1,2,3,4,5\}$, z may correspond to any of the three remaining, etc. Then $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ **one-to-one correspondences**.

(b) There would be $1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = \mathbf{6}$ **one-to-one correspondences**.

(c) The set $\{x, y, z\}$ could correspond to the set $\{1,3,5\}$ in $3 \cdot 2 \cdot 1 = 6$ ways. The set $\{u, v\}$ could correspond with the set $\{2,4\}$ in $2 \cdot 1 = 2$ ways. There would then be $6 \cdot 2 = \mathbf{12}$ **one-to-one correspondences**.

6. **(*i*)** $\boldsymbol{A = C}$. The order of the elements does not matter.

(*ii*) $\boldsymbol{E = H}$; they are both the null set.

(*iii*) $\boldsymbol{I = L}$. Both represent the numbers $1,3,5,7,\ldots$.

7. **(a)** Assume an arithmetic sequence with $a_1 = 201, a_n = 1100,$ and $d = 1$. Thus $1100 = 201 + (n-1) \cdot 1$; solving, $n = 900$. The cardinal number of the set is therefore **900**.

(b) Assume an arithmetic sequence with $a_1 = 1, a_n = 101,$ and $d = 2$. Thus $101 = 1 + (n-1) \cdot 2$; solving, $n = 51$. The cardinal number of the set is therefore **51**.

(c) Assume a geometric sequence with $a_1 = 1, a_n = 1024,$ and $r = 2$. Thus $1024 = 1 \cdot 2^{n-1} \Rightarrow 2^{10} = 2^{n-1} \Rightarrow n - 1 = 10 \Rightarrow n = 11$. The cardinal number of the set is therefore **11**.

(d) If $k = 1,2,3,\ldots,100$, the cardinal number of the set $\{x | x = k^3, k = 1,2,3,\ldots,100\} = \mathbf{100}$, since there are 100 elements in the set.

8. $\overline{A}$ represents all elements in U that are not in A, or the set of all college students with at least one grade that is not an A.

9. **(a)** A proper subset must have at least one less element than the set, so the maximum $\boldsymbol{n(B) = 7}$.

(b) Since $B \subset C$, and $n(B) = 8$ then C could have any number of elements in it, so long as it was greater than eight.

10. **(a)** The sets arc equal, so $\boldsymbol{n(D) = 5}$.

(b) Answers vary.For example, the sets are equal; the sets are also equivalent

11. **(a)** A has 5 elements, thus $2^5 = \mathbf{32}$ **subsets**.

(b) Since A is a subset of A and A is the only subset of A that is not proper, A has $2^5 - 1 =$ **31 proper subsets**.

(c) Let $B = \{b,c,d\}$. Since $B \subset A$, the subsets of B are all of the subsets of A that do not contain a and e. There are $2^3 = 8$ of these subsets. If we join (union) a and e to each of these subsets there are still **8 subsets**.

Alternative. Start with $\{a,e\}$. For each element $b, c,$ and d there are two options: include the element or don't include the element. So there are $2 \cdot 2 \cdot 2 = 8$ ways to create subsets of A that include a and e.

12. If there are n elements in a set, 2^n subsets can be formed. This includes the set itself. So if there are 127 proper subsets, then there are 128 subsets. Since $2^7 = 128$, the set has **7 elements**.

13. In roster format, $A = \{3,6,9,12,\ldots\}, B = \{6,12,18,24,\ldots\}$, and $C = \{12,24,36,\ldots\}$. Thus, $C \subset A, C \subset B$, and $B \subset A$.

Alternatively: $12n = 6(2n) = 3(4n)$. Since $2n$ and $4n$ are natural number $\boldsymbol{C \subset A, C \subset B}$, and $\boldsymbol{B \subset A}$.

14. **(a)** $\notin$. There are no elements in the empty set.

(b) $\in$. $1024 = 2^{10}$ and $10 \in N$.

(c) $\in$. $3(1001) - 1 = 3002$ and $1001 \in N$.

(d) $\notin$. For example, $x = 3$ is not an element because for $3 = 2^n$, $n \notin N$.

15. **(a)** $\not\subseteq$. 0 is not a set so cannot be a subset of the empty set, which has only one subset, $\varnothing$.

(b) $\not\subseteq$. 1024 is an element, not a subset.

(c) $\not\subseteq$. 3002 is an element, not a subset.

(d) $\not\subseteq$. x is an element, not a subset.

16. **(a)** **Yes**. Any set is a subset of itself, so if $A = B$ then $A \subseteq B$.

(b) **No**. A could equal B; then A would be a subset but not a proper subset of B.

(c) **Yes**. Any proper subset is also a subset.

(d) **No**. Consider $A = \{1, 2\}$ and $B = \{1,2,3\}$.

17. **(a)** Let $A = \{1,2,3,\ldots,100\}$ and $B = \{1,2,3\}$. Then $n(A) = 100$ and $n(B) = 3$. Since $B \subset A$, $\boldsymbol{n(B) = 3 < 100 = n(A)}$.

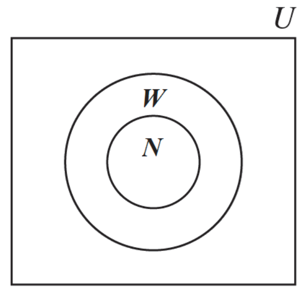
(b) $n(\emptyset) = 0$. Let $A = \{1,2,3\} \Rightarrow n(A) = 3$. $\emptyset \subset A$, which implies that there is at least one more element in A than in $\emptyset$. Thus $\mathbf{0 < 3}$.

18. There are seven senators to choose from and 3 will be chosen. Consider the ways to form subsets with only three members. If we pick the first member, there are 7 senators to choose from. To pick the second member, there are only 6 to choose from, since 1 member has already been chosen. For the third seat, there are 5 to choose from. This yields $7 \cdot 6 \cdot 5$. However, this calculation counts {Able, Brooke, Cox} as a different committee that {Brooke, Able, Cox}. In fact, for any 3 names, there are $3 \cdot 2 \cdot 1$ ways to arrange the names. Thus, the number of unique committees is $7 \cdot 6 \cdot 5 / 3 \cdot 2 \cdot 1 = 7 \cdot 5 = \mathbf{35}$.

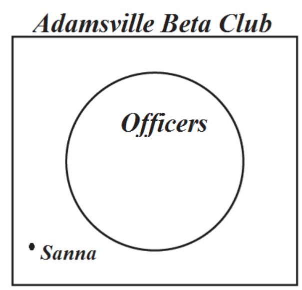
19. Answers vary. For example, the set of all odd natural numbers and the set of all even natural numbers are two infinite sets that are equivalent but not equal.. Another possibility is the set of all natural numbers and the set of all whole numbers.

20. Each even natural number $2n$ can be paired with each odd natural number $2n - 1$ in a one-to-one correspondence.

21.



22.



23. Answers vary. Example: All members of the Adamsville Beta Club are officers.

Assessment 2-2B

1. **(a)** Either a list or set-builder notation may be used: $\{\boldsymbol{a,l,g,e,b,r}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x}$ **is a letter in the word *algebra*}**

(b) $\{\mathbf{1,2,3,4,5,6,7,8,9}\}$ or $\{\boldsymbol{x} \mid \boldsymbol{x}$ **is a natural number and $\boldsymbol{x<10}$}** or $\{\boldsymbol{x} \mid \boldsymbol{x \in N}$ **and $\boldsymbol{x<10}$}.**

2. **(a)** $\boldsymbol{Q = \{q, r, s\}}$

(b) $\mathbf{\{1,3\} = \{3,1\}}$. The symbol $=$ refers to the sets being equal (containing the same elements).

(c) $\mathbf{\{1,3\} \not\subset \{1,4,6\}}$. The symbol $\not\subset$ refers to "not a proper subset."

3. **(a)** **Yes.** $\{1,2,3,4\} \sim \{w,c,y,z\}$ because both sets have the same number of elements and thus exhibit a one-to-one correspondence.

(b) **Yes**, because both sets have the same number of elements.

(c) **No**. $\{x|x$ is a letter in the word *geometry*$\}$ $\nsim \{1,2,3,4,...,8\}$; there are only seven unduplicated letters in geometry.

4. **(a)** The first element of the first set can be paired with any of the eight in the second set, leaving seven possible pairings for the second element, six for the third, five for the fourth, etc. Thus, there are $8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 =$ **40,320** one-to-one correspondences.

(b) There are $\boldsymbol{(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot3\cdot2\cdot1}$ possible one-to-one correspondences. The first element of the first set can be paired with any of the $n-1$ elements of the second set; there are $n-2$ ways the second element of the first set can be paired with any element of the second set; which means there are $n-3$ ways the third element of the first set can be paired with any element of the third set; and so on. The Fundamental Counting Principle states that the choices can be multiplied to find the total number of correspondences

5. **(a)** If b must correspond to 3, then a may correspond to any of the three remaining elements of $\{1,2,3,4\}$, and c may correspond to any of the two remaining, etc. Then $1\cdot3\cdot2\cdot1 =$ **6 one-to-one correspondences.**

(b) **T**here would be a $1\cdot1\cdot2\cdot1 =$ **2 one-to-one correspondences.**

(c) The set $\{a,c\}$ could correspond to the set $\{2,4\}$ in $2\cdot1=2$ ways. The set $\{b,d\}$ could correspond with the set $\{1,3\}$ in $2\cdot1=2$ ways. There would then be $2\cdot2 =$ **4 one-to-one correspondences.**

6. $\boldsymbol{A = C}$.

7. **(a)** Assume an arithmetic sequence with $a_1 = 19,\ a_n = 99,$ and $d = 1.$ Thus $99 = 19 + (n-1)\cdot1;$ solving, $n = 81.$ The cardinal number of the set is therefore **81**.

(b) Assume an arithmetic sequence with $a_1 = 2,\ a_n = 1002,$ and $d = 2.$ Thus $1002 = 2 + (n-1)\cdot2;$ solving, $n = 501.$ The cardinal number of the set is therefore **501**.

(c) Assume an arithmetic sequence with $a_1 = 1,\ a_n = 99,$ and $d = 2.$ Thus $99 = 1 + (n-1)\cdot2;$ solving, $n = 50.$ The cardinal number of the set is therefore **50**.

(d) There are no natural numbers that have the property $x = x + 1,$ so the set is empty and the cardinal number is **0**.

8. $\overline{G}$ represents all elements in U that are not in G, or the set of all women who are not alumni of Georgia State University. In set-builder notation, $\overline{G} = \{x|x$ is a woman who is not a graduate of Georgia State University$\}$.

9. **(a)** Since the empty set is a subset of any set, A could be the empty set; then the minimum number of elements in A would be **0**.

(b) **Yes**. Since A is not assumed to be a proper subset, A and B could be equal . Thus, both sets could be empty.

10. To be subsets of each other, the two sets must be **equal** and **equivalent**.

11. **(a)** Since $A = \{1,2,3,4,5,6,7,8,9\}$ has 9 elements, A has $2^9 = \mathbf{512}$ **subsets**.

(b) Since A is a subset of A and not proper, A has $2^9 - 1 = \mathbf{511}$ **subsets**.

12. If there are n elements in a set, 2^n subsets can be formed. Thus, $16 = 2^n$ and $n = \mathbf{4}$ **elements**.

13. In roster format, $A = \{4,7,10,13,16,...\}$, $B = \{7,13,18,...\}$, and $C = \{13,25,38,...\}$. Thus, $C \subset A,\ C \subset B,$ and $B \subset A.$

Alternative: $12n + 1 = 6(2n) + 1 = 3(4n) + 1.$ Since $2n$ and $4n$ are natural numbers $\boldsymbol{C \subset A}$, $\boldsymbol{C \subset B}$, and $\boldsymbol{B \subset A}$.

14. **(a)** $\in$. The set containing the empty set has one element; the empty set.

(b) $\in$. $1022 = 2^{10} - 2$ and $10 \in N$.

(c) $\in$. $3(1001) + 1 = 3004$ and $1001 \in N$.

(d) $\in$. 17 is an element of the natural numbers.

15. **(a)** **No**. For example, if $A = \{1\}$ and $B = \{1,2\}$, then $A \subseteq B$ but $A \neq B$.

(b) **No**. $A \subset B$ implies that A must have at least one less element than B.

(c) **No**. A and B must have the same number of elements, but not necessarily be equal; for example, if $A = \{1,2\}$ and $B = \{a,b\}$.

(d) **No**. See part (c).

16. **(a)** $n(\emptyset) = 0$. Let $A = \{1,2\} \Rightarrow n(A) = 2$. $\emptyset \subset A$, which implies that there is at least one more element in A than in $\emptyset$. Thus $\mathbf{0 < 2}$.

(b) Let $A = \{1,2,3,\ldots 99\}$ and $B = \{1,2,3,\ldots, 100\}$. $n(A) = 99$ and $n(B) = 100$, but $A \subset B$ so $\boldsymbol{n(A) = 99 < 100 = n(B)}$.

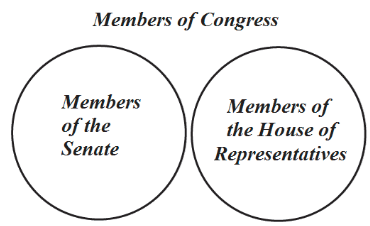
17. **(a)** There are 4 flavors from which to pick the first scoop, leaving 3 from which to pick the second, 2 from which to pick the third, and only 1 from which to pick the last. Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ **ways to pick the four flavors**.

(b) Each of the four scoops may be picked in four different ways, thus there are $4^4 = \mathbf{256}$ **ways to pick the four flavors**.

18. 6. Note that $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. The Fundamental Counting Principle states that the product of choices gives the number of ways that the one-to-one correspondence can happen; since there are 720 one-to-one correspondences, then there must be six elements in each set.

19. Let every natural number n be paired with the number $n - 1$ from the set of whole numbers. The pairing is a one-to-one correspondence between the set of natural numbers and the set of whole numbers.

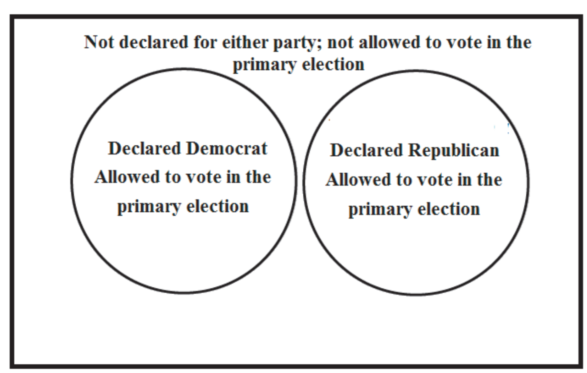
20. The Congress of the United States consists of all members of the Senate, and all members of the House of Representatives; however, no member of the Senate is also a member of the House of Representatives, and vice-versa. Therefore, a Venn diagram consisting of two disjoint circles will work for this problem, as shown below:



21. Since both sets are equal, that means both sets contain all of the same elements, by defintion of equal sets. So, you can infer that every swimmer in the 100 meter butterfly race is from the Maryville Swim Team.

22. For voting in a primary election, some states make voters declare a party affiliation (so that you vote in that parties' primary only). No state would allow a voter to declare as both a Republican and a Democrat, so the Venn diagram would consist of two disjoint circles, as shown below:

Voting in Primary Elections



23. Answers will vary. Some possibilities: Since set B has one more element in it than set A, set A could be a subset of set B. However, the number of elements doesn't necessarily imply that the set with the smaller number of elements is a subset of the set with the larger number of elements, so set A might not be a subset of set B. Similar reasoning could be used to state that set A could be a proper subset of set B, or not. Assuming the sets are finite, the sets are not equivalent nor equal.

24. **(a)** Answers will vary. For example, let the universal set be the whole numbers. Now, let set A consist of all whole numbers greater than ten. That set is infinite. Its complement, $\overline{A}$, would be the natural numbers less than or equal to ten. That set is finite.

(b) Answers will vary. For example, let the universal set be the set of whole numbers. Let A be the set of odd whole numbers. This set is infinite. Then $\overline{A}$ would be the set of even whole numbers, also an infinite set.

Mathematical Connections 2-2: Review Problems

16. **(a)** False. In order for a conjunction to be true, both parts of the conjunction must be true. Since p is false, $p \wedge q$ is false.

(b) False. Since q is true, $\sim q$ is false.

(c) True. Since p is false, $\sim p$ is true. This makes both parts of the conjunction true, so therefore the conjunction $\sim p \wedge q$ is true.

(d) True. In part (a) above, we found out that $p \wedge q$ was false. Therefore, $\sim (p \wedge q)$ must be true.

(e) False. In order for a conjunction to be true, both parts of the conjunction must be true. Since q is true, $\sim q$ is false; so $\sim q \wedge \sim p$ is false.

17. **(a)** False. In order for a conjunction to be true, both parts of the conjunction must be true. Since q is false, $p \wedge q$ is false.

(b) True. Since q is false, $\sim q$ is true.

(c) False. In order for a conjunction to be true, both parts of the conjunction must be true. In this conjunction, neither side is true; so, $\sim p \wedge q$ is false.

(d) True. In part (a) above, we found out that $p \wedge q$ was false. Therefore, $\sim (p \wedge q)$ must be true.

(e) False. In order for a conjunction to be true, both parts of the conjunction must be true. Since p is true, $\sim p$ is false; so $\sim q \wedge \sim p$ is false.

18.

p	q	$\sim q$	$p \vee \sim q$
T	T	F	**T**
T	F	T	**T**
F	T	F	**F**
F	F	T	**T**

Assessment 2-3A: Other Set Operations and Their Properties

1. $A = \{1, 3, 5, \ldots\}; B = \{2, 4, 6, \ldots\};$ $C = \{1, 3, 5, \ldots\}$

(a) $\boldsymbol{A}$ or $\boldsymbol{C}$. Every element in C is either in A or C.

(b) $\boldsymbol{N}$. Every natural number is in either A or B.

(c) $\emptyset$. There are no natural numbers in both A and B.

2. For example, let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{1, 5, 6\}, B = \{1, 4, 5, 6, 7\}$, and $C = \{1, 2, 3, 4\}$

(a) **Yes**. $A - A$ means the set of all elements that are in A that are also not in A; or more formally, $A - A = \{x | x \in A \text{ and } x \notin A\} = \varnothing$.

(b) **Yes**. $B - A = \{4, 7\}$; $\overline{A} = \{2, 3, 4, 7\}$ which means $B \cap \overline{A} = \{4, 7\}$ so the sets are equal.

(c) **No**. $B - A = \{4,7\}$;
$B \cap A = \{1,5,6\}$
so the sets are not equal.

(d) **Yes**. $A \cup A = \{1,5,6\}$; $A \cup \emptyset = \{1,5,6\}$, so the sets are equal. In general, combining a set to itself via set union will not change the elements in the set; combining a set via set union to the empty set will not change the elements in the set.

3. **(a)** **True**. Let $A = \{1,2\}$. $A \cup \varnothing = \{1,2\}$. In general, combining any set with the empty set via set union contributes no extra elements to the set than what is already in the set.

(b) **False**. Let $A = \{1,2\}$ and $B = \{2,3\}$.
$A - B = \{1\}; B - A = \{3\}$.

(c) **False**. Let $U = \{1,2\}$; $A = \{1\}$; $B = \{2\}$.
$\overline{A \cap B} = \{1,2\}; \overline{A} \cap \overline{B} = \varnothing$.

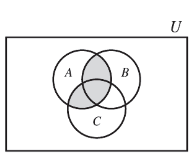
(d) **False**. Let $A = \{1,2\}$; $B = \{2,3,4\}$.
$(A \cup B) - A = \{1,2,3,4\} - \{1,2\} = \{3,4\} \neq B$.

(e) **False**. Let $A = \{1,2\}$; $B = \{2,3\}$.
$(A - B) \cup A = \{1\} \cup \{1,2\} = \{1,2\}$;
$(A - B) \cup (B - A) = \{1\} \cup \{3\} = \{1,3\}$.

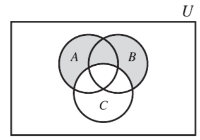
4. **(a)** If $B \subseteq A$, all elements of B must also be elements of A, but there may be elements of A that are not elements of B, so $\boldsymbol{A \cap B = B}$.

(b) If $B \subseteq A$, all elements of B must also be elements of A, but there may be elements of A that are not elements of B, so $\boldsymbol{A \cup B = A}$.

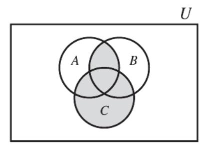
5. **(a)** $(A \cap B) \cup (A \cap C)$



(b) $(A \cup B) \cap \overline{C}$



(c) $(A \cap B) \cup C$



6. **(a)** $S \cup \overline{S} = \{x | x \in S \text{ or } x \in \overline{S}\} = \boldsymbol{U}$.

(b) If U is the universe the complement of U can have no elements, thus $\boldsymbol{\overline{U} = \emptyset}$.

(c) There are no elements common to S and $\overline{S}$, so $\boldsymbol{S \cap \overline{S} = \varnothing}$.

(d) Since there are no elements in the empty set there are none common to it and S, so $\boldsymbol{\varnothing \cap S = \varnothing}$.

7. **(a)** If $A \cap B = \varnothing$ then A and B are disjoint sets and any element in A is not in B, so $A - B = \{x | x \in A \text{ and } x \notin B\} = \boldsymbol{A}$.

(b) Since B is the empty set, there are no elements to remove from A, so $A - B = \boldsymbol{A}$.

(c) If $B = U$ there are no elements in A which are not in B, so $A - B = \varnothing$.

8. **Yes**. By definition, $A - B$ is the set of all elements in A that are not in B. If $A - B = \varnothing$ this means there are no elements in A that are not in B, thus $A \subseteq B$.

More formally, suppose $A \not\subseteq B$. Then there must be an element in A that is not in B, which implies that it is in $A - B$. This implies that $A - B$ is not empty, which is a contradiction. Thus $A \subseteq B$.

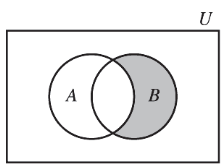
9. Answers may vary.

(a) $\boldsymbol{B \cap \overline{A}}$ or $\boldsymbol{B - A}$; i.e., $\{x | x \in B$ but $x \notin A\}$.

(b) $\overline{\boldsymbol{A \cup B}}$ or $\boldsymbol{\overline{A} \cap \overline{B}}$; i.e., $\{x | x \notin A$ or $B\}$.

(c) $(\boldsymbol{A \cap B}) \cap \overline{\boldsymbol{C}}$ or $(\boldsymbol{A \cap B}) - \boldsymbol{C}$; i.e., $\{x | x \in A$ and B but $x \notin C\}$.

10. **(a)** $\overline{A}$ is the set of all elements in U that are not in A. $\overline{A} \cap B$ is the set of all elements common to $\overline{A}$ and B:

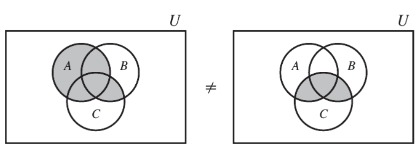


$\overline{A} \cap B$

(b) Answers vary. $\overline{A} \cap B = B - (A \cap B)$. or $\overline{A} \cap B = B - A$. are two common responses.

11. **(a)** **False**:

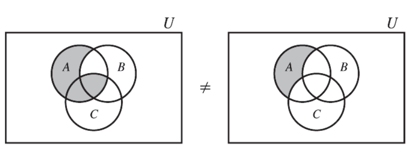
$A \cup (B \cap C) \quad \neq \quad (A \cup B) \cap C$



$A \cup (B \cap C)$ $\qquad$ $(A \cup B) \cap C$

(b) **False**:

$A - (B - C) \quad \neq \quad (A - B) - C$



$A - (B - C)$ $\qquad$ $(A - B) - C$

12. **(a)** Yes. Answers will vary. Note that in general, $\boldsymbol{A \cap B \subseteq A \cup B}$ because all elements of $A \cap B$ are included in $A \cup B$. Example: Let $A = \{1,2,3,4,5\}$ and $B = \{4,5,6,7,8\}$. Then $A \cap B = \{4,5\}$ and $A \cup B = \{1,2,3,4,5,6,7,8\}$.

(b) Yes. Answers will vary. Example: Let $A = \{1,2,3,4,5\}$ and $B = \{4,5,6,7,8\}$, where the universal set is $\{1,2,3,4,5,6,7,8\}$ Then $A - B = \{1,2,3\}$ which means $\overline{A - B} = \{4,5,6,7,8\}$, which does contain two elements of A, namely 4 and 5.

13. **(a)** **(*i*)** Greatest $n(A \cup B) = n(A) + n(B) = \mathbf{5}$ if A and B are disjoint.

(*ii*) Greatest $n(A \cap B) = n(B) = \mathbf{2}$ if $B \subseteq A$.

(*iii*) Greatest $n(B - A) = n(B) = \mathbf{2}$ if A and B are disjoint.

(*iv*) Greatest $n(A - B) = n(A) = \mathbf{3}$ if A and B are disjoint.

(b) **(*i*)** Greatest $n(A \cup B) = \boldsymbol{n + m}$ if A and B are disjoint.

(*ii*) Greatest $n(A \cap B) = \boldsymbol{m}$, if $B \subseteq A$, or $\boldsymbol{n}$, if $A \subseteq B$.

(*iii*) Greatest $n(B - A) = \boldsymbol{m}$ if A and B are disjoint.

(*iv*) Greatest $n(A - B) = \boldsymbol{n}$ if A and B are disjoint.

14. **(a)** **(*i*)** Greatest $n(A \cup B \cup C) = n(A) + n(B) + n(C) = 4 + 5 + 6 = \mathbf{15}$, if A, B, and C are disjoint.

(*ii*) Least $n(A \cup B \cup C) = n(C) = \mathbf{6}$, if $A \subset B \subset C$.

(b) **(*i*)** Greatest $n(A \cap B \cap C) = n(A) = \mathbf{4}$, if $A \subset B \subset C$.

(*ii*) Least $n(A \cap B \cap C) = \mathbf{0}$, if A, B, and C are disjoint.

15. **(a)** 319.2 million non-Muslims. Subtract the Muslim population (2.6 million) away from the total population (321.8 million) of the United States.

(b) 148 million. Subtract the Muslim population (2.6 million) away from the number of people identified as being of some faith (150.6 million).

16. Constructing a Venn diagram will help in visualization:

(a) $B \cap S$ is the set of college **basketball players more than 200 cm tall**.

(b) $\overline{S}$ is the set of humans who are **not college students** or who are **college students less than or equal to 200 cm tall**.

(c) $B \cup S$ is the set of humans who are **college basketball players** or who are **college students taller than 200 cm**.

(d) $\overline{B \cup S}$ is the set of all humans who are **not college basketball players and who are not college students taller than 200 cm**.

(e) $\overline{B} \cap S$ is the set of all **college students taller than 200 cm who are not basketball players**.

(f) $B \cap \overline{S}$ is the set of all **college basketball players less than or equal to 200 cm tall**.

17. Use a three-set Venn diagram, labeling the sets B (for basketball), V (volleyball), and S (soccer):

(i) Enter 2 in the region representing $B \cap V \cap S$ (i.e., there were two who played all three sports);

(ii) Enter 1 in the region representing $(B \cap V) - S$ (i.e., there was one who played basketball and volleyball but not soccer);

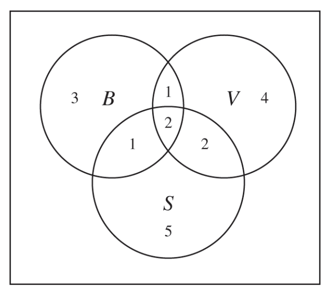
(iii) Enter 1 in the region representing $(B \cap S) - V$ (i.e., there was one who played basketball and soccer but not volleyball);

(iv) Enter 2 in the region representing $(V \cap S) - B$ (i.e., there were two who played volleyball and soccer but not basketball);

(v) Enter $7 - (1 + 1 + 2) = 3$ in the region representing $B - (V \cup S)$ (i.e., of the seven who played basketball, one also played volleyball, one also played soccer, and two also played both volleyball and soccer—leaving three who played basketball only);

(vi) Enter $9 - (1 + 2 + 2) = 4$ in the region representing $V - (B \cup S)$ (i.e., of the nine who played volleyball, one also played basketball, two also played soccer, and two also played both basketball and soccer—leaving four who played volleyball only);

(vii) Enter $10 - (1 + 2 + 2) = 5$ in the region representing $S - (B \cup V)$ (i.e., of the ten who played soccer, one also played basketball, two also played volleyball, and two also played both basketball and volleyball—leaving five who played soccer only).



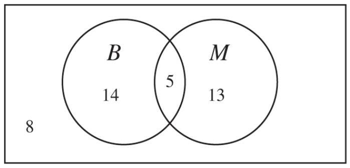
There are then $3 + 4 + 5 + 1 + 1 + 2 + 2 = \mathbf{18}$ who played one or more sports.

18. In the Venn diagram below:

(i) There were 5 members who took both biology and mathematics;

(ii) Of the 18 who took mathematics 5 also took biology, leaving 13 who took mathematics only;

(*iii*) 8 took neither course, so of the total of 40 members there were $40 - (5 + 13 + 8) = \mathbf{14}$ who took biology but not mathematics.



19. **(a)** If all bikes needing new tires also need gear repairs, i.e., if $\{TIRES\} \subset \{GEARS\}$, then $n[\{TIRES\} \cap \{GEARS\}] = \mathbf{20\ bikes}$.

(b) Adding the separate repairs gives $20 + 30 = 50$ bikes, which is the total number of bikes; so it is possible that **zero bikes** needed both repairs, if every bike there needed exactly one repair.

(c) If the maximum number of bikes needed both repairs then all 20 receiving tires would also receive gear work. That would leave 10 additional bikes needing gear work only, leaving **20 bikes** that needed no service.

20. Generate the following Venn diagram in this order:

(*i*) The 4 who had A, B, and Rh antigens;

(*ii*) The $5 - 4 = 1$ who had A and B antigens, but who were Rh negative;

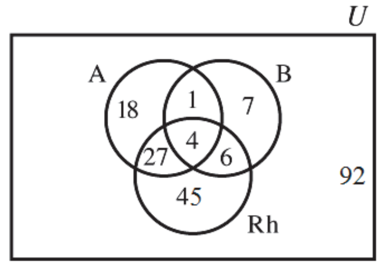
(*iii*) The $31 - 4 = 27$ who had A antigens and were Rh positive;

(*iv*) The $10 - 4 = 6$ who had B antigens and were Rh positive;

(*v*) The $50 - 27 - 4 - 1 = 18$ who had A antigens only;

(*vi*) The $18 - 6 - 4 - 1 = 7$ who had B antigens only, and;

(*vii*) The $82 - 27 - 4 - 6 = 45$ who were O positive.



$n(A \cup B \cup \text{Rh}) = 18 + 1 + 7 + 27 + 4 + 6 + 45 = 108$. Thus the set of people who are O-negative is $200 - 108 = \mathbf{92}$.

21. Let M be the set of students taking mathematics, C be the set of students taking chemistry, and P be the set of students taking physics. Generate the following Venn diagram in this order:

(*i*) The 8 who had all three subjects;

(*ii*) The $10 - 8 = 2$ students who took mathematics and physics, but not chemistry;

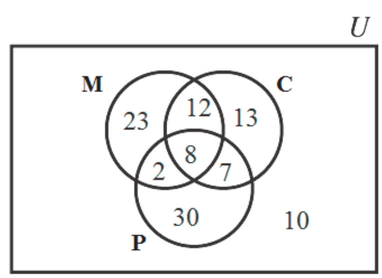
(*iii*) The $15 - 8 = 7$ students who took chemistry and physics, but not mathematics;

(*iv*) The $20 - 8 = 12$ students who took mathematics and chemistry, but not physics;

(*v*) The $45 - 12 - 8 - 2 = 23$ students who took mathematics only;

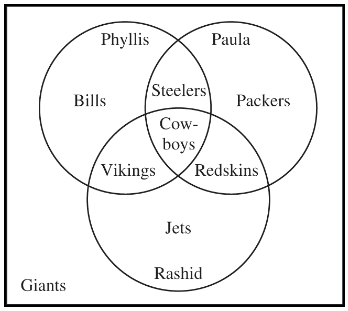
(*vi*) The $40 - 12 - 8 - 7 = 13$ students who took chemistry only;

(*vii*) The $47 - 2 - 8 - 7 = 30$ students who took physics only



Combined with the 10 students who didn't take any of the three courses, this Venn diagram yields a total number of students as 105, which contradicts John's reported total of 100. So, given that there must be errors somewhere in John's data gathering, John probably shouldn't be hired for the job.

22. The following Venn diagram helps in isolating the choices:



All picked the Cowboys to win their game, so their opponent cannot be among any of the other choices; the only team not picked was the Giants.

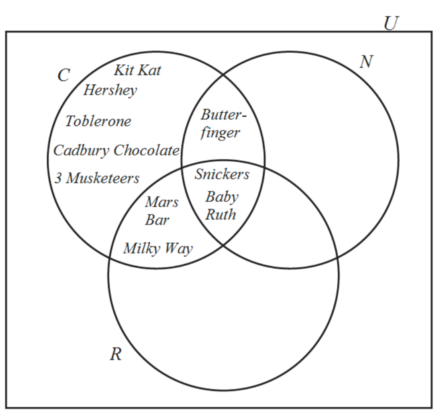
Phyllis and Paula both picked the Steelers, so their opponent cannot be among their other choices. This leaves the Jets.

Phyllis and Rashid both picked the Vikings which leaves the Packers as the only possible opponent.

Paula and Rashid both picked the Redskins which leaves the Bills as the only possible opponent.

Thus we have **Cowboys vs Giants**, **Vikings vs Packers**, **Redskins vs Bills**, **and Jets vs Steelers**.

23. **(a)**



(b) ***(i)*** **none**. Since all candy bars have chocolate, there aren't any in the set of bars without chocolate.

(b) ***(ii)*** **none**. Same reason as (i).

(iii) **Chocolate** is the most popular ingredient; **nuts/peanut butter** is the least popular ingredient.

(iv) Answers will vary.

24. **57.** Of the 324 first class passengers, you know there were $146 + 4 = 150$ total women and children, and 117 men who were lost. To find the number of men who survived, find $324 - 150 - 117 = 57$

25. **Abby, Harry, and Dick are in one family; Tom, Jane, and Mary are in the other family.** First, note that Abby and Harry have the same characteristics, blue eyes and blond hair. Also note that Mary is their opposite, with brown eyes and brown hair; therefore, Mary is certainly in a different family than Abby and Harry. Now note that Jane and Tom both have blue eyes and brown hair, while Dick has brown eyes and blond hair. Dick cannot be in the same family as Jane and Tom. So, since each family has 3 children, Dick is in the family with Abby and Harry (where they all have blond hair), while Tom, Jane, and Mary are in the other family (where they all have brown hair).

26. If a Cartesian product is the set of all ordered pairs such that the first element of each pair is an element of the first set and the second element of each pair is an element of the second set:

(a) $A \times B = \{(\boldsymbol{x},\boldsymbol{a}),(\boldsymbol{x},\boldsymbol{b}),(\boldsymbol{x},\boldsymbol{c}),(\boldsymbol{y},\boldsymbol{a}),(\boldsymbol{y},\boldsymbol{b}),(\boldsymbol{y},\boldsymbol{c})\}.$

(b) $B \times A = \{(\boldsymbol{a},\boldsymbol{x}),(\boldsymbol{a},\boldsymbol{y}),(\boldsymbol{b},\boldsymbol{x}),(\boldsymbol{b},\boldsymbol{y}),(\boldsymbol{c},\boldsymbol{x}),(\boldsymbol{c},\boldsymbol{y})\}.$

(c) **No**. For sets to be equal, all the elements of one must be elements of the other. But (a,x) for example is not a member of $A\times B.$ For ordered pairs to be equal, their first coordinates must be equal and their second coordinates must be equal.

27. **(a)** The first element of each ordered pair is $a,$ so $\boldsymbol{C} = \{\boldsymbol{a}\}.$ The second elements in the ordered pairs are, respectively, $b,c,d,$ and $e,$ so $\boldsymbol{D} = \{\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e}\}.$

(b) The first element in the first three ordered pairs is 1; in the second three is 2, so $\boldsymbol{C} = \{\mathbf{1,2}\}$. The second element in the ordered pairs is, respectively, 1, 2, and 3, so $\boldsymbol{D} = \{\mathbf{1,2,3}\}$.

(c) The numbers 0 and 1 appear in each ordered pair, so $\boldsymbol{C} = \boldsymbol{D} = \{\mathbf{0,1}\}$. (The order of the numbers in these sets is irrelevant.)

Assessment 2-3B

1. Given $W = \{0,1,2,3,\ldots\}$ and $N = \{1,2,3,\ldots\}$ then $A = \{1,3,5,7,\ldots\}$ and $B = \{0,2,4,6,\ldots\}$:

(a) $\boldsymbol{B}$, or the set of all elements in W that are not in A.

(b) $\varnothing$. There are no elements common to both A and B.

(c) $\boldsymbol{N}$. All elements are common to both except 0.

2. (a) **No**. For example, $A = \{l,i,t\}$ and $B = \{l,i\}$. Then $A - B = \{t\}$ and $B - A = \varnothing$.

(b) **Yes**. By definition $A - B = \{x | x \in A$ and $x \notin B\}$. Since $x \notin B$ describes $\overline{B}$, $A - B = \{x | x \in A$ and $x \in \overline{B}\} = A \cap \overline{B}$.

(c) **No**. By definition, $B - A = \{x | x \in B$ and $x \notin A\}$. So, $B - A$ cannot contain an element that is in A unless it is also in their intersection. But, $B \cup A = A$, if $B \subseteq A.$, which means everything in B must be in A. So, the two sets cannot be equal

(d) **Yes**. Since $\varnothing$ is empty, $B \cup \varnothing = B$. Since $B \cap B$ is all the elements common to B and B, $B \cap B = B$.

3. (a) **False**. Let $A = \{1,2\}$ and $B = \{2\}$. Then $A - B = \{1\}$, but $A - \varnothing = \{1,2\}$.

(b) **False**. Let $U = \{1,2,3\}, A = \{1,2\}$, and $B = \{2,3\}$. Then $A \cup B = \{1,2,3\}$ and $\overline{A \cup B} = \varnothing$. But $\overline{A} = \{3\}$ and $\overline{B} = \{1\}$, making $\overline{A} \cup \overline{B} = \{1,3\}$.

(c) **False**. Let $A = \{1,2\}, B = \{2,3\}$, and $C = \{1,2,4\}$. Then $B \cup C = \{1,2,3,4\}$ $\Rightarrow A \cap (B \cup C) = \{1,2\}$ but $A \cap B = \{2\} \Rightarrow (A \cap B) \cup C = \{1,2,4\}$.

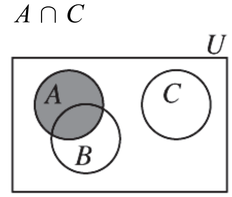
(d) **False**. Let $A = \{1,2\}$ and $B = \{2,3\}$. Then $A - B = \{1\} \Rightarrow (A - B) \cap A = \{1\}$, but $A = \{1,2\}$.

(e) **False.** Let $A = \{1,2,3\}, B = \{1,2,4\}$, and $C = \{1,2,3,4\}$. Then $B \cap C = \{1,2,4\}$, $A - B = \{3\}$, and $A - C = \varnothing$. Thus $A - (B \cap C) = \{3\}$ but $(A - B) \cap (A - C) = \varnothing$.

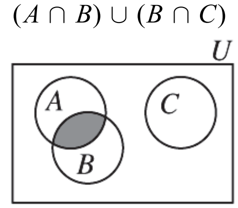
4. (a) Since $X \subseteq Y$, there are no elements in X, that are not also in Y so $X - Y = \varnothing$.

(b) If $X \subseteq Y$, then there are no elements in $\overline{Y}$ that are also in X, so $X \cap \overline{Y} = \varnothing$.

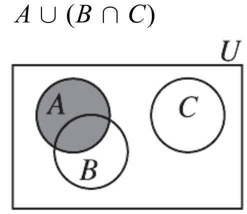
5. (a) $A \cap \overline{C}$



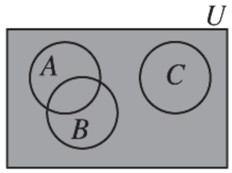
(b) $(A \cap B) \cup (B \cap C)$



(c) $A \cup (B \cap C)$



(d) $A \cup \overline{(B \cap C)}$



6. (a) $A \cup U = \boldsymbol{U}$. There are no elements in A that are not also in U.

(b) $U - A = \overline{\boldsymbol{A}}$. The set of all elements in U that are not in A is the complement of A.

(c) $A - \varnothing = \boldsymbol{A}$. There are no elements in $\varnothing$, so there are no elements which are not in A.

(d) $\overline{\varnothing} \cap A = \boldsymbol{A}$. $\overline{\varnothing} = U$, so $U \cap A = A$.

7. (a) If $A = B$ there are no elements in one set which are not also in the other, so $B - A = \varnothing$.

(b) If $B \subseteq A$ then all elements of B must also be in A, so $B - A = \{x | x \in B$ and $x \notin A\} = \varnothing$.

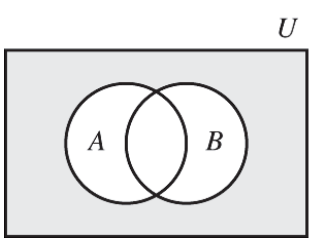
8. Answers may vary.

(a) $\boldsymbol{A \cap C}$; i.e., $\{x | x \in A \text{ and } C\}$.

(b) $\boldsymbol{(A \cup B) \cap C}$ or $\boldsymbol{C - (\overline{A \cup B})}$ or $\boldsymbol{(A \cap C) \cup (B \cap C)}$; i.e., $\{x | x \in A \text{ or } B, \text{ and } x \in C\}$.

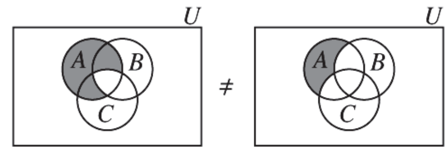
(c) In the diagram, the elements in the shaded region are in B or C but not in A. Thus, the shaded region is $\boldsymbol{(B \cup C) - A}$ **or** $\boldsymbol{(B \cup C) \cap \overline{A}}$

9.



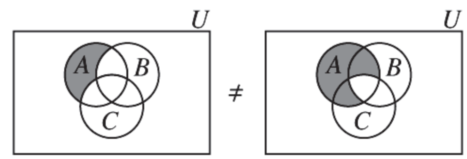
(b) Answers may vary. For example, $\overline{A \cup B} = \overline{A} \cap \overline{B}$; also $\overline{A \cup B} = U - (A \cup B)$

10. (a) False.



$A - (B \cap C)$ $\qquad$ $(A - B) \cap (A - C)$

(b) False.



$A - (B \cup C)$ $\qquad$ $(A - B) \cup (A - C)$

11. The first description should have fewer people fitting it. The first description is a subset of the second description, since it includes the fact that the suspect had a beard (in addition to having blond hair and green eyes). There would be less elements of that set, since it would be expected that some blond hair, green-eyed people do not have beards.

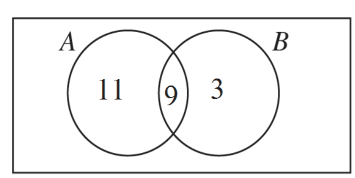
12. (a) Use a Venn diagram:

(i) Enter 9 as $n(A \cap B)$;

(ii) $n(B) = 12$, but 9 of these are in $A \cap B$, so there are 3 elements in B but not A;

(iii) $n(A \cup B) = 23$, but 12 are accounted for so there are 11 elements in A but not in B; so

(iv) $\boldsymbol{n(A) = 11 + 9 = 20}$.



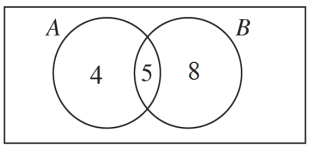
(b) Use a Venn diagram:

(*i*) Enter 5 as $n(A \cap B)$;

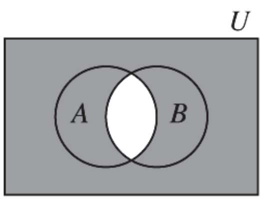
(*ii*) $n(A) = 9$, but 5 of these are in $A \cap B$, so there are 4 elements in A but not B;

(*iii*) $n(B) = 13$, but 5 of these are in $A \cap B$, so there are 8 elements in B but not in A, so;

(*iv*) $\boldsymbol{n(A \cup B) = 4 + 5 + 8 = 17.}$



13. (a) $\overline{\boldsymbol{A \cap B}} = \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$



$\overline{A \cap B} = \overline{A} \cup \overline{B}$

(b) Let $U = \{a,b,c,d\}, A = \{a,b\}, B = \{b,c\}$:

$\overline{A \cap B} = \overline{\{b\}} = \{a,c,d\}$;

$\overline{A} \cup \overline{B} = \{c,d\} \cup \{a,d\} = \{a,c,d\}$.

14. (a) The set of all Paxson 8th graders who are **members of the band but not the choir**, or $B - C$.

(b) The set of all Paxson 8th graders who are **members of both the band and the choir**, or $B \cap C$.

(c) The set of all Paxson 8th graders who are **members of the choir but not the band**, or $C - B$.

(d) The set of all Paxson 8th graders who are **neither members of the band nor of the choir**, or $\overline{B \cup C}$.

15. Enter numbers in each region in the following order:

(*i*) $n(A \cap B \cap C) = 3$;

(*ii*) $n(A \cap B) = 10$; removing the 3 from $(A \cap B \cap C)$ leaves 7 in the region $n(A \cap B \cap \overline{C})$

(*iii*) Similarly, $n(A \cap C \cap \overline{B}) = 8 - 3 = 5$; and

(*iv*) $n(B \cap C \cap \overline{A}) = 12 - 3 = 9$;

So, now what is left inside the Venn Diagram is to find (a) the number of elements in A, but not in B and C; (b) the number of elements in B, but not in A and C; and (c) the number of elements in C that are not in A and B. Those calculations are shown below

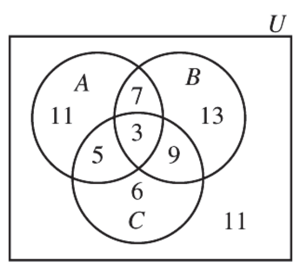
(*v*) $(a) = 26 - 7 - 3 - 5 = 11$;

(*vi*) $(b) = 32 - 7 - 3 - 9 = 13$;

(*vii*) $(c) = 23 - 5 - 3 - 9 = 6$;

Now, $11 + 7 + 3 + 5 + 13 + 9 + 6 = 54$ represents the total number of elements inside the Venn Diagram. Since $n(U) = 65$, find the number of elements outside the Venn Diagram but in the Universe:

(*viii*) $65 - 54 = 11$.



16. In the Venn diagram below:

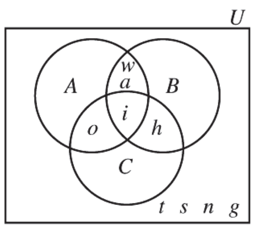
(*i*) "I" is the only letter contained in the set $A \cap B \cap C$ (i.e., the only letter common to *Iowa*, *Hawaii*, and *Ohio*);

(*ii*) "W" and "A" are the only letters contained in the set $(A \cap B) - C$ (i.e., the letters contained in both *Iowa* and *Hawaii* other than "I");

(*iii*) "O" is the only letter contained in the set $(A \cap C) - B$ (i.e., the letter contained in both *Iowa* and *Ohio* other than "I");

(*iv*) "H" is the only letter contained in the set $(B \cap C) - A$ (i.e., the letter contained in both *Hawaii* and *Ohio* other than "I");

(*v*) "T", "S", "N", and "G" are the letters in *Washington* not used in *Iowa*, *Hawaii*, or *Ohio*.



17. **(a)** Elements in regions $\boldsymbol{c, f, g,}$ **and** $\boldsymbol{h}$ represent students who do not take algebra.

(b) Elements in regions $\boldsymbol{b, c, d, e, f,}$ **and** $\boldsymbol{g}$ represent students who take biology or chemistry.

(c) Elements in regions $\boldsymbol{b}$ **and** $\boldsymbol{e}$ represent students who took both algebra and biology.

(d) Since (a) is the part of A with no elements in B or C, we could say **the students at Hellgate who took algebra but did not take biology and did not take chemistry**.

(e) Students at Hellgate who took biology and chemistry but did not take algebra.

(f) $\boldsymbol{(B \cap C) - A}$ **or** $\boldsymbol{(B \cap C) \cap \overline{A}}$

(g) Since regions (d) and (g) represent students who took chemistry, but did not take biology, we write $\boldsymbol{C - B}$ **or** $\boldsymbol{C \cap \overline{B}}$.

(h) Region (g) represents students who took chemistry but did not take biology and did not take algebra. We write $\boldsymbol{C - (A \cup B)}$ **or** $\boldsymbol{C \cap \overline{A \cup B}}$**.**

18. Generate the following Venn diagram in this order:

(*i*) The 50 who used all three cards;

(*ii*) The 60 who used **only** Super and Thrift cards;

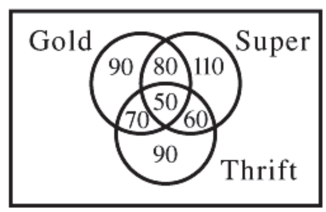
(*iii*) The 70 who used **only** Gold and Thrift cards;

(*iv*) The 80 who used **only** Gold and Super cards;

(*v*) The $290 - (80 + 50 + 70) = 90$ who used **only** Gold cards;

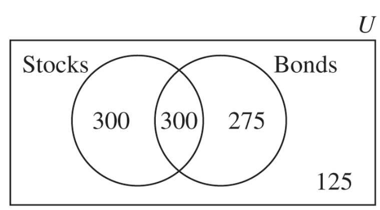
(*vi*) The $300 - (80 + 50 + 60) = 110$ who used **only** Super cards; and

(*vii*) The $270 - (70 + 50 + 60) = 90$ who used **only** Thrift cards.



The diagram indicates only 550 cardholders accounted for, so either there is some other type credit card used by the remaining 50 people or **the editor was right**.

19. Complete the Venn diagram by first completing the region that represents investors who invested in stocks and bonds.



(a) **300**

(b) **875**

(c) **125**

20. **(a)** It is possible that no student took both algebra and biology. **0**

(b) It is possible that all **30** biology students also took algebra.

(c) If all 30 biology students also took algebra, then the numbers of students who took neither is $150 - 90 = \mathbf{60}$ students.

21. **(a)** **146.4 million.** Subtract those who identify themselves as religious Jews (4.2 million) away from the number of people identified as being of some faith (150.6 million).

(b) **171.2 million**. Subtract those who identify themselves as being of some faith (150.6 million) from the total population of the United States (321.8 million).

22. **93.** From the data given, we know there are 44 total Canadian citizens. We also know there were a total of 34 American women and males on the tour. So, all we need to find the number of American girls to complete the problem. Because there are 29 women total, and 17 are American, we know there were 12 Canadian women on the tour. Since the total number of Canadian females is 26, we can deduce that 14 Canadian females were girls. Since there were 29 girls on the trip, we know that 15 girls must be American girls. Therefore $44 + 34 + 15 = \mathbf{93}$

23. **(a)** **False**. These are ordered pairs, thus order is relevant.

(b) **False**. The left side is an ordered pair, while the right side is a set.

24. **(a)** Each of the five elements in A are paired with each of the four in B so there are $5 \cdot 4 = \mathbf{20}$ **elements**.

(b) Each of the m elements in A are paired with each of the n in B so there are $\boldsymbol{m \cdot n}$ **elements**.

(c) $A \times B$ has $m \cdot n$ elements, each of which are paired with the p elements in C so there are $\boldsymbol{m \cdot n \cdot p}$ **elements**.

Mathematical Connections 2-3: Review Problems

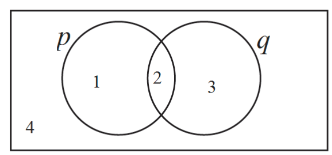
14. The contrapositive of the contrapositive of a statement is the statement itself. If the original statement is $p \to q$, then by definition its contrapositive is $\sim q \to \sim p$. To take the contrapositive of that statement, you would end up with $p \to q$.

15. **(a)** Therefore, Mary will change the lunch menu. (Modus Ponens, or law of detachment)

(b) Therefore, Samuel stays after school.

(c) Therefore, the lake is not frozen. (Modus Tollens)

16. Answers will vary. Consider the numbered Venn diagram below:



The area in numbered region 1 represents the area where p is true but q is false; the area in numbered region 3 represents the area where q is true but p is false; the area in numbered region 2 (the intersection) is where both p and q are true; and the area in numbered region 4 (outside the two circles) is the area where both p and q are false. All cases, therefore, are covered.

17. **(a)** Set-builder notation describes the elements of the set, rather than listing them; i.e., $\{\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{N} \text{ and } \mathbf{3} < \boldsymbol{x} < \mathbf{10}\}$, where N represents the set of natural numbers, allows the set to be built.

(b) This set has only three elements; i.e., $\{\mathbf{15, 30, 45}\}$ thus the elements may easily be listed.

18. **(a)** **6**; c, o, m, n, r, and e are the elements in the set. c and m are duplicated once,and o is duplicated twice.

(b) **6**; c, o, m, i, t, e are the elements in the set. $m, t,$ and e are duplicated once.

19. **(a)** These are all the subsets of $\{2,3,4\}$. There are $2^3 = \mathbf{8}$ such subsets.

(b) **8**. Every subset either contains 1 or it does not, so exactly half the $2^4 = 16$ subsets contain 1.

(c) **Twelve** subsets. There are 4 subsets of $\{3,4\}$. Each of these subsets can be appended with 1, 2, or 1 and 2 to each. By the Fundamental Counting Principle then, there are $3 \cdot 4 = 12$ possibilities. (It is also possible to simple systematically list all the possible subsets with 1 or 2 and count them.)

(d) There are **four** subsets containing neither 1 nor 2, since 12 do contain 1 or 2.

(e) **16**. B has $2^5 = 32$ subsets; half contain 5 and half do not.

20. Answers may vary. Some possibilities:

(a) The set of 18 holes on a golf course, and the set of 18 flagsticks on a golf course.

(b) The set of letters in the English alphabet (26) with the set of letters in the Greek alphabet (24).

21. The number of combinations is equivalent to the Cartesian product of $\{SLACKS\}$, $\{SHIRTS\}$, and $\{SWEATERS\}$ so the number of elements is $n(\{SLACKS\}) \cdot n(\{SHIRTS\}) \cdot n(\{SWEATERS\}) =$ $4 \cdot 5 \cdot 3 = \mathbf{60}$ **combinations**.

Chapter 2 Review

1. Answers will vary. Some statements: i). Arizona is a state; ii) The president of the United States is Barack Obama; iii) $2 + 12 = 24$; $2 \times 12 = 24$. Some non-statements: i) Las Vegas is a fun city; ii) Teaching is a tough job; iii) How old are you?; iv) Zane Grey is the best writer ever.

2. In statement (i), every student in the class earned an A, B, or C grade on the final exam; in statement (ii), there was at least one student who earned an A, B, or C grade on the final exam. In statement (i), no student could have received a grade of D or F on the final exam; while in statement (ii), it is possible that a student (or several students) did receive a grade of D or F on the final exam.

3. **(a)** **Yes**. Even though it is false, it is still a statement, by the definition of a statement.

(b) **No**. This sentence is neither true or false, since it depends upon the value of the variable n. Since its truth value cannot be definitely ascertained, it is not a statement.

(c) **Yes**. It can definitely be determined whether or not this statement is true or false.

4. **(a)** **Some women smoke**. The original statement means that there exists no women who smoke. So to negate it, one would have to write a statement that implied there exists at least one woman who smokes.

(b) $3 + 5 = 8$. The original statement is false; to negate it, write a statement that is true.

(c) **Bach wrote some music that was not classical**. The reasoning behind this answer is similar to the answer in part (a). The original statement implies that Bach only wrote classical music. To negate it, a statement must be written to imply that there exists at least one piece of Bach's music that was not classical.

5. The original statement is in the form $p \to q$. Converse $(q \to p)$:If someone will read a tweet, the whole world is tweeting. Inverse $(\sim p \to \sim q)$: If the whole world is not tweeting, no one will read a tweet. Contrapositive $(\sim q \to \sim p)$: If no one will read a tweet, the whole world is not tweeting.

6.

p	q	$\sim p$	$\sim q$	$\sim p \to q$	$\sim q \to p$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

7. (a)

p	q	$\sim q$	$p \vee \sim q$	$p \vee q$	$(p \vee \sim q) \wedge (p \vee q)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	F	T	F
F	F	T	T	F	F

(b)

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge \sim q$	$[(p \vee \sim q) \wedge \sim q] \rightarrow p$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	F

8. (a) $\therefore$ Alfinia loves Mom and apple pie. This is true by direct reasoning. The first statement could be read as "If a person is a Eurasian, then that person loves Mom and apple pie". We can then make p be the statement "a person is a Eurasian" and q the statement "A person loves Mom and apple pie." So, the first statement is denoted in logical notation as $p \rightarrow q$; the second statement, Alfinia is a Eurasian, is denoted as p. So, it makes sense that the conclusion should be q.

(b) $\therefore$ The Washington Monument will eventually crack. This is true by direct reasoning, similar to (a). The first statement could be written in $p \rightarrow q$ form as "if an object is made of marble and granite, then it will eventually crack."

(c) $\therefore$ Therefore, Josef passed the math for elementary teachers course. The first sentence consists of two statements "Josef passed the math for elementary teachers course" is one statement; "Josef dropped out of school" is the other statement. The "or" between the two statements implies that one or the other is true. The second sentence "Josef did not drop out of school" would imply that the second part of that first sentence isn't true…so the first part must be.

9. Let p be the statement "you passed your classes"; q be the statement "your parents will allow you to go to the dance"; and r be the statement "you sit in the corner." So, the sentences can be written as follows: first sentence: $p \rightarrow q$; second sentence $q \rightarrow \sim r$. Third sentence: $\sim(\sim r) = r$; so the conclusion is $\sim p$. This is a valid argument by transitivity and modus tollens.

10. This argument is valid by modus tollens. Let p be the statement "Bob passes the course" and q be the statement "Bob scored at least a 75 on the final exam". Then the first sentence is $p \rightarrow q$; the second sentence is $\sim q$; therefore the conclusion is $\sim p$.

11. A set with n elements has 2^n subsets. $2^4 = 16$. This includes A. There are $2^4 - 1 =$ **15 proper subsets**.

12. There are 4 elements in the set, thus there are $2^4 = \mathbf{16}$ **subsets**: $\{\}, \{m\}, \{a\}, \{t\}, \{h\}, \{m,a\}, \{m,t\}, \{m,h\}, \{a,t\}, \{a,h\}, \{t,h\}, \{m,a,t\}, \{m,a,h\}, \{m,t,h\}, \{a,t,h\}, \{m,a,t,h\}$.

13. (a) $A \cup B = \{r,a,v,e\} \cup \{a,r,e\} = \{r,a,v,e\} = \boldsymbol{A}$.

(b) $C \cap D = \{l,i,n,e\} \cap \{s,a,l,e\} = \boldsymbol{\{l,e\}}$.

(c) $\overline{D} = \overline{\{s,a,l,e\}} = \boldsymbol{\{u,n,i,v,r\}}$.

(d) $A \cap \overline{D} = \{r,a,v,\text{e}\} \cap \{u,n,i,v,r\} = \boldsymbol{\{r,v\}}$.

(e) $\overline{B \cup C} = \overline{\{a,r,\text{e}\} \cup \{l,i,n,e\}}$
$= \overline{\{a,e,i,l,n,r\}}$
$= \boldsymbol{\{s,u,v\}}$.

(f) $\text{B} \cup \text{C} = \{a,e,i,l,n,r\} \Rightarrow (B \cup C) \cap D$
$= \{a,e,i,l,n,r\} \cap \{s,a,l,\text{e}\}$
$= \boldsymbol{\{a,l,e\}}$.

(g) $\overline{A} \cup B = \{u,n,i,s,l\} \cup \{a,r,e\}$
$= \{u,n,i,e,r,s,a,l\}$.

$C \cap \overline{D} = \{l,i,n,e\} \cap \{u,n,i,v,r\} = \{i,n\}. \Rightarrow (\overline{A} \cup B) \cap (C \cap \overline{D}) = \boldsymbol{\{i,n\}}$.

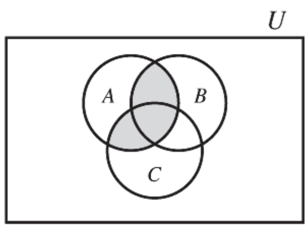
(h) $(C \cap D) \cap A = (\{l,i,n,e\} \cap \{s,a,l,e\}) \cap \{r,a,v,e\} = \{l,e\} \cap \{r,a,v,e\} = \mathbf{\{e\}}$.

(i) $n(B - A) = \mathbf{0}$.

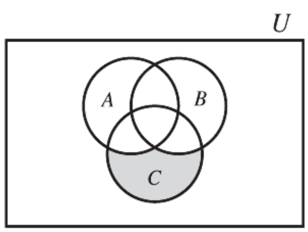
(j) $n(\overline{C}) = n\{u,v,r,s,a\} = \mathbf{5}$.

(k) $n(C \times D) = 4 \cdot 4 = \mathbf{16}$. Each of the four elements in C can be paired with each of the four in D.

14. (a) $A \cap (B \cup C)$ includes the elements in A that are common to the union of B and C:



(b) $(\overline{A \cup B}) \cap C$ is the same as $C - (A \cup B)$:

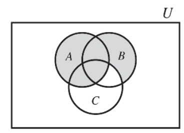


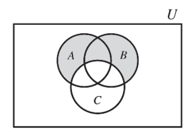
15. Since all 5 letters are distinct, consider seven "slots" in which to put the letters. There are 5 letters which could go in the first slot, then 4 left which could go in the second slot, and so on. So, the number of possible arrangements is then $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

16. (a) Answers may vary; **one possible** correspondence is $\boldsymbol{t \leftrightarrow e}$, $\boldsymbol{h \leftrightarrow n}$, **and** $\boldsymbol{e \leftrightarrow d}$.

(b) There are three possible one-to-one correspondences between D and the e in E, two possible between D and the n in E, and then only one possible remaining for the d in E. Thus $3 \cdot 2 \cdot 1 = \mathbf{6}$ one-to-one correspondences are possible.

17. $A \cup (B - C) \neq (A \cup B) - C$

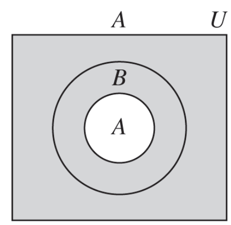




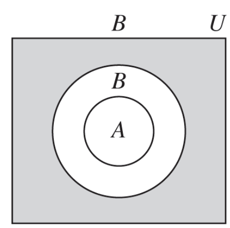
18. (a) **False**. Let $U = \{1,2,3,4\}$, $A = \{1,2\}$, and $B = \{1,2,3\}$ Then $\overline{A} = \{3,4\}$ and $\overline{B} = \{4\}$. So $\overline{A} \not\subseteq \overline{B}$.

(b) **True**.

$\overline{A}$



$\overline{B}$



(c) **True**. Since $A \subseteq B$, every element in A is in B, So combining A and B does not include new elements to B.

(d) **True**. Since $A \subseteq B$ and $A \cap B = \{x | x \in A \text{ and } x \in B\}$, the intersection, the elements common to A and B, is A.

(e) **True**. Since $\overline{B} \subseteq \overline{A}$, this property was established in d.

(f) **True,** This is because $\overline{B} \subseteq \overline{A}$, as established in b.

19. (a) False. For example, let $U = \{1,2,3,4,5\}$; $A = \{1,2,3,4\}$; $B = \{3,4,5\}$ and $C = \{3,4\}$. Then $A - B = \{1,2\} = A - C$; but $B \neq C$

(b) True

20. (a) A **.** By the distributive property of set intersection over set union $(A \cap B) \cup (A \cap \overline{B})$ can be written as $A \cap (B \cup \overline{B})$. Since $(B \cup \overline{B}) = U$, we have $A \cap U$, which equals A.

(b) $A \cup \overline{B}$ **.**By the distribution property of set union over set intersection, $(A \cap B) \cup \overline{B}$ can be rewritten as $(A \cup \overline{B}) \cap (B \cup \overline{B})$. Since $(B \cup \overline{B}) = U$, we have $(A \cup \overline{B}) \cap U$, which equals $A \cup \overline{B}$

21. (a) False. The sets could be disjoint.

(b) False. The empty set is not a proper subset of itself.

(c) False. $A \sim B$ only requires the same number of elements—not necessarily the same elements.

(d) False. The set is in one-to-one correspondence with the set of natural numbers, so it increases without limit.

(e) False. For example, the set $\{5,10,15,...\}$ is a proper subset of the natural numbers and is equivalent since there is a one-to-one correspondence.

(f) False. Let $B = \{1,2,3\}$ and $A =$ the set of natural numbers.

(g) True. If $A \cap B \neq \emptyset$, then the sets are not disjoint.

(h) False. The sets may be disjoint but not empty.

22. (a) True. Venn diagrams show that $A - B$, $B - A$, and $A \cap B$ are all disjoint sets, so $n(A - B) + n(B - A) + n(A \cap B) = n(A \cup B)$.

(b) True. Venn diagrams show that $A - B$ and B are disjoint sets, so $n(A - B) + n(B) = n(A \cup B)$. Likewise, Venn diagrams show that $(B - A)$ and A are disjoint, so $n(B - A) + n(A) = n(A \cup B)$.

23. (a) 17, if $P = Q$.

(b) 34, if P and Q are disjoint.

(c) 0, if P and Q are disjoint.

(d) 17, if $P = Q$.

24. $A \times B \times C$ is the set of ordered triples (a,b,c), where $a \in A$, $b \in B$, and $c \in C$. There are 3 possibilities for a, the first entry, 4 possibilities for the second, and 2 for the third. So $n(A \times B \times C) =$ **24**.

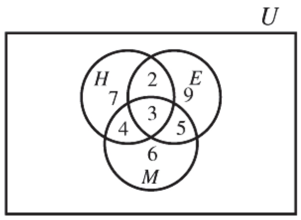
25. $n(\text{Crew}) + n(\text{Swimming}) + n(\text{Soccer}) = 57$. The 2 lettering in all three sports are counted three times, so subtract 2 twice, giving 53. $n(\text{Awards}) = 46$, so $53 - 46 = 7$ were counted twice; i.e., **7 lettered in exactly two sports**.

26. Use the following Venn diagram place values in appropriate areas starting with the fact that 3 students liked all three subjects; 7 liked history and mathematics, but of those 7, three also liked English so 4 is placed in the $\text{H} \cap \text{M}$ region; etc.

(a) A total of **36 students** were in the survey.

(b) 6 students liked only mathematics.

(c) **5 students** liked English and mathematics but not history.



27. Answers may vary. Possibilities are:

(a) The shaded areas show $\boldsymbol{B \cup (A \cap C)}$

(b) The shaded areas show $\boldsymbol{B - C}$ or $\boldsymbol{B \cap \overline{C}}$

28. 2 slacks $\times$ 3 blouses $\times$ 2 sweaters $=$ **12** outfits.

29. **(a)** Let $A = \{1, 2, 3, \ldots, 13\}$ and $B = \{1, 2, 3\}$. $B \subset A$, so B has fewer elements than A. Then $n(B) < n(A)$ and thus $3 < 13$.

(b) Let $\text{A} = \{1, 2, 3, \ldots, 12\}$ and $B = \{1, 2, 3, \ldots, 9\}$. $B \subset A$, so A has more elements than B. Then $n(A) > n(B)$ and thus $12 > 9$.

CHAPTER 3

NUMERATION SYSTEMS AND WHOLE NUMBER OPERATIONS

Assessment 3-1A: Numeration Systems

1. Using the following place value table,

Billions	Hundred Millions	Ten Millions	Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units

we can write the numerals in words.
(a) Fifty-six million, two hundred eighty-three thousand, nine hundred fourteen.
(b) Five billion, three hundred sixty-five million, two hundred ninety-five thousand, two hundred thirty-four.

2. Using the place value table in (1):
(a) The hundreds place value is 5.
(b) The ten thousands place value is 3.

3. **(a)** **Hundreds**. From the decimal point moving to the left: units $\rightarrow$ tens $\rightarrow$ hundreds.
(b) **Tens**. From the decimal point moving to the left: units $\rightarrow$ tens.

4. **(a)** $3{,}000{,}000 + 4000 + 5 =$ **3,004,005**.
(b) $20{,}000 + 1 =$ **20,001**.

5. If □, △ and ○ each represent different non-zero base-ten digits then ordering them from least to greatest is based on the number of digits each number has.
(a) □ (one-digit); □ △ (two-digit); △ □ ○ (three-digit)
(b) □ □ (two-digit); △ □ ○ (three-digit); □ □ △ △ (four-digit)

6. There are 15 digits in the standard form of $625 \bullet 10^{12}$: $625{,}000{,}000{,}000{,}000$.

7. Either **811** or **910**. Each number satisfies the conditions that the hundreds digit must be 8 or 9, the tens digit must be odd, and the sum of the digits must be 10.

8. **(a)** **$\overline{\overline{\text{M}}}$CDXXIV**. The double bar over the M represents $1000 \cdot 1000 \cdot 1000$ while a single bar over the M would represent only $1000 \cdot 1000$.
(b) **46,032**. The 4 in 46,032 represents 40,000 while the 4 in 4632 represents only 4000.
(c) < ▼▼. < in the first number represents 10, while in the second number it represents $10 \cdot 60$ because of the space.
(d) ∩ |. has a place value of 1000 while has a place value of only 100.
(e) . represents three groups of 20 plus zero l's, or 60, while represents three 5's and three l's, or 18.

9. **(a)** MCMXLIX represents 1949; thus one more is 1950, or **MCML**; one less is 1948, or **MCMXLVIII**.
(b) < < < ∇ is $20 \cdot 60 + 11 = 1211$; thus one more is 1212, or < < < ∇∇ one less is 1210, or < < <.
(c) is $1000 + 100 + 100 = 1200$; thus one more is 1201, or ; one less is 1199, or ∩∩∩∩ |||| ∩∩∩∩∩ |||||.
(d) is $7 \cdot 20 + 13 \cdot 1 = 153$; thus one more is 154, or ; one less is 152, or .

10. MCMXXII is $1000 + 900 + 20 + 2$, or the year **1922**.

11. **(a)** CXXI, or $100 + 20 + 1$.
(b) XLII, or $50 - 10 + 2$.
(c) XCI, or $100 - 10 + 1$.
(d) MMXIV, or $2000 + 10 + 5 - 1$.

12. **(a)** Hindu-Arabic: **72**

Babylonian: $60 + 10 + 2$, or ∇ < ∇∇

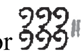
Egyptian: $70 + 2$, or 

Roman: **LXXII**

Mayan: $3 \cdot 20 + 12$, or 

(b) Hindu-Arabic: **602**

Babylonian: $10 \cdot 60 + 2$, or < ∇∇

Egyptian: $6 \cdot 100 + 2$, or 

Roman: **DCII**

Mayan: $1 \cdot 360 + 12 \cdot 20 + 2$, or 

(c) Hindu-Arabic: **1223**

Babylonian: $2 \cdot 10 \cdot 60 + 23$, or

< < < < ∇∇∇

Egyptian: 

Roman: **MCCXXIII**

Mayan: $3 \cdot 360 + 7 \cdot 20 + 3$, or 

13. **(a)** $(3 \cdot 25) + (2 \cdot 5) + (1 \cdot 1) = \mathbf{86}$.

(b) $(1 \cdot 8) + (0 \cdot 4) + (1 \cdot 2) + (1 \cdot 1) = \mathbf{11}$.

14. The top row of blocks contains one set of 64 blocks plus four sets of 16 each, or $2 \cdot 4^3$ blocks; the second row of blocks contains one set of 16 blocks, or $1 \cdot 4^2$, plus four single blocks, or $1 \cdot 4^1$ blocks, plus two single blocks, or $2 \cdot 4^0 = \mathbf{2112}_{\boldsymbol{four}}$.

15. **(a)** Place values represent powers of 2; e.g.,

$1 \cdot 2^0 = 1_{two}$

$1 \cdot 2^1 + 0 \cdot 2^0 = 10_{two}$

$1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 100_{two}$, etc.

Thus the first 15 counting numbers are: $(1,10,11,100,101,110,111,1000,1001,$ $1010,1011,1100,1101,1110,1111)_{two}$.

(b) Place values represent powers of 4; e.g.,

$1 \cdot 4^0 = 1_{four}$

$1 \cdot 4^1 + 0 \cdot 4^0 = 10_{four}$

$1 \cdot 4^2 + 0 \cdot 4^1 + 0 \cdot 4^0 = 100_{four}$, etc.

Thus the first 15 counting numbers are: $(1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33)_{four}$.

16. **20** digits. One digit is needed for each of the units from 1 to 19, and one digit for 0.

17. $2032_{four} = (2 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 2 \cdot 10^0)_{four}$. 2032_{four} may also be expanded as $(2 \cdot 4^3 + 0 \cdot 4^2 + 3 \cdot 4^1 + 2 \cdot 4^0)_{ten} = \mathbf{142}$.

18. **(a)** $\mathbf{111}_{\boldsymbol{two}}$. 1 is the largest units digit in base two.

(b) $\mathbf{EEE}_{\boldsymbol{twelve}}$. E is the largest units digit in base twelve.

19. **(a)** $EE0_{twelve} = 11 \cdot 12^2 + 11 \cdot 12^1 + 0 \Rightarrow$

(***i***) $(EE0 - 1)_{twelve} =$ $11 \cdot 12^2 + 10 \cdot 12^1 + 11 = \boldsymbol{ETE}_{twelve}$.

(***ii***) $(EE0 + 1)_{twelve} =$ $11 \cdot 12^2 + 11 \cdot 12^1 + 1 = \boldsymbol{EE1}_{twelve}$.

(b) (***i***) $(100000 - 1)_{two} = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 - 1 =$ $1 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 1 =$ $\mathbf{11111}_{two}$ [analogous to $(100,000 - 1)_{ten} = 99,999_{ten}$].

(***ii***) $(100000 + 1)_{two} = 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^3 + 0 \cdot 2^1 + 0 + 1 =$ $\mathbf{100001}_{two}$.

(c) (***i***) $(555 - 1)_{six} = 5 \cdot 6^2 + 5 \cdot 6^1 + 5 - 1 = \mathbf{554}_{\boldsymbol{six}}$.

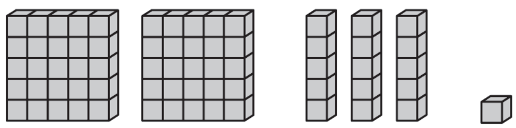
(***ii***) $(555 + 1)_{six} = 5 \cdot 6^2 + 5 \cdot 6^1 + 5 + 1 = 1 \cdot 6^3 + 0 \cdot 6^2 + 0 \cdot 6^1 + 0 = 1000_{six}$ [analogous to $(999 + 1)_{ten} = 1000_{ten}$].

20. **(a)** There is **no numeral 4 in base four**. The numerals are 0, 1, 2, and 3.

(b) There are **no numerals 6 or 7 in base five**. The numerals are 0, 1, 2, 3, and 4.

21. There are 3 groups of 4^3, 1 group of 4^2, 1 group of 4^1, and 2 groups of 4^0 in 214, so $214 = 3112_{four}$. Thus, where a block is $4 \times 4 \times 4$, a flat is 4×4, and a long is 1×4, **three** blocks, **one** flat, **one** long, and **two** units is the least number of base four blocks.

22. 231_{five} is represented by two flats $(2 \cdot 5^2)$, three longs $(3 \cdot 5^1)$, and one unit.



23. **(a)** **(*i*)** 8 pennies may be changed into 1 nickel and 3 pennies. The total is now 2 quarters, 10 nickels, and 3 pennies.

(*ii*) 10 nickels may be changed into 2 quarters. The total is now **4 quarters, 0 nickels, and 3 pennies**.

(b) 73¢, in the fewest number of coins, can be obtained with 2 quarters, 4 nickels, and 3 pennies, or $(2 \cdot 5^2 + 4 \cdot 5 + 3)$¢. This is $\mathbf{243}_{five}$.

24. **(a)** Since both numerals have the same groups of 64s, we only need to compare lesser place values.

$$100_{four} = (1 \cdot 4^2 + 0 \cdot 4 + 0 \cdot 1)_{ten} = 16$$

$$030_{four} = (0 \cdot 4^2 + 3 \cdot 4 + 0 \cdot 1)_{ten} = 12.$$

So, 3030_{four} is the lesser.

(b) Again, we only need to compare TE_{twelve} to ET_{twelve}.

TE_{twelve} has ten groups of 12 and 11 groups of 1.

ET_{twelve} has 11 groups of 12 and 10 groups of 1.

Thus, $\boldsymbol{EOTE_{twelve}}$ **is lesser**.

25. **(a)** 10 flats = 1 **block**. $1 \cdot 10^3 = \mathbf{1000}$.

(b) 20_{twelve} flats = 12 flats + 8 flats = **1 block + 8 flats**. When written in base twelve notation, it looks like $1 \cdot 12^3 + 8 \cdot 10^2 = \mathbf{1800}_{twelve}$.

26. **(a)**

$$\begin{array}{r|r|l} 125 & 456 & 3 \\ & -375 & \\ 25 & 81 & 3 \\ & -75 & \\ 5 & 6 & 1 \\ & -5 & \\ & 1 & \end{array}$$

$456 = \mathbf{3311}_{five}$

How many groups of 125 in 456?
How many groups of 25 in 81?
How many groups of 5 in 6?

(b)

$$\begin{array}{r|r|l} 1728 & 1782 & 1 \\ & -1728 & \\ 12 & 54 & 4 \\ & -48 & \\ & 6 & \end{array}$$

$1782 = \mathbf{1046}_{twelve}$

How many groups of $1728 = 12^3$ in 1782?
How many groups of 12 in 54?

(c)

$$\begin{array}{r|r|l} 32 & 32 & 1 \\ & -32 & \\ & 0 & \end{array}$$

$32 = \mathbf{100000}_{two}$

How many groups of $2^5 = 32$ in 32?

27. **(a)** $432_{five} = 4 \cdot 5^2 + 3 \cdot 5^1 + 2 = 100 + 15 + 2 = \mathbf{117}$.

(b) $101101_{two} = 1 \cdot 2^5 + 1 \cdot 2^3 + 1 \cdot 2^2 + 1 = 32 + 8 + 4 + 1 = \mathbf{45}$.

(c) $92E_{twelve} = 9 \cdot 12^2 + 2 \cdot 12 + 11 = 1296 + 24 + 11 = \mathbf{1331}$.

28. Using the place value in base six,

6^3	6^2	6^1	1

we have $a \bullet 6^3 + b \bullet 6^2 + c \bullet 6^1 + d \bullet 1$.

29. To find out the base, add the units digits: $2_b + 6_b$. In base 10 that equals 8 but in base b it equals a numbers with units digit 1. Take out a group of b and have one left over. The only choice is base seven and then $2_7 + 6_7 = 11_7$ and

$$\begin{aligned} 12_7 + 26_7 &= (10_7 + 20_7) + (2_7 + 6_7) \\ &= (30_7) + (11_7) \\ &= 41_7 \end{aligned}$$

30. **(a)** Using the base sixteen place value:

$$\begin{aligned} 256_{sixteen} &= 2 \bullet 16^2 + 5 \bullet 16^1 + 6 \bullet 1 \\ &= 512 + 80 + 6 \\ &= 598 \end{aligned}$$

(b) Using the base sixteen place value:

$$\begin{aligned} 3ACD_{sixteen} &= 3 \bullet 16^3 + A \bullet 16^2 + C \bullet 16^1 + D \bullet 1 \\ &= 3 \bullet 16^3 + 10 \bullet 16^2 + 12 \bullet 16^1 + 13 \bullet 1 \\ &= 12{,}288 + 2560 + 192 + 13 \\ &= 15{,}053 \end{aligned}$$

31. To convert 584 from base ten to base sixteen, ask:
How many groups of $16^3 (= 4096)$ are there in 584?
None 4096>584.
How many groups of $16^2 (= 256)$ are there in 584?
2 groups ($2 \cdot 16^2 = 512$) with 72 left over ($584 - 512 = 72$).
How many groups of $16^1 (= 16)$ are there in 72?
4 groups ($4 \cdot 16 = 64$) with 8 left over ($72 - 64 = 8$).
How many groups of $16^0 (= 1)$ are there in 8?
8 groups ($8 \cdot 16^0 = 8$) with none left over ($8 - 8 = 0$).
So $584_{10} = 248_{\text{sixteen}}$.

32. To give the fewest number of prizes, the dollar amount of each must be maximized. Thus $900 =

1 prize or $625 with $275 left over;
2 prizes of $125 with $25 left over; and
1 prize of $25 with nothing left over.
I.e., $900 = 12100_{five}$.

33. **(a)** There are 8 groups of 7 days (1 week) with 2 days left over, so 58 days = **8 weeks and 2 days** (or $58 = 82_{seven}$).

(b) There is 1 group of 24 hours (1 day) with 5 hours left over, so 29 hours = **1 day and 5 hours** (or $29 = 15_{twenty\ four}$).

34. **(a)** $b = \mathbf{6}$. There are 6 groups of 7 in 44.

(b) $b = \mathbf{1}$. Subtracting 5 groups of 144 from 734 leaves 14; there is 1 group of 12 in 14.

35. **(a)** $3 \cdot 5^4 + 3 \cdot 5^2 = 3 \cdot 5^4 + 0 \cdot 5^3 + 3 \cdot 5^2 + 0 \cdot 5^1 + 0 \cdot 5^0 = \mathbf{30300}_{\boldsymbol{five}}$.

(b) $2 \cdot 12^5 + 8 \cdot 12^3 + 12 = 2 \cdot 12^5 + 0 \cdot 12^4 + 8 \cdot 12^3 + 0 \cdot 12^2 + 1 \cdot 12^1 + 0 \cdot 12^0 = \mathbf{208010}_{\boldsymbol{twelve}}$.

36. Answers may vary; some possibilities might be:

(a) Subtract a number with a 2 in the thousands place and a 2 in the tens place, or 2020. Then $32{,}420 - 2020 = 30{,}400$.

(b) Subtract a number with a 5 in the tens place, or 50. Then $67{,}357 - 50 = 67{,}307$.

Assessment 3-1B

1. Using the following place value table,

Billions	Hundred Millions	Ten Millions	Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Units

we can write the numerals in words.

(a) Two hundred fifty-three million, two hundred forty-three thousand, eight hundred ten.

(b) Sixty-two billion, three hundred fifty-five million, two hundred eighty-eight thousand, four hundred thirty-two.

2. Using the place value table in (1):

(a) The thousands place value is 5.

(b) The hundred thousands place value is 3.

3. **(a)** **Thousands**. From the decimal point moving to the left: units → tens → hundreds → thousands → ten thousands.

(b) **Hundred thousands**. From the decimal point moving to the left: units → tens → hundreds → thousands → ten thousands → hundred thousands.

4. **(a)** $3000 + 500 + 60 =$ **3560**.

(b) $9{,}000{,}000 + 90 + 9 =$ **9,000,099**.

5. If □, △ and ○ each represent different non-zero base-ten digits then ordering them from least to greatest is based on the number of digits each number has.

(a) (one-digit); ○ ○ (two-digit); □ △ ○ (three-digit).

(b) △ △ (two-digit); □ △ ○ (three-digit); □ □ □ □ (four-digit)

6. There are 19 digits in the standard form of $5250 \bullet 10^{15}$: $5{,}250{,}000{,}000{,}000{,}000{,}000$.

7. Let u = units digit and t = tens digit.

Then $u = t - 4$

$t = 2u$

So $u = 2u - 4 \rightarrow u = 4$ and $t = 8$.

The number is **84**.

8. **(a)** $\overline{\textbf{M}}$**DCXXIV**. D is greater than CD.

(b) **30,456**. The 3 in 30,456 represents 30,000 while the 3 in 3456 represents only 3000.

(c) . The space indicates that < is multiplied by 60; the < without the space represents just 10.

(d) , which represents 300; the other figure represents 211.

(e) , which represents $2 \cdot 20$. The other figure represents 13.

9. **(a)** $\overline{\textbf{M}}$**I** represents 1,000,001. The preceding number is 1,000,000, or $\overline{\textbf{M}}$; the succeeding number is 1,000,002, or $\overline{\textbf{M}}$**II**.

(b) CMXCIX represents 999. The preceding number is 998, or **CMXCVIII**; succeeding number is 1000, or **M**.

(c) < <∇ represents $10 \cdot 60 + 11 = 611$. The preceding number is 610, or < <; the succeeding number is 612, or < < ∇∇.

(d) represents $1000 + 100 = 1100$. The preceding number is 1099, or ; the succeeding number is 1101, or .

(e) represents $6 \cdot 20 + 8 = 128$. The preceding number is 127, or ; the succeeding number is 129, or .

10. Since no numeral of lesser value is to the left of a numeral of greater value, the numerals can be combined in an additive process.

MDCCLXXVI represents $1000 + 500 + 100 + 100 + 50 + 10 + 10 + 5 + 1 =$ **1776**.

11. **(a)** LXXXIX or $50 + 30 + 9$.

(b) $\overline{\text{V}}$CCII, or $1000 \cdot 5 + 200 + 2$.

(c) XXXII, or $30 + 2$.

(d) CII, or $100 + 2$

12. **(a)** Hindu-Arabic: 78

Babylonian: $60 + 10 + 8$, or

Egyptian:

Roman: **LXXVIII**

Mayan: $3 \cdot 20 + 18$, or

(b) Hindu-Arabic: **601**

Babylonian: $10 \cdot 60 + 1$, or < ∇

Egyptian:

Roman: **DCI**

Mayan: $360 + 12 \cdot 20 + 1$, or

(c) Hindu-Arabic: **1111**

Babylonian: $18 \cdot 60 + 31$, or < ∇∇∇∇∇∇∇∇ < < < ∇

Egyptian:

Roman: **MCXI**

Mayan: $3 \cdot 360 + 20 + 11$, or

13. **47**, or $1 \cdot 27 + 2 \cdot 9 + 2 \cdot 1 = 47$.

14. The row of blocks contains one set of 27 blocks, or $1 \cdot 3^3$, plus two sets of 9 each, or $2 \cdot 3^2$ blocks, plus two sets of 3 each, or $2 \cdot 3^1$ blocks, plus two single blocks, or $2 \cdot 3^0$. The representation is thus $\mathbf{1222}_{\boldsymbol{three}}$.

15. **(a)** Place values represent powers of 3; e.g.,

$1 \cdot 3^0 = 1_{three}$

$1 \cdot 3^1 + 0 \cdot 3^0 = 10_{three}$

$1 \cdot 3^2 + 0 \cdot 3^1 + 0 \cdot 3^0 = 100_{three}$, etc.

Thus the first 10 counting numbers are: $\mathbf{(1,2,10,11,12,20,21,22,100,101)}_{\boldsymbol{three}}$.

(b) Place values represent powers of 8; e.g.,

$1 \cdot 8^0 = 1_{eight}$

$1 \cdot 8^1 + 0 \cdot 8^0 = 10_{eight}$

$1 \cdot 8^2 + 0 \cdot 8^1 + 0 \cdot 8^0 = 100_{eight}$, etc.

Thus the first 10 counting numbers are: $\mathbf{(1,2,3,4,5,6,7,10,11,12)}_{\boldsymbol{eight}}$.

16. **18** digits. One digit is needed for each of the units from 1 to 17, and one digit for 0.

17. $2022_{three} = \mathbf{(2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 2 \times 10^0)}_{\boldsymbol{three}}$. 2022_{three} may alternatively be expanded as $\mathbf{(2 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0)}_{\boldsymbol{ten}} = \mathbf{62}$.

18. **(a)** $\mathbf{222}_{\boldsymbol{three}}$. 2 is the largest units digit in base three.

(b) 666_{seven}. 6 is the largest units digit in base seven.

19. **(a)** **(*i*)** $(100 - 1)_{seven} = 1 \cdot 7^2 + 0 \cdot 7^1 + 0 \cdot 7^0 - 1 = 6 \cdot 7^1 + 6 \cdot 7^0 = \mathbf{66}_{\boldsymbol{seven}}$.

(*ii*) $(100 + 1)_{seven} = 1 \cdot 7^2 + 0 \cdot 7^1 + 0 \cdot 7^0 + 1 = \mathbf{101}_{\boldsymbol{seven}}$.

(b) **(*i*)** $(1000 - 1)_{two} = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 - 1 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = \mathbf{111}_{\boldsymbol{two}}$.

(*ii*) $(1000 + 1)_{two} = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 + 1 = \mathbf{1001}_{\boldsymbol{two}}$.

(c) **(*i*)** $(101 - 1)_{two} = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 - 1 = 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = \mathbf{100}_{\boldsymbol{two}}$.

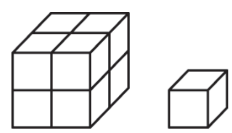
(*ii*) $(101 + 1)_{two} = 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 + 1 = \mathbf{110}_{\boldsymbol{two}}$.

20. **(a)** There is **no numeral 6** in base four. The numerals are 0, 1, 2, and 3.

(b) There are **no numerals 2 or 3** in base two. The numerals are 0 and 1.

21. There are 2 groups of 3^3, 2 groups of 3^2, 2 groups of 3^1, and 1 group of 3^0 in 79, so $79 = 2221_{three}$. Thus, where a block is $3 \times 3 \times 3$, a flat is 3×3, and a long is 1×3, **two** blocks, **two** flats, **two** longs, and **one** unit, so, **seven** is the fewest number of base three blocks.

22. 1001_{two} is represented by 1 block, 0 flats, 0 longs, and 1 unit, or



23. $277 = 1 \cdot 12^2 + 11 \cdot 12^1 + 1 \cdot 12^0$, or **1 gross, 11 dozens, and 1 unit**.

24. **(a)** $\boldsymbol{E0T9E}_{\boldsymbol{twelve}} < \boldsymbol{EET9E}_{\boldsymbol{twelve}}$. $EE000_{twelve}$ is greater than $E0000_{twelve}$.

(b) $\mathbf{101011}_{\boldsymbol{two}} < \mathbf{1011011}_{\boldsymbol{two}}$. Any seven-digit number is greater than any six-digit number.

(c) $\mathbf{50555}_{\boldsymbol{six}} < \mathbf{51000}_{\boldsymbol{six}}$. 1000_{six} is greater than 555_{six}.

25. **(a)** 10_{four} longs $= 40$ units, which can be represented by **2 flats and 2 longs**, or four pieces.

(b) 10_{three} longs $= 30$ units, which can be represented by **1 block and 1 long**, or two pieces.

26. **(a)** Place values in base 4 are $4^3 = 64, 4^2 = 16, 4^1 = 4,$ and $4^0 = 1$; the largest needed here is 4^3.

(***i***) There are 3 groups of 64 (or 4^3) in 234, with remainder 42;

(***ii***) There are 2 groups of 16 (or 4^2) in 42, with remainder 10;

(***iii***) There are 2 groups of 4 (or 4^1) in 10, with remainder 2;

(***iv***) There are 2 groups of 1 in 2, with remainder 0.

Thus $234 = \mathbf{3222}_{\boldsymbol{four}}$.

(b) Place values in base 12 are $12^3 = 1728, 12^2 = 144, 12^1 = 12,$ and $12^0 = 1$.

(***i***) There is 1 group of 1728 in 1876, with remainder 148;

(***ii***) There is 1 group of 144 in 148, with remainder 4;

(***iii***) There are 0 groups of 12 in 4, with remainder 4;

(***iv***) There are 4 groups of 1 in 4, with remainder 0.

Thus $1876 = \mathbf{1104}_{\boldsymbol{twelve}}$.

(c) Place values in base 3 are $3^5 = 243, 3^4 = 81, 3^3 = 27, 3^2 = 9, 3^1 = 3,$ and $3^0 = 1$.

(***i***) There is 1 group of 243 in 303, with remainder 60;

(***ii***) There are 0 groups of 81 in 60, with remainder 60;

(***iii***) There are 2 groups of 27 in 60, with remainder 6;

(***iv***) There are 0 groups of 9 in 6, with remainder 6;

(***v***) There are 2 groups of 3 in 6, with remainder 0;

(***vi***) There 0 groups of 1 in 0, with remainder 0.

Thus $303 = \mathbf{102020}_{\boldsymbol{three}}$.

(d) Place values in base 2 are $2^4 = 16, 2^3 = 8, 2^2 = 4, 2^1 = 2,$ and $2^0 = 1$.

(***i***) There is 1 group of 16 in 22, with remainder 6;

(***ii***) There are 0 groups of 8 in 6, with remainder 6;

(***iii***) There is 1 group of 4 in 6, with remainder 2;

(***iv***) There is 1 groups of 2 in 2, with remainder 0;

(***v***) There are 0 groups of 1 in 0, with remainder 0.

Thus $22 = \mathbf{10110}_{\boldsymbol{two}}$.

27. **(a)** $432_{six} = 4 \cdot 6^2 + 3 \cdot 6^1 + 2 \cdot 6^0 = \mathbf{164}$.

(b) $11011_{two} = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = \mathbf{27}$.

(c) $E29_{twelve} = 11 \cdot 12^2 + 2 \cdot 12^1 + 9 \cdot 12^0 = \mathbf{1617}$.

28. Using the place value in base eight,

8^3	8^2	8^1	1

we have $a \bullet 8^3 + b \bullet 8^2 + c \bullet 8^1 + d \bullet 1$.

29. To find out the base, add the units digits: $4_b + 5_b$. In base 10 that equals 9 but in base b it equals a numbers with units digit 2. Take out a group of b and have two left over. The only choice is base seven and then $4_7 + 5_7 = 12_7$ and

$$\begin{aligned} 54_7 + 45_7 &= (50_7 + 40_7) + (4_7 + 5_7) \\ &= (120_7) + (12_7) \\ &= 132_7 \end{aligned}$$

30. **(a)** Using the base sixteen place value:

$$\begin{aligned} 864_{sixteen} &= 8 \bullet 16^2 + 6 \bullet 16^1 + 4 \bullet 1 \\ &= 2{,}048 + 96 + 4 \\ &= 2{,}148 \end{aligned}$$

(b) Using the base sixteen place value:

$$\begin{aligned} 2CDE_{sixteen} &= 2 \bullet 16^3 + C \bullet 16^2 + D \bullet 16^1 + E \bullet 1 \\ &= 2 \bullet 16^3 + 12 \bullet 16^2 + 13 \bullet 16^1 + 14 \bullet 1 \\ &= 8{,}192 + 3072 + 208 + 14 \\ &= 11{,}486 \end{aligned}$$

31. To convert 256 from base ten to base sixteen, ask:
How many groups of 16^3(= 4096) are there in 256? None 4096>584.
How many groups of 16^2(= 256) are there in 256? 1 group ($1 \bullet 16^2 = 256$) with none left over (256 – 256 = 0). There are no groups of 16^1(= 16) and no groups of 16^0(= 1).
So $256_{10} = 100_{\text{sixteen}}$.

32. To give the fewest number of prizes, the dollar amount of each must be maximized. Thus $\$900 =$

1 prize of $512 with $388 left over;

1 prize of $256 with $132 left over;

1 prize of $128 with $4 left over, and

1 prize of $4, with nothing left over,

i.e., $\$900 = \$1110000100_{two}$. Thus, the minimum number of prizes is **four**.

33. There are $4 = 2^2$ cups per quart and $2 = 2^1$ cups per pint. Thus 1 cup, 1 pint, and 1 quart $= \mathbf{111_{two}}$.

34. **(a)** 7. $b3_{four} = 4b + 3 = 31$

$\Rightarrow 4b = 28 \Rightarrow b = 7.$

But there is no digit 7 in base 4 so this is not possible.

(b) Not possible. $1534_{six} = 418_{ten}$ and $1b2_{twelve} = (1 \cdot 10^2 + b + 2)_{twelve} = (1 \cdot 12^2 + b \cdot 12 + 2)_{ten} = 146 + 12b.$

Then $418 = 146 + 12b \Rightarrow b$ is not an integer, so no solution exists.

35. If each key may be used only once, the largest four-digit number would be **9876.**

36. Answers vary. For example,

$MID = 1000 + 500 - 1$.

$= 1499$

Assessment 3-2A: Addition and Subtraction of Whole Numbers

1. The student is adding 13 and 4.Start with thirteen and add one each time until seventeen then notice that four numbers were added.

2. **(a)** **True**.

$n(A) = 3; n(B) = 2; n(A) + n(B) = 5.$

$A \cup B = \{a, b, c, d, e\}; n(A \cup B) = 5.$

$\boldsymbol{n(A) + n(B) = n(A \cup B)}$ because the sets are disjoint.

(b) **False**.

$n(A) = 3; n(B) = 2; n(A) + n(B) = 5.$

$A \cup B = \{a, b, c\}; n(A \cup B) = 3.$

$\boldsymbol{n(A) + n(B) \neq n(A \cup B)}$ because the sets are not disjoint.

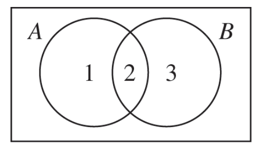
(c) **True**.

$n(A) = 3; n(B) = 0; n(A) + n(B) = 3.$

$A \cup B = \{a, b, c\}; n(A \cup B) = 3.$

$\boldsymbol{n(A) + n(B) = n(A \cup B)}$ because the sets are disjoint.

3. The sets are not disjoint because $n(A) + n(B) \neq n(A \cup B)$. $n(A \cap B)$ must therefore equal $n(A) + n(B) - n(A \cup B) = 3 + 5 - 6 = 2$. In the following Venn diagram:



$n(A) = 3, n(B) = 5, n(A \cup B) = 6$, and $\boldsymbol{n(A \cap B) = 2}$.

4. If A and B are not disjoint:

Let $A = \{1, 2\}, B = \{2, 3\}$, so $A \cup B = \{1, 2, 3\}$.

Then $n(A) = 2, n(B) = 2, n(A \cup B) = 3.$

But $n(A) + n(B) = 4 \neq 3 = n(A \cup B).$

5. **(a)** ***(i)*** $\boldsymbol{n(B) = 3}$ if the sets are disjoint.

(ii) $\boldsymbol{n(B) = 4}$ if $n(A \cap B) = 1.$

(iii) $\boldsymbol{n(B) = 5}$ if $n(A \cap B) = 2.$

(iv) $\boldsymbol{n(B) = 6}$ if $n(A \cap B) = 3.$

(b) If $n(A \cap B) = \emptyset$ the sets are disjoint and $\boldsymbol{n(B) = 3}$.

6. **(a)** **Closed**. $0 + 0 = 0$, and $0 \in \{0\}$.

(b) **Closed**. Assuming the arithmetic sequence $0, 3, 6, \ldots,$ any element of T added to any other element in T is whole and divisible by 3.

(c) **Closed**. N is the set of natural numbers and any natural number added to any other natural number is an element of N.

(d) **Not Closed**. $3 + 7 \notin \{3, 5, 7\}$.

(e) **Closed**. $\{W\}$ is the set of whole numbers and any whole number greater than 10 added to any other whole number greater than 10 is an element of W.

(f) **Not Closed.** $1 + 1 \notin \{0, 1\}$.

7. Answers may vary. A possible answer is
$2 + 2 = \mathbf{4}$,
$4 + 2 = \mathbf{6}$, $4 + 4 = \mathbf{8}$, $2 + 3 = \mathbf{5}$, $5 + 2 = \mathbf{7}$,
and $7 + 2 = \mathbf{9}$.

8. If set A is closed under addition then all whole numbers greater than or equal to 2 are in A. The only possible missing numbers are 0 and 1.

9. **(a)** 1, commutative property of addition.
(b) 7, commutative property of addition.
(c) 0, additive identity.
(d) 7, associative and commutative properties of addition.

10. **(a)** **Commutative property of addition**; i.e., if a and b are any whole numbers, then $a + b = b + a$.
(b) **Associative property of addition**; i.e., if $a, b,$ and c are any whole numbers, then $(a + b) + c = a + (b + c)$.
(c) **Commutative property of addition**; i.e., $(6 + 3) = (3 + 6)$.
(d) **Identity property of whole numbers**; i.e., for any whole number $a, a + 0 = a$.
(e) **Commutative property of addition**.
(f) **Associative property of addition**.
(g) **Closure property of addition**.

11. **No**. If $k = 0$ (a whole number) and $a = b = 0$, then the $a = b + k \Rightarrow a < b$, which would imply that $0 < 0$, a contradiction.

12. **(*i*)** For any whole numbers a and $b, a < b$ if and only if there exists a natural number k such that $b - k = a$ (or equivalently, if and only if $b - a$ is a natural number).
(*ii*) For any whole numbers a and $b, a > b$ if and only if there exists a natural number k such that $a - k = b$ (or equivalently, if and only if $a - b$ is a natural number).

13. **(a)** Each term is found by adding 5 to the previous term. Thus the next three are $28 + 5 = \mathbf{33}$, $33 + 5 = \mathbf{38}$, $38 + 5 = \mathbf{43}$.
(b) Each term is found by subtracting 7 from the previous term. Thus the next three are $63 - 7 = \mathbf{56}$, $56 - 7 = \mathbf{49}$, $49 - 7 = \mathbf{42}$.

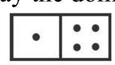
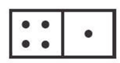
14. In a magic square each row, column, and diagonal must have the same sum. $8+5+2=15$ so each row, column, and diagonal must add to 15.

8	**3**	**4**
1	**5**	**9**
6	**7**	**2**

or

8	**1**	**6**
3	**5**	**7**
4	**9**	**2**

15. **(a)** **9**. A number greater than 9 would have two digits.
(b) **8**. If A were greater, C would have two digits.
(c) **3**. A and B must be 1 and 2 (in any order); no lesser single digit numbers are available.
(d) **6 or 8**. A and B must be 2 and 4 or 2 and 6; greater even numbers would make C have two digits.
(e) **5**. If $A + B = C$ and C is 5 more than A, then $A + 5 = C$.
(f) **4 or 8**. B could be 1 and A must then be 3; or B could be 2 and then A must be 6.
(g) **9**. B must be 2 and A must be 7.

16. Since the way the domino is positioned doesn't matter, i.e.,  is the same domino as , each number put on the left side gets paired with each of the 9 choices for the right

Number printed on left	Number of choices for right
0	9
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1

Sum the right column and we have $\frac{10(9)}{2} = 45$ dominos.

17. **(a)** Order the players as follows:
$\boxed{\text{Kent}} < \boxed{\text{Mischa}} < \boxed{\text{Sally}} < \boxed{\text{Vera}}$
Kent is shortest; Vera is tallest.

(b) Answers may vary, as long as the player's heights increase in the order:

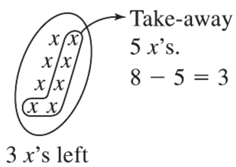
Kent $\rightarrow$ Mischa $\rightarrow$ Sally $\rightarrow$ Vera.

18. **(a)** Add 7 to each side: $\boldsymbol{9 = 7 + x}$.

(b) Add 6 to each side: $\boldsymbol{x = 3 + 6}$

(c) Add x to each side: $\boldsymbol{9 = x + 2}$.

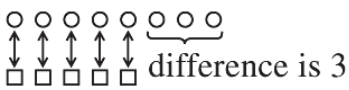
19. **(a)** Take away:



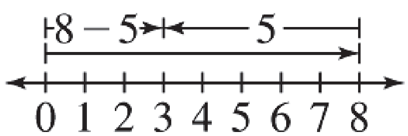
(b) Missing addend:

$\square + 5 = 8 \Rightarrow \square = 3$.

(c) Comparison:



(d) Number line:



20. **(a)** **Answers vary.** For example, "Kaden has 12 cookies and Tristan takes away 7 cookies. How many cookies does Kaden have left?"

(b) **Answers vary.** For example, "Jasmine has 7 yards of fabric and makes a blanket for her sister Ebony with 3 yards of **fabric. How much fabric is left over?"**

(c) **Answers vary.** For example, "Luke has seventeen marbles. His brother Zach has 12 marbles. How many more marbles does Luke have than Zach?"

(d) **Answers vary.** For example, "Isabelle has \$25 and is saving for a video game called Minecraft. If the game costs \$30, how much more does Isabelle need to save?"

21. **(a)** Solve for x:

i. $15 - 9 = 15 - 10 + x$

$6 = 5 + x$

$1 = x$

$27 - 9 = 27 - 10 + x$

ii. $18 = 17 + x$

$1 = x$

$32 - 9 = 32 - 10 + x$

iii. $23 = 22 + x$

$1 = x$

(b) A strategy for subtracting 9 from a whole number would be subtracting 10 then adding 1.

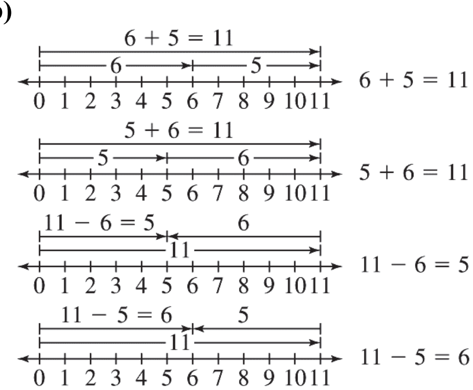
22. **(a)** The other three members of the fact family are:

$5 + 6 = 11$

$11 - 6 = 5$

$11 - 5 = 6$

(b)



23. **(a)** $\boldsymbol{a \geq b}$; if $b > a$ then $a - b$ would not be a whole number.

(b) $a - (b - c) \geq 0 \Rightarrow a \geq (b - c) \Rightarrow$ $a + c \geq b$ and $b \geq c$ so that each difference is a whole number.

24. **(a)** By the associative property of addition, $x + (y + z) = (x + y) + z$. By the commutative property of addition**,** $(x + y) + z = z + (x + y)$.

(b) By the commutative property of addition, $x + (y + z) = (y + z) + x$.

By the associative property of addition, $(y + z) + x = y + (z + x)$.

By the commutative property of addition, $y + (z + x) = y + (x + z)$.

25. **(a)** $3 + (4 + 7) = (3 + x) + 7$
$(3 + 4) + 7 = (3 + x) + 7$
$3 + 4 = 3 + x$
$\mathbf{4 = x}$.

(b) $8 + 0 = x \Rightarrow \boldsymbol{x = 8}$.

(c) $5 + 8 = 8 + x$
$5 + 8 = x + 8$
$\mathbf{5 = x}$.

(d) $x + 8 = 12 + 5$
$x + 8 = 17$
$x + 8 - 8 = 17 - 8$
$\boldsymbol{x = 9}$.

(e) $x + 8 = 5 + (x + 3)$
$x + 8 = x + 8$
$\boldsymbol{x} = x$.
The solution is all whole numbers.

(f) $x - 2 = 9$
$\boldsymbol{x = 11}$.

(g) $\boldsymbol{x} - 3 = \boldsymbol{x} + 1$
$-3 = 1$
There are no solutions
in set of whole numbers.

(h) $0 + x = x + 0$
$\boldsymbol{x = x}$.
All whole numbers
are solutions.

26. Kelsey has $\boldsymbol{a - (b + c)}$ more marbles than Gena and Noah combined.

27. Yes. Both sets are closed with respect to addition. Each set represents the multiples of 5. A multiple of 5 plus a multiple of 5 is always a multiple of 5.

Assessment 3-2B

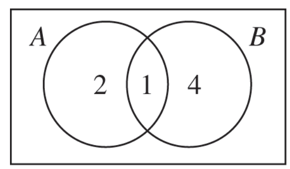
1. The student is adding 15 and 3.Start with fifteen and add one each time until eighteen then notice that three numbers were added.

2. **(a)** **True**.
$n(A) = 2; n(B) = 2; n(A) + n(B) = 4$.
$A \cup B = \{a, b, d, e\}; n(A \cup B) = 4$.
$\boldsymbol{n(A) + n(B) = n(A \cup B)}$ because the sets are disjoint.

(b) **False**.
$n(A) = 3; n(B) = 3; n(A) + n(B) = 6$.
$A \cup B = \{a, b, c, d\}; n(A \cup B) = 4$.
$\boldsymbol{n(A) + n(B) \neq n(A \cup B)}$ because the sets are not disjoint.

(c) **True**.
$n(A) = 1; n(B) = 0; n(A) + n(B) = 1$.
$A \cup B = \{a\}; n(A \cup B) = 1$.
$\boldsymbol{n(A) + n(B) = n(A \cup B)}$ because the sets are disjoint.

3. The sets are not disjoint because $n(A \cap B) \neq 0$. $n(A \cup B)$ must therefore equal $n(A) + n(B) - n(A \cap B) = 3 + 5 - 1 = \mathbf{7}$. In the following Venn diagram:



$n(A) = 3, n(B) = 5, n(A \cap B) = 1$, and
$\boldsymbol{n(A \cup B) = 7}$.

4. $A - B =$ all the elements in A that are not in B. Then if $n(A) - n(B)$ is defined as $n(A - B)$ there can be no elements in B that are also not in A. So, that would mean that $n(A - B) = 0$, or $A \cap B = B$.

5. **(a)** Since $n(A \cup B) = 6$ and $n(B) = 4$, A could contain 2 elements not in B or A could contain some or all of the elements of B. For example, if $B \subset A$, then A would have 6 elements. So there could be **2, 3, 4, 5, or 6** elements in A.

(b) If $n(A \cap B) = 0$, then $A \cap B = \emptyset$. So A would have only **2** elements.

6. **(a)** **Not Closed**. $1 + 1 = 2$, and $2 \notin \{0, 1\}$.

(b) **Closed**. Assuming the arithmetic sequence $0, 4, 8, \ldots$, any element of T added to any other element in T is whole and divisible by 4.

(c) **Closed.** Any element of E added to any other element of E is > 5 and is thus $\in E$.

(d) **Closed**. W is the set of whole numbers and any whole number greater than 100 added to any other whole number greater than 100 is $\in W$.

7. **(a)** Answers may vary. For example, if $A = \{2, 5, 7, 8, 9, 10, 11, 12, 13, \ldots\}$, then possible sums of 2, 5, and 8 would be elements of the set.

(b) The set must contain **the rest of the whole numbers greater than 1** to be closed.

8. If set A is closed under addition and 2 and 4 are included then all other even whole numbers must be included for the set to be closed.

9. **(a)** $3 + 4 = \mathbf{4} + 3$; **commutative property of addition.**

(b) $5 + (4 + 3) = (4 + 3) + \mathbf{5}$; **commutative property of addition**.

(c) $8 + \mathbf{0} = 8$; **identity property of addition**.

(d) $3 + (4 + 5) = (3 + \mathbf{4}) + 5$; **associative property of addition**.

10. **(a)** **Commutative property of addition**; i.e., if a and b are any whole numbers, then $a + b = b + a$.

(b) **Identity property of whole numbers**; i.e., for any whole numbers $a + b, (a + b) + 0 = a + b$.

(c) **Commutative property of addition**; i.e., $(6 + 8) = (8 + 6)$.

(d) **Associative property of addition**; i.e., if $a, b,$ and c are any whole numbers, then $(a + b) + c = a + (b + c)$.

11. In this chapter we are only considering whole numbers a and b. So k must be a **whole** number.

12. $a \geq b$ if and only if $a - b$ is a whole number. Alternative: $a \geq b$ if and only if $a - k = b$ for some whole number k.

13. **(a)** Each term is found by adding 7 to the previous term. Thus the next three are $33 + 7 = \mathbf{40}$, $40 + 7 = \mathbf{47}$, $47 + 7 = \mathbf{54}$.

(b) Each term is found by subtracting 4 from the previous term. Thus the next three are $47 - 4 = \mathbf{43}$, $43 - 4 = \mathbf{39}$, $39 - 4 = \mathbf{35}$.

14. In a magic square each row, column, and diagonal must have the same sum. $6 + 1 + 8 = 15$ so each row, column, and diagonal must add the 12.

6	**7**	**2**
1	**5**	**9**
8	**3**	**4**

15. **(a)** **1**. Addition of any two different single-digit numbers will result in a sum no greater than 17.

(b) **No**. $C = 1$ and D must be different.

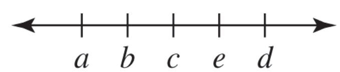
(c) **9 or 8**. If A is 9, B must be 8; if A is 8, B must be 9.

(d) **2**. A must be 9; B must be 3.

16. **(a)** The number 0 can be paired with all digits from 0 to 6 (7 pairs), the number 1 can additionally be paired with all digits from 1 to 6 (6 pairs), then 2 can be paired with all digits from 2 to 6, . . ., 6 is paired with 6. The total of unique pairings is thus $7 + 6 + 5 + 4 + 3 + 2 + 1 = \mathbf{28}$.

(b) If the domino is horizontal and the sum of the dots in the right square plus the sum of the dots in the left square is known, and the domino is then turned 180°, the sum is the same.

17. A strategy would be to construct a number line with the information.



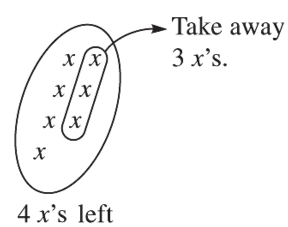
So, $\boldsymbol{a, b, c, e, d}$ are ordered from least to greatest.

18. **(a)** Add 3 to each side: $\mathbf{9 = x + 3}$.

(b) Add 5 to each side: $\mathbf{x = 8 + 5}$.

(c) Add x to each side: $\mathbf{11 = 2 + x}$.

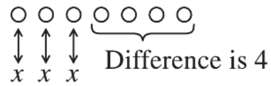
19. **(a)** Take-away:



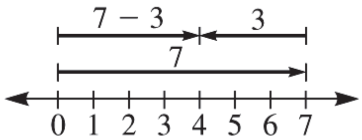
(b) Missing addend:

$\square + 3 = 7 \Rightarrow \square = 4$

(c) Comparison:



(d) Number line:



20. **(a)** **Answers vary.** For example, "Kaden has 12 cookies and Tristan takes away 7 cookies. How many cookies does Kaden have left?"

(b) **Answers vary.** For example, "Jasmine has 7 yards of fabric and makes a blanket for her sister Ebony with 3 yards of **fabric. How much fabric is left over?"**

(c) **Answers vary.** For example, "Luke has seventeen marbles. His brother Zach has 12 marbles. How many more marbles does Luke have than Zach?"

(d) **Answers vary.** For example, "Isabelle has $25 and is saving for a video game called Minecraft. If the game costs $30, how much more does Isabelle need to save?"

21. Using the doubles plus strategy:

(a)
$$\begin{aligned}11+7&=7+11\\&=7+(7+4)\\&=(7+7)+4\\&=14+4\\&=18\end{aligned}$$

(b)
$$\begin{aligned}24+28&=24+(24+4)\\&=(24+24)+4\\&=48+4\\&=52\end{aligned}$$

(c)
$$\begin{aligned}35+37&=35+(35+2)\\&=(35+35)+2\\&=70+2\\&=72\end{aligned}$$

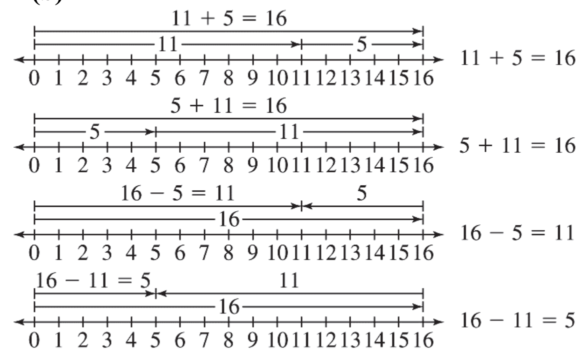
22. **(a)** The other three members of the fact family are:

$5+11=16$

$16-11=5$

$16-5=11$

(b)



23. **(a)** For $b - a$ to be a whole number, $\boldsymbol{b \geq a}$.

(b) If we assert that the expression $a - 3$ must be a whole number, then $\boldsymbol{a \geq 3}$. Then, b must be greater than or equal to $a - 3$, $\boldsymbol{b \geq a - 3}$.

24. **(a)** $a + (b + c) = (a + b) + c$: Associative property of addition.

$(a + b) + c = c + (a + b)$: Commutative property of addition.

(b) $a + (b + c) = a + (c + b)$: Commutative property of addition.

$a + (c + b) = (c + b) + a$: Commutative property of addition.

25. **(a)**
$$\begin{aligned}&12 - x = x + 6\\&12 - x + x = x + x + 6\\&12 = 2x + 6\\&12 - 6 = 2x + 6 - 6\\&6 = 2x \Rightarrow \tfrac{6}{2} = \tfrac{2}{2}x \Rightarrow \boldsymbol{x = 3}.\end{aligned}$$

(b)
$$\begin{aligned}&9 - x - 6 = 1\\&3 - x = 1\\&3 - x + x = 1 + x\\&3 = 1 + x\\&3 - 1 = 1 - 1 + x \Rightarrow \boldsymbol{x = 2}.\end{aligned}$$

(c) $3 + x = x + 3$

$x + 3 = x + 3 \Rightarrow$

All whole numbers.

(d) $11 - x = 0$

$11 - x + x = 0 + x$

$11 = x \Rightarrow \boldsymbol{x = 11}$

(e) $14 - x = 7 - x$

$14 - x + x = 7 - x + x$

$14 = 7 \Rightarrow$ **No solution**.

(f) $x - 3 = 17$

$x - 3 + 3 = 17 + 3$

$\boldsymbol{x = 20}$

(g) $x + 3 = x - 1$

$x + 3 - x = x - 1 - x$

$3 = -1 \Rightarrow$ **No solution**.

(h) $0 + x = x - 0$

$x = x \Rightarrow$ **All whole numbers**.

26. The number of Kelly's pencils must be subtracted from the number of Rob's pencils, or ***(iii)***, $11 - 5 = 6$.

27. Yes. Both sets are closed with respect to addition. Each set represents the multiples of 7. A multiple of 7 plus a multiple of 7 is always a multiple of 7.

Mathematical Connections 3-2: Review Problems

25. In base 10, $5280 = \mathbf{5 \cdot 10^3 + 2 \cdot 10^2 + 8 \cdot 10^1 + 0 \cdot 10^0}$.

26. **1410**. $\boldsymbol{M} = 1000, \boldsymbol{CD} = 500 - 100$, and $\boldsymbol{X} = 10$.

27. **(a)** $E0T_{twelve} = 11 \cdot 12^2 + 0 \cdot 12^1 + 10 \cdot 12^0 = \mathbf{1594}$.

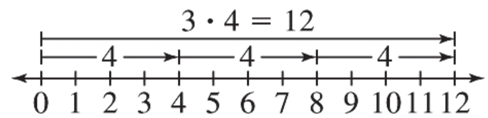
(b) $1011_{two} = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = \mathbf{11}$.

(c) $43_{five} = 4 \cdot 5^1 + 3 \cdot 5^0 = \mathbf{23}$.

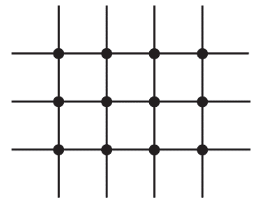
28. $12^4 + 12^2 + 13 = 1 \cdot 12^4 + 0 \cdot 12^3 + 1 \cdot 12^2 + 1 \cdot 12 + 1 = \mathbf{10111_{twelve}}$.

Assessment 3-3A: Multiplication and Division of Whole Numbers

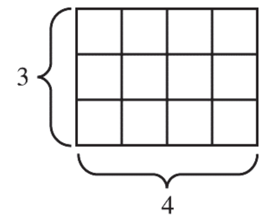
1. **(a)**



(b)



(c)



(d) $A = \{a, b, c\}$

$B = \{w, x, y, z\}$

$A \times B = \{(a,w),(a,x),(a,y),(a,z),$
$(b,w),(b,x),(b,y),(b,z),$
$(c,w),(c,x),(c,y),(c,z)\}$

$n(A) = 3, n(B) = 4$

$n(A \times B) = 3 \cdot 4 = 12.$

2. **(a)** $A = \{a, b\}$

$B = \{x, y, z\}$

$A \times B = \{(a,x),(a,y),(a,z),$
$(b,x),(b,y),(b,z)\}$

(b) $n(A) = 2, n(B) = 3$

$n(A \times B) = 2 \cdot 3 = 6$

(c) $2 \cdot 3 = 6$ **.**

3. **(a)** Illustrated is 4 groups of 2 xs: $\mathbf{4 \cdot 2} = 8$.

(b) Illustrated is a 4 by 2 array or a 2 by 4 array: $\mathbf{2 \cdot 4 = 8}$ **or** $\mathbf{4 \cdot 2 = 8}$.

4. **(a)** Use the repeated addition model: $3 + 3 + 3 + 3 + 3 = 15$; there are five threes, so $3 \cdot \boxed{5} = 15$.

(b) $18 - 6 = 6 - 6 + 3 \cdot \square$ (note the order of operations specifying that $6 + 3$ is not permitted) $\Rightarrow 12 = 3 \cdot \square$. Now use the repeated addition model: $3+3+3+3=12$; there are four threes, so $18 = 6 + 3 \cdot \boxed{4}$.

(c) The distributive property of multiplication over addition, where n is **any whole number,** specifies that $a(b + c) = ab + ac$. Thus $\boxed{n} \cdot (5 + 6) = \boxed{n} \cdot \mathbf{5} + \boxed{n} \cdot \mathbf{6}$.

5. **(a)** **Closed**. $0 \cdot 0 = 0, 0 \cdot 1 = 0, 1 \cdot 0 = 0,$ and $1 \cdot 1 = 1$ are all products contained in $\{0,1\}$.

(b) **Closed**. The product of any two even numbers is also an even number.

(c) **Closed**. Since the set can be written as $\{3n+1 \mid n \in W\}$, for any whole numbers m and n, $(3m+1)(3n+1) = 3(3mn+m+n)+1$ is in the set since the whole numbers are closed under multiplication and addition.

6. **(a)** **No**. $2 + 3 = 5$.

(b) **Yes**. There will be no numbers in the set that will multiply to give a product of 5.

7. **(a)** Applying the distributive property gives $(a + b)(c + d) = a(c + d) + b(c + d) = \boldsymbol{ac + ad + bc + bd}$.

(b) $\square(\Delta + \mathrm{O}) = \square \cdot \boldsymbol{\Delta} + \square \cdot \mathbf{O}$.

(c) $a(b + c) - ac = \boldsymbol{ab + ac - ac} = ab$.

8. **(a)** $\mathbf{(5 + 6) \cdot 3} = 33$. Without parentheses, the result would be 23.

(b) **No parentheses** are needed; the order of operations specifies that addition and subtraction are performed in order from left to right.

(c) **No parentheses** are needed; the order of operations specifies that division is performed before addition or subtraction.

(d) $\mathbf{(9 + 6) \div 3} = 5$. Without parentheses, the result would be 11.

9. **(a)** $xy + y^2 = \boldsymbol{y(x + y)}$, factoring y from each term.

(b) $xy + x = \boldsymbol{x(y + 1)}$, factoring x from each term.

(c) $a^2b + ab^2 = \boldsymbol{ab(a + b)}$, factoring ab from each term.

10. **(a)** $18 \div 3 = \boxed{6}$ (since $3 \cdot 6 = 18$).

(b) $\boxed{0} \div 76 = 0$ (since $0 \cdot 76 = 0$).

(c) $28 \div \boxed{4} = 7$ (since $7 \cdot 4 = 28$).

11. Use the Cartesian-product model, where s is shirts, p is pants, and v is vests. Then $n(s) = 6$, $n(p) = 4$, and $n(v) = 3$. The Fundamental Counting Principle indicates that the number of ordered triplets in $s \times p \times v = n(s) \cdot n(p) \cdot n(v) = 6 \cdot 4 \cdot 3 = \mathbf{72}$ **possible outfits**.

12. **(a)** Associative property of multiplication of whole numbers; i.e., $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(b) Commutative property of multiplication of whole numbers; i.e., $a \cdot b = b \cdot a$.

(c) Commutative property of multiplication.

(d) Identity property of multiplication of whole numbers; i.e., $1 \cdot a = a$.

(e) Zero multiplication property of whole numbers; i.e., $a \cdot 0 = 0$.

(f) Distributive property of multiplication over addition for whole numbers; i.e., $(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$.

13. **(a)** Closure property of multiplication of whole numbers; i.e., for any whole numbers a and b, $a \cdot b$ is a *unique* whole number.

(b) Zero multiplication property of whole numbers; i.e., $a \cdot 0 = 0$.

(c) Identity property of multiplication of whole numbers; i.e., $a \cdot 1 = a$.

14. **(a)** Distributive property of multiplication over addition for whole numbers; i.e., $a(b + c) = ab + ac$.

(b) ***(i)*** $12 \cdot 32 = 12(30 + 2) = 12 \cdot 30 + 12 \cdot 2 = 360 + 24 = \mathbf{384}$, or

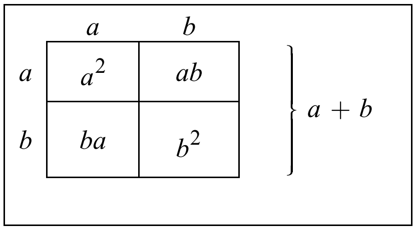
(ii) $32 \cdot 12 = 32(10 + 2) = 32 \cdot 10 + 32 \cdot 2 = 320 + 64 = \mathbf{384}$.

15. **(a)** $9(10 - 2) = 9 \cdot 10 - 9 \cdot 2 = 90 - 18 = \mathbf{72}$.

(b) $20(8 - 3) = 20 \cdot 8 - 20 \cdot 3 = 160 - 60 = \mathbf{100}$.

16. **(a)** $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2.$

(b)



The area of a square with sides $a+b$ can be expressed as $(a+b)\cdot(a+b)$, and also as the sum of areas of the four regions: two squares, $a\cdot a$ and $b\cdot b$, and two rectangles $a\cdot b$ and $b\cdot a$. Thus:

$$(a+b)^2 = a\cdot a + a\cdot b + b\cdot a + b\cdot b$$
$$= a^2 + 2ab + b^2.$$

17. The question is really to show that the area of the large square minus the area of the small square is the same as the area of the four rectangles.

The area of the large square is $(a+b)^2$. The area of the small square is $(a-b)^2$. The difference between the two areas is the four rectangles, each with area ab; the total area of the four is $4ab$.

Therefore $(a+b)^2 - (a-b)^2 = 4ab.$

18. **(a)** $51^2 = (50+1)(50+1)$
$= 50^2 + 2{\bullet}50{\bullet}1 + 1^2$
$= 2500 + 100 + 1 = 2601$

(b) $102^2 = (100+2)(100+2)$
$= 100^2 + 2{\bullet}100{\bullet}2 + +2^2$
$= 10{,}000 + 400 + 4 = 10{,}404$

19. **(a)** ***(i)*** $(ab)c = c(ab)$ by the commutative property of multiplication of whole numbers.

(ii) $c(ab) = (ca)b$ by the associative property of multiplication of whole numbers.

(b) ***(i)*** $(a+b)c = c(a+b)$ by the commutative property of multiplication.

(ii) $c(a+b) = c(b+a)$ by the commutative property of addition.

20. To factor is to reverse the process of the distributive property of multiplication over addition or subtraction. The result is to find the factors which, when multiplied together, will yield a given product.

(a) $xy - y^2 = \boldsymbol{y(x-y)}$; i.e., if y and $x-y$ were to be multiplied, using the distributive property of multiplication over subtraction, the product would be $xy - y^2$.

(b) $47\cdot 101 - 47 = \mathbf{47(101-1)}$. i.e., 47 is a factor of both $47\cdot 101$ and $47\cdot 1$.

(c) $ab^2 - ba^2 = \boldsymbol{ab(b-a)}$.

21. **(a)** $40 \div 8 = 5 \Rightarrow \mathbf{40 = 8\cdot 5}$.

(b) $326 \div 2 = x \Rightarrow \mathbf{326 = 2\cdot} \boldsymbol{x}$.

22. If $108/a = b$ then $108 = a{\bullet}b$ and $108/b = a$.

23. The complete fact family:

$72/8 = 9$
$72/9 = 8$
$8{\bullet}9 = 72$
$9{\bullet}8 = 72$

24. The process always results in the original numbers.

Step 1: "Think of a number." Name this number x.
Step 2: $5x$
Step 3: $5x+5$
Step 4: $\frac{5x+5}{5} = x+1$
Step 5: $x+1-1 = x$.

25. Answers may vary, but for example:

(a) There is no associative property in division; e.g., $(8\div 4)\div 2 \neq 8\div(4\div 2)$.

(b) There is no distributive property in division; e.g., $8\div(2+2) \neq (8\div 2)+(8\div 2)$.

26. Suppose there are two bags of marbles containing a marbles in one and b marbles in the other. It is desired to divide the marbles equally among c boys. Then the number of marbles that each boy receives can be found in two different ways:

Put all the marbles in one bag. There will be $a+b$ marbles and each boy will receive $(a+b)\div c$ marbles.

Divide the marbles in the first bag first and then the second. Each boy would receive $(a \div c) + (b \div c)$ marbles.

Let $a \div c = x$ and $b \div c = y$.
Then $a = cx$ and $b = cy$.
So $a + b = cx + cy = c(x + y)$.
By the definition of division:
$$(x + y) = (a + b) \div c.$$
Substituting for x and y:
$$(a \div c) + (b \div c) = (a + b) \div c.$$

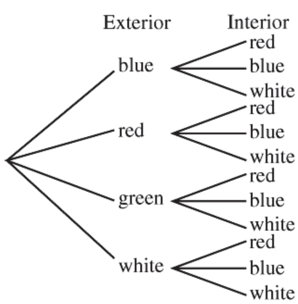
27. **(a)** $5x + 2 = 22 \Rightarrow 5x + 2 - 2 = 22 - 2 \Rightarrow 5x = 20 \Rightarrow \frac{5}{5}x = \frac{20}{5} \Rightarrow \boldsymbol{x = 4}$.

(b) $3x + 7 = x + 13 \Rightarrow 3x - x + 7 = x - x + 13 \Rightarrow 2x + 7 = 13 \Rightarrow 2x + 7 - 7 = 13 - 7 \Rightarrow 2x = 6 \Rightarrow \frac{2}{2}x = \frac{6}{2} \Rightarrow \boldsymbol{x = 3}$.

(c) $3(x + 4) = 18 \Rightarrow 3x + 12 = 18 \Rightarrow 3x + 12 - 12 = 18 - 12 \Rightarrow 3x = 6 \Rightarrow \frac{3}{3}x = \frac{6}{3} \Rightarrow \boldsymbol{x = 2}$.

(d) $(x - 5) \div 10 = 9 \Rightarrow \left(\frac{x-5}{10}\right)10 = 9(10) \Rightarrow x - 5 = 90 \Rightarrow x - 5 + 5 = 90 + 5 \Rightarrow \boldsymbol{x = 95}$.

28. Answers may vary; one possibility is:



Resulting in $4 \cdot 3 = \mathbf{12}$ **color schemes**.

29. **Yes**. The natural numbers must be $14, 24, 34, \ldots, 94$ to leave a remainder of 4 when divided by 10. Then $\mathbf{64} \div 47 = 1$ with a remainder of 17.

30. There were 10 teams with 12 on each team. So, there were $10(12) = 120$ people. Divide them into teams of 8 people and there are $120 \div 8 = \mathbf{15}$ **teams**.

31. **(a)** **Subtract 18 from 45**. Order of operations specifies operations within parenthese first, then multiplication before addition or subtraction.

(b) **Divide 54 by 9**. Order of operations specifies operations within parentheses first.

(c) **Add 11 to 48**. Operations within parentheses first.

(d) **Add 8 to 61**. Multiplication or division before addition.

32. These numbers will be multiples of 4 plus 1, $\{4n + 1 | n \in W\}$ or $\{1, 5, 9, 13, \ldots\}$.

33. **(a)** **Yes**, since only elements at $\{a, b, c\}$ appear in the table.

(b) **Yes**, $a \odot c = c \odot a$; $a \odot b = b \odot a$; and $b \odot c = c \odot b$.

(c) **Yes**, the first row and first column tell us that $a \odot b$, for example, is b.

(d) Answers vary. For example $(a \odot b) \odot c = b \odot c = a$ and $a \odot (b \odot c) = a \odot a = a$. All combinations would have to be checked to prove the operation is associative.

34. If $a \# b = a + b + 4$ then, for example, $3 \# 2 = 3 + 2 + 4 = 9$ and $2 \# 3 = 2 + 3 + 4 = 9$, so it is commutative.
In general $a \# b = a + b + 4 = b + a + 4 = b \# a$.

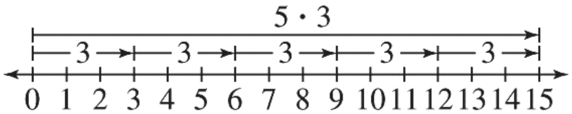
35. **(a)** The number of people who have entered after the 25[th] ring of the bell is the sum of the first 25 terms. It is an arithmetic sequence 1, 3, 5, … with the n[th] term equal to $2n$-1. So the 25[th] term is $2(25) - 1 = 49$. And the sum is $1 + 3 + 5 + ... + 49 = 625$. So 625 people entered.

(b) After the n^{th} ring
$$\frac{(1 + (2n - 1)) \bullet n}{2} = \frac{(2\text{n}) \bullet \text{n}}{2} = \frac{2n^2}{2} = n^2$$ people have entered.

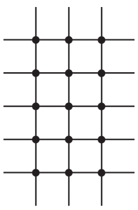
(c) After $n^2 = 1000 \Rightarrow n = \sqrt{1000} \Rightarrow n \approx 32$. After 32 rings there will be at least 1000 people.

Assessment 3-3B

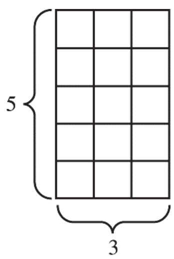
1. (a)



(b)



(c)



(d) $A = \{a, b, c, d, e\}$

$B = \{x, y, z\}$

$A \times B = \{(a, x), (a, y), (a, z),$
$(b, x), (b, y), (b, z),$
$(c, x), (c, y), (c, z),$
$(d, x), (d, y), (d, z),$
$(e, x), (e, y), (e, z)\}$

$n(A) = 5, n(B) = 3$

$n(A \times B) = 5 \cdot 3 = 15.$

2. (a) $A = \{a, b, c, d\}$

$B = \{x, y, z\}$

$A \times B = \{(a, x), (a, y), (a, z),$
$(b, x), (b, y), (b, z),$
$(c, x), (c, y), (c, z),$
$(d, x), (d, y), (d, z)\}$

(b) $n(A) = 4, n(B) = 3$

$n(A \times B) = 4 \cdot 3 = 12$

(c) $4 \cdot 3 = 12$.

3. (a) Illustrated is 4 groups of 3 xs: $4 \cdot 3 = 12$.

(b) Illustrated is a 3 by 7 array or a 7 by 3 array: $3 \cdot 7 = 21$ *or* $7 \cdot 3 = 21$.

4. (a) Use the repeated addition model: $8 + 8 + 8 = 24$; there are three eights, so $8 \cdot \boxed{3} = 24$.

(b) $28 - 4 = 4 - 4 + 6 \cdot \boxed{}$ (note the order of operations indicates that multiplication is performed before addition, so that $4 + 6$ is not permitted) $\Rightarrow 24 = 6 \cdot \boxed{}$. Now use the repeated addition model: $6 + 6 + 6 + 6 = 24$; there are four sixes, so $28 = 4 + 6 \cdot \boxed{4}$.

(c) The distributive property of multiplication over addition, where n is any whole number, specifies that $a(b + c) = ab + ac$. Thus

$\boxed{\boldsymbol{n}} \cdot \mathbf{(8 + 6)} = \boxed{\boldsymbol{n}} \cdot \mathbf{8} + \boxed{\boldsymbol{n}} \cdot \mathbf{6}$.

5. (a) **No**, $2 \cdot 2 = 4$ which is not in $\{1, 2\}$.

(b) **Yes**, the expression $2k + 1, k \in W$, represents any odd number.

Let $2\ell + 1$ be another odd number, then $(2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$ which is an odd number.

(c) **Yes**, a power of 2 multiplied by a power of 2 is still a power of 2.

6. (a) **No**, $1 + 1 = 2$.

(b) **Yes**, the product of two whole numbers greater than 1 is greater than one and still a whole number.

7. (a) $3(x + y + 5) = 3x + 3y + 3 \cdot 5 = \mathbf{3}\boldsymbol{x} + \mathbf{3}\boldsymbol{y} + \mathbf{15}$.

(b) $(x + y)(x + y + z) = x(x + y + z) + y(x + y + z) = xx + xy + xz + yx + yy + yz = \boldsymbol{x^2 + 2xy + xz + y^2 + yz}$. (Note: $xy = yx$.)

(c) $x(y + 1) - x = \boldsymbol{xy + x - x} = xy$.

8. (a) $\mathbf{(4 + 3) \cdot 2 = 14}$. Without parentheses the result would be $4 + 6 = 10$.

(b) $\mathbf{9 \div 3 + 1 = 4}$. Parentheses are unnecessary.

(c) $\mathbf{(5 + 4 + 9) \div 3 = 6}$. Without parentheses the result would be $5 + 4 + 3 = 12$.

(d) $\mathbf{3 + 6 - 2 \div 1 = 7}$. Parentheses are unnecessary.

9. **(a)** $47 \cdot 99 + 47 = 47 \cdot 99 + 47 \cdot 1 = \mathbf{47 \cdot (99 + 1)}$

(b) $(x + 1)y + (x + 1) = (x + 1)y + (x + 1)1 = \mathbf{(x + 1)(y + 1)}$

(c) $x^2y + zx^3 = \boldsymbol{x^2(y + zx)}$

10. **(a)** $27 \div 9 = \boxed{3}$ (since $3 \cdot 9 = 27$).

(b) $\boxed{52} \div 52 = 1$ (since $1 \cdot 52 = 52$).

(c) $13 \div \boxed{1} = 13$ (since $13 \cdot 1 = 13$).

11. 5 exterior colors $\times 3$ interior colors $\Rightarrow 5 \cdot 3 =$ **15** possibilities.

12. **(a)** Zero multiplication property of whole numbers; i.e., $a \cdot 0 = 0$.

(b) Commutative property of multiplication of whole numbers; i.e., $a \cdot b = b \cdot a$.

(c) Commutative property of multiplication; i.e., $a(bc) = (bc)a$.

(d) Identity property of multiplication of whole numbers; i.e., $a \cdot 1 = a$.

(e) Distributive property of multiplication over addition for whole numbers; i.e. $(a + b)c = ac + bc$.

(f) Distributive property of multiplication over addition for whole numbers; i.e., $(a + b)(c + d) = (a + b) \cdot c + (a + b) \cdot d$.

13. **(a)** Commutative property of multiplication of whole numbers; i.e., $a \cdot b = b \cdot a$.

(b) Distributive property of multiplication over addition for whole numbers; i.e., $9 \cdot 6 = 9 \cdot 5 + 9 \cdot 1 = 45 + 9 = 54$.

14. **(a)** Distributive property of multiplication over addition for whole numbers; i.e., $a(b + c) = ab + ac$.

(b) $8 \cdot 34 = 8(30 + 4) = 8 \cdot 30 + 8 \cdot 4 = 240 + 32 = \mathbf{272}$.

15. **(a)** $15(10 - 2) = 15 \cdot 10 - 15 \cdot 2 = 150 - 30 = \mathbf{120}$.

(b) $30(9 - 2) = 30 \cdot 9 - 30 \cdot 2 = 270 - 60 = \mathbf{210}$.

16. The area of the shaded region is $a(b - c)$, but it can also be expressed as the area of the large rectangle minus the area of the small rectangle $\Rightarrow a(b - c) = ab - ac$.

17. The first two figures on the left illustrates a^2, viewed as the area of the large square, minus b^2, viewed as the area of the small square on the bottom right. The figure on the right illustrates how the pieces can be rearranged to form a rectangle with height $a + b$ and length $a - b$.

18. **(a)** $19 \cdot 21 = 21 \cdot 19 = (20 + 1)(20 - 1) = 20^2 - 1 = 400 - 1 = \mathbf{399}$.

(b) $25 \cdot 15 = (20 + 5)(20 - 5) = 20^2 - 5^2 = 400 - 25 = \mathbf{375}$.

(c) $99 \cdot 101 = 101 \cdot 99 = (100 + 1)(100 - 1) = 100^2 - 1^2 = 10000 - 1 = \mathbf{9999}$.

(d) $101^2 - 99^2 = (101 + 99)(101 - 99) = (200)(2) = \mathbf{400}$.

19. **(a)** ***(i)*** $(ab)c = (ba)c$ by the commutative property of multiplication of whole numbers.

(ii) $(ba)c = b(ac)$ by the associative property of multiplication of whole numbers.

(b) ***(i)*** $a(b + c) = ab + ac$ by the distributive property of multiplication over addition.

(ii) $ab + ac = ac + ab$ by the commutative property of addition.

20. To factor is to reverse the process of the distributive property of multiplication over addition or subtraction. The result is to find the factors which, when multiplied together, will yield a given product.

(a) $xy - y = \boldsymbol{y(x - 1)}$. Note that $xy - y$ can be written as $xy - 1 \cdot y$.

(b) $(x + 1)y - (x + 1) = \boldsymbol{(x + 1)(y - 1)}$. $(x + 1)$ is a factor of both $(x + 1)y$ and $(x + 1) \cdot 1$.

(c) $a^2b^3 - ab^2 = \boldsymbol{ab^2(ab - 1)}$. a and b^2 are factors of each term.

21. **(a)** $48 \div x = 16 \Rightarrow \mathbf{48 = x \cdot 16}$.

(b) $x \div 5 = 17 \Rightarrow \mathbf{x = 5 \cdot 17}$.

22. If $64 / a = b$ then $64 = a{\bullet}b$ and $64 / b = a$.

23. The complete fact family:

$30 / 6 = 5$

$30 / 5 = 6$

$5{\bullet}6 = 30$

$6{\bullet}5 = 30$

24. **(a)** **The original number is returned**.

(b) **Yes**. Let n be the original number.

Then $\frac{2n+2}{2} - 1 = \frac{2(n+1)}{2} - 1 =$
$n + 1 - 1 = n.$

So n is returned, regardless of the value of n.

25. Answers may vary, but for example:

(a) There is no commutative property in division; e.g., $(8 \div 4) \neq (4 \div 8)$.

(b) There is no commutative property in subtraction; e.g., $8 - 4 \neq 4 - 8$.

26. Write $(a - b) \div c = d$, where d is a whole number. This can be written as a multiplication problem $a - b = cd$. Similarly, $b \div c$ can be written as a multiplication problem $b = ck$. Combining this, we have $a - b = cd \Rightarrow a = cd + b \Rightarrow a = cd + ck \Rightarrow a = c(d + k)$
$\Rightarrow a \div c = d + k.$

Recall that $b \div c = k$. So,
$a \div c = d + k \Rightarrow a \div c - k = d$
$\Rightarrow a \div c - b \div c = d = (a - b) \div c.$

27. **(a)** $5x + 8 = 28 \Rightarrow 5x + 8 - 8 = 28 - 8 \Rightarrow 5x = 20 \Rightarrow \frac{5}{5}x = \frac{20}{5} \Rightarrow \mathbf{x = 4}$.

(b) $5x + 6 = x + 14 \Rightarrow 5x - x + 6 = x - x + 14 \Rightarrow 4x + 6 = 14 \Rightarrow 4x + 6 - 6 = 14 - 6$
$\Rightarrow 4x = 8 \Rightarrow \frac{4}{4}x = \frac{8}{4} \Rightarrow \mathbf{x = 2.}$

(c) $5(x + 3) = 35 \Rightarrow 5x + 15 = 35 \Rightarrow 5x + 15 - 15 = 35 - 15 \Rightarrow 5x = 20$
$\Rightarrow \frac{5}{5}x = \frac{20}{5} \Rightarrow \mathbf{x = 4}.$

(d) $(x - 6) \div 3 = 1 \Rightarrow x - 6 = 3 \Rightarrow \mathbf{x = 9}$.

28. Look for a pattern:

2 nails on each axis $\rightarrow$ 1 intersection.

3 nails on each axis $\rightarrow$ 3 intersections $\rightarrow$ 2 new.

4 nails on each axis $\rightarrow$ 6 intersections $\rightarrow$ 3 new.

5 nails on each axis $\rightarrow$ 10 intersections $\rightarrow$ 4 new.

$\vdots$

6 nails $\rightarrow$ 15 intersections.

7 nails $\rightarrow$ 21 intersections.

8 nails $\rightarrow$ 28 intersections.

9 nails $\rightarrow$ 36 intersections.

10 nails $\rightarrow$ **45 intersections**.

29. If Jonah adds the borrowed 5 marbles to his last row, he will have a row of 13 marbles. Then $13 - 5 = \mathbf{8}$, **the remainder** when his original number is divided by 13.

30. 8 teams $\times$ 9 players per team $=$ 72 players $\Rightarrow$
72 players $\div$ 6 players per team $=$ **12 teams**.

31. **(a)** **Add 2** to $30 - 12$. Order of operations specifies multiplication before addition or subtraction.

(b) **Add 7 to 3**. Order of operations specifies multiplication and division before addition or subtraction.

(c) **Subtract 12 from** 15. $6 \div 2 \cdot 4 = 12$.

(d) **Add 18 to 5**. Operations within parentheses first, then multiplication.

32. Given any whole numbers a and $b (b \neq 0)$, there exist unique whole numbers q (quotient) and r (remainder) such that $a = bq + r \Rightarrow 5n + 3 \in W$. Those numbers are $\mathbf{3, 8, 13, \ldots}$.

33. **(a)** **Yes**. $x \odot y \in S$ for any values of $x \in S$ and $y \in S$.

(b) **Yes**. $a \odot b = b \odot a, a \odot c = c \odot a, \ldots$ for all possible combinations of terms.

(c) **Yes**. Since $x \odot a = x$ and $a \odot x = x$ for all x in S, a is the identity on S.

(d) **Yes**. $a \odot (b \odot c) = (a \odot b) \odot c$ for all possible combinations of terms.

34. Yes, if both a and b are odd then 3#5=0=5#3, if one is odd then 3#2=0=2#3, and if both are even then 2#4=1=4#2.

35. In rows of 5 one member remained so it is a number of rows multiplied by.5 plus 1: 6, 11, 16, 21, 26, 31, … In rows of 6 one member remained so it is the number of rows multiplied by 6 plus 1: 7, 13, 19, 25, … In rows of 7 no member remained so it is the number of rows multiplied by 7: 7, 14, 21, 28, 35, … The least number of members is then 91.

Mathematical Connections 3-3: Review Problems

19. Answers may vary; an example would be $\{2, 6, 10, 14, \ldots\}$.

20. **No**. For example, $5 - 2 \neq 2 - 5$.

21. **No**. The number "two" exists in base two as 10_{two} but there is no single symbol representing "two."

22. $\mathbf{10{,}000}_{three} = 3^4 = 81$. Any power of 3 in base ten can be written as a 1 followed by an appropriate number of zeros in base three.

Assessment 3-4A: Addition and Subtraction Algorithms, Mental Computation, and Estimation

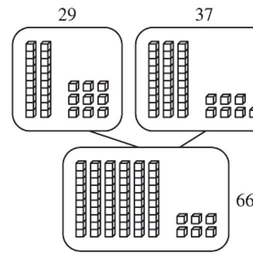
1. **(a)** In the units column, $1 + \underline{1} = 2$.
In the tens column, $\underline{8} + 2 = 10$ (regroup).
In the hundreds column, $1 + \underline{9} + 4 = 14$.

$$\begin{array}{cccc} & \underline{\mathbf{9}} & \underline{\mathbf{8}} & 1 \\ + & 4 & 2 & \underline{1} \\ \hline \underline{\mathbf{1}} & 4 & 0 & 2 \end{array}$$

(b) In the units column, $5 + 6 + 8 = 19$ (regroup).
In the tens column, $1 + 2 + \underline{9} + 4 = 16$ (regroup).
In the hundreds column, $1 + 0 + 1 + 1 = \underline{3}$.
In the thousands column, $\underline{2} + 1 + 3 = 6$.

$$\begin{array}{ccccc} & \underline{\mathbf{2}} & 0 & 2 & 5 \\ & 1 & 1 & \underline{\mathbf{9}} & 6 \\ + & 3 & 1 & 4 & 8 \\ \hline & 6 & \underline{\mathbf{3}} & 6 & \underline{\mathbf{9}} \end{array}$$

2. Base ten blocks:



3. **(a)** Answers may vary: for the unique greatest sum, the larger numbers must be in the hundreds column:

$$\begin{array}{cccc} & \boxed{7} & \boxed{6} & \boxed{2} \\ + & \boxed{8} & \boxed{5} & \boxed{3} \\ \hline 1 & 6 & 1 & 5 \end{array}$$

(b) Answers may vary; for the unique least sum the smaller numbers must be in the hundreds column:

$$\begin{array}{cccc} & \boxed{2} & \boxed{6} & \boxed{7} \\ + & \boxed{3} & \boxed{5} & \boxed{8} \\ \hline & 6 & 2 & 5 \end{array}$$

4. The ones digit must be a sum of 9, so the options are 3-6, 4-5, or 8-1. There is no way that a regrouping will take place, so the tens column must sum to 5 or 15. But 15 is not an option with the given digits, so it must sum to 5; the only options are 4-1 and 2-3. Since 15 was not an option, there is no regrouping into the hundreds column, meaning the sum must be 10. The only option for the hundreds column is 4-6. Placing these values in the correct boxes eliminates the options for the ones and tens columns, so the ones column must be 8-1 and the tens column must be 2-3. This leaves only 3-5 for the thousands column.

$$\begin{array}{ccccc} & \boxed{3} & \boxed{4} & \boxed{2} & \boxed{8} \\ + & \boxed{5} & \boxed{6} & \boxed{3} & \boxed{1} \\ \hline & 9 & 0 & 5 & 9 \end{array}$$

5. (a)

$$\begin{array}{r} 93 \\ -\ 37 \\ \hline \end{array} \Rightarrow \begin{array}{r} 93+3 \\ -\ 37+3 \\ \hline \end{array} \Rightarrow \begin{array}{r} 96 \\ -\ 40 \\ \hline 56 \end{array}$$

(b)

$$\begin{array}{r} 321 \\ -\ 38 \\ \hline \end{array} \Rightarrow \begin{array}{r} 321+2 \\ -\ 38+2 \\ \hline \end{array} \Rightarrow \begin{array}{r} 323 \\ -\ 40 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{r} 323+60 \\ -\ 40+60 \\ \hline \end{array} \Rightarrow \begin{array}{r} 383 \\ -\ 100 \\ \hline 283 \end{array}$$

6. (a) (*i*)

$$\begin{array}{rrrr} & 6 & 8 & 7 \\ + & 5 & 4 & 9 \\ \hline & & 1 & 6 \\ & 1 & 2 & \\ 1 & 1 & & \\ \hline 1 & 2 & 3 & 6 \end{array}$$

(*ii*)

$$\begin{array}{rrrr} & 3 & 5 & 9 \\ + & 6 & 7 & 3 \\ \hline & & 1 & 2 \\ & 1 & 2 & \\ & 9 & & \\ \hline 1 & 0 & 3 & 2 \end{array}$$

(b) The algorithm works because the placement of partial sums under their addends accounts for place value.

7. Answers may vary; some possibilities are:

(a) $8+5=13$ and 13 was written down with no regrouping. $2+7=9$ and 9 was simply placed in front of the 13.

(b) $8+5=13$, but instead of writing 3 and regrouping with the 1, the 1 was written and the 3 was regrouped.

(c) Only the difference in the units $(9-5=4)$, tens $(5-0=5)$, and the hundreds $(3-2=1)$ was recorded, without taking into account the signs of the numbers.

(d) Three hundreds was regrouped as 2 hundreds and 10 tens, but 10 tens was not regrouped as $9 \cdot 10 + 15$ in order to obtain $15-9=6$ in the ones place.

8. By dinner time Tom had consumed $90+120+119+185+110+570=1194$ calories. Subtracting 1194 from 1500 gives $1500-1194=306$. Tom may have fish or salad, but **not both**. (He may have tea with either.)

9. Step 1 → Expanded form;

Step 2 → Commutative and associative properties of addition;

Step 3 → Distributive property of multiplication over addition;

Step 4 → Closure property of addition

Step 5 → Expanded form condensed.

10. (a)

$$\begin{aligned} 66+23 &= (6\cdot 10+6)+(2\cdot 10+3) \\ &= (6\cdot 10+2\cdot 10)+(6+3) \\ &= (6+2)\cdot 10+(6+3) \\ &= 8\cdot 10+9 \\ &= 89. \end{aligned}$$

(b)

$$\begin{aligned} 124+235 &= (1\cdot 100+2\cdot 10+4) \\ &\quad +(2\cdot 100+3\cdot 10+5) \\ &= (1\cdot 100+2\cdot 100) \\ &\quad +(2\cdot 10+3\cdot 10)+(4+5) \\ &= (1+2)\cdot 100+(2+3)\cdot 10 \\ &\quad +(4+5) \\ &= 3\cdot 100+5\cdot 10+9 \\ &= \mathbf{359}. \end{aligned}$$

11. (a)

$$\begin{array}{ccccc} & 4 & 3 & 5 & 8 \\ + & 3 & 8 & 6 & 4 \\ \hline & \boxed{{}^0/_7} & \boxed{{}^1/_1} & \boxed{{}^1/_1} & \boxed{{}^1/_2} \\ & 8 & 2 & 2 & 2 \end{array}$$

(b)

$$\begin{array}{ccccc} & 4 & 9 & 2 & 3 \\ + & 9 & 8 & 9 & 7 \\ \hline & \boxed{{}^1/_3} & \boxed{{}^1/_7} & \boxed{{}^1/_1} & \boxed{{}^1/_0} \\ 1 & 4 & 8 & 2 & 0 \end{array}$$

12. Remember that in base five the number 11, for example, means $1 \cdot 5 + 1$, corresponding to 6 in base ten.

(a)

$$\begin{array}{rrrl} & 1 & & \\ & 4 & 3 & _{five} \\ + & 2 & 3 & _{five} \\ \hline 1 & 2 & 1 & _{five} \end{array}$$

Alternative approach: An example of a different algorithm for checking our answers would be to write in expanded form.

$$\begin{aligned}43_{five} + 23_{five} &= (4 \cdot 5 + 3 \cdot 5^0) \\ &\quad + (2 \cdot 5^1 + 3 \cdot 5^0) \\ &= 6 \cdot 5^1 + 6 \cdot 5^0 \\ &= (5+1)5^1 + (5+1)5^0 \\ &= 5^2 + 5 + 5 + 1 \\ &= 1 \cdot 5^2 + 2 \cdot 5 + 1 \cdot 5^0 \\ &= (121)_{five}\end{aligned}$$

(b)
$$\begin{array}{rrrl} & 4 & 3 & _{five} \\ - & 2 & 3 & _{five} \\ \hline & 2 & 0 & _{five}\end{array}$$

(c)
$$\begin{array}{rrrrl} & \overset{1}{4} & \overset{1}{3} & 2 & _{five} \\ + & & 2 & 3 & _{five} \\ \hline 1 & 0 & 1 & 0 & _{five}\end{array}$$

(d)
$$\begin{array}{rrrl} & \overset{3}{\cancel{4}} & \overset{12}{\cancel{2}} & _{five} \\ - & 2 & 3 & _{five} \\ \hline & 1 & 4 & _{five}\end{array}$$

(e)
$$\begin{array}{rrrrl} & \overset{1}{1} & 1 & 0 & _{two} \\ + & & 1 & 1 & _{two} \\ \hline 1 & 0 & 0 & 1 & _{two}\end{array}$$

(f)
$$\begin{array}{rrrrrl} & \overset{1}{\cancel{10}} & \overset{1}{\cancel{10}} & 10 & & \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & 1 & _{two} \\ - & & 1 & 1 & 1 & _{two} \\ \hline & 1 & 0 & 1 & 0 & _{two}\end{array}$$

13. For an example of how to use the table, move down the rows in the $+$ column to 3 and then across that row to the column headed by 6 . This will give the sum of $3 + 6$, or 11_{eight}.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

(a)
$$\begin{array}{rrrrl} & \overset{4}{\cancel{5}} & \overset{16}{\cancel{7}} & \overset{13}{\cancel{3}} & _{eight} \\ - & & 7 & 7 & _{eight} \\ \hline & 4 & 7 & 4 & _{eight}.\end{array}$$

We used the table to complete the subtraction. For example $13_{eight} - 7_{eight}$ can be found by observing that $4_{eight} + 7_{eight} = 13_{eight}$ in the table.

(b)
$$\begin{array}{rrrrl} & \overset{6}{\cancel{7}} & \overset{15}{\cancel{6}} & \overset{15}{\cancel{5}} & _{eight} \\ - & & 7 & 6 & _{eight} \\ \hline & 6 & 6 & 7 & _{eight}\end{array}$$

14. **(a)**
$$\begin{array}{rlrlrl} & \overset{1}{3} \text{ hours} & \overset{1}{36} & \text{minutes} & 58 & \text{seconds} \\ + & 5 \text{ hours} & 56 & \text{minutes} & 27 & \text{seconds} \\ \hline & 9 \text{ hours} & 33 & \text{minutes} & 25 & \text{seconds}\end{array}$$

(b)
$$\begin{array}{rlrlrl} & \overset{4}{\cancel{5}} \text{ hours} & \overset{95}{\overset{35}{\cancel{36}}} & \text{minutes} & \overset{98}{\cancel{38}} & \text{seconds} \\ - & 3 \text{ hours} & 56 & \text{minutes} & 58 & \text{seconds} \\ \hline & 1 \text{ hour} & 39 & \text{minutes} & 40 & \text{seconds}\end{array}$$

15. **(a)** The sum of each row, column, and diagonal is **34**. For example, the sum of the entries in row 1 is $1 + 15 + 14 + 4 =$ **34**.

(b) $6 + 7 + 10 + 11 =$ **34**.

(c) $1 + 4 + 16 + 13 =$ **34**.

(d) **Yes**. Adding five to each number in the square will increase the sum of any four numbers in the square by 20.

(e) **Yes**. Subtracting 1 from each number in the square will decrease the sum of any four numbers in the square by 4.

16. **(a)** Using the scratch algorithm, whenever a sum is 10 or more, scratch a line through the last digit added and write the number of units next to the scratched digit; count the number of scratches in each column and add to the column to the left.

$$\begin{array}{rcccc} & & ^{1} & ^{1} & \\ & & 4 & 3 & 2 \\ & & \cancel{9}_4 & \cancel{7}_1 & 6 \\ + & ^{1}1 & 4 & 1 & \cancel{8}_6 \\ \hline & 2 & 8 & 2 & 6 \end{array}$$

(b) A base five addition table might be helpful:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

$$\begin{array}{rccl} & & ^{3} & \\ ^{3} & & \cancel{3}_1 & 2\ _{five} \\ & 1 & \cancel{3}_0 & \ _{five} \\ & 2 & 2 & \ _{five} \\ & \cancel{4}_3 & \cancel{3}_0 & \ _{five} \\ & \cancel{2}_0 & 3 & \ _{five} \\ + & 1 & \cancel{2}_0 & \ _{five} \\ \hline 3 & 1 & 0 & \ _{five} \end{array}$$

17. There is **no numeral 5** in base five; $2_{five} + 3_{five} = 10_{five}$; $22_{five} + 33_{five} = 110_{five}$.

18. **(a)** Subtract 2132_{five} from 3423_{five}.

$$\begin{array}{rcccc l} & 3 & 4 & 2 & 3 & _{five} \\ - & 2 & 1 & 3 & 2 & _{five} \\ \hline & \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{1} & \mathbf{_{five}} \end{array}$$

(b) Subtract 11011_{two} from $100,000_{two}$.

$$\begin{array}{rcccccc l} & 1 & 0 & 0 & 0 & 0 & 0 & _{two} \\ - & & 1 & 1 & 0 & 1 & 1 & _{two} \\ \hline & & & & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{_{two}} \end{array}$$

(c) Subtract 1 from TEE_{twelve}:

$$\begin{array}{rccc l} & T & E & E & _{twelve} \\ - & & & 1 & _{twelve} \\ \hline & \mathbf{T} & \mathbf{E} & \mathbf{T} & \mathbf{_{twelve}} \end{array}$$

(d) Subtract 1000_{five} from $10,000_{five}$.

$$\begin{array}{rccccc l} & 1 & 0 & 0 & 0 & 0 & _{five} \\ - & & 1 & 0 & 0 & 0 & _{five} \\ \hline & & \mathbf{4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{_{five}} \end{array}$$

19. The information in (a) and (b) complete the first column.

Teams	1
Hawks	14
Elks	18 = 14 + 4

The information in (c) and (d) complete the second column.

Teams	1	2
Hawks	14	22 = 23 − 1
Elks	18	23 = 18 + 5

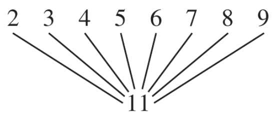
The information in (g) and (h) complete the third column.

Teams	1	2	3
Hawks	14	31	36 = 2(18)
Elks	18	23	45 = 14 + 31

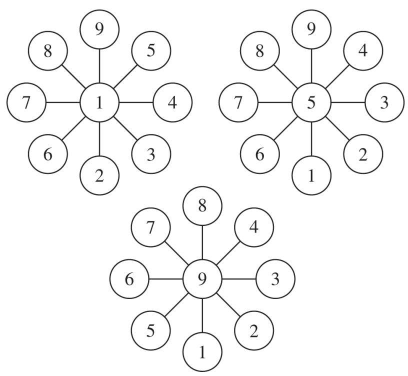
The information in (f) tells us that $14 + 22 + 36 + (4^{th}\text{ quarter score}) = 120$.
So the 4^{th} quarter score for the Hawks is 39. The information in (g) tells us that the Elks scored $48 + 6$ points in the 4^{th} quarter.

	Quarter				
Teams	1	2	3	4	Final
Hawks	14	31	36	39	120
Elks	18	23	45	45	131

20. **(a)** If 1 is placed in the middle then we can pair the numbers 2 through 9 to sum to eleven so that the sum in each of the four directions is twelve.



Similar arrangements can be made for 5 and 9 in the middle.



(b) The three $\{1,5,9\}$ are the only numbers that can be placed in the middle. To observe that 2, for example, can't be placed in the middle, pair 9 with each of the remaining numbers and observe that there is no way to pair the other numbers to form a common sum.

21. **(a)** $93 + 39 = 132; 132 + 231 = 363,$ which is a palindrome.

(b) $588 + 885 = 1473; 1473 + 3741 = 5214;$ $5214 + 4125 = 9339,$ which is a palindrome.

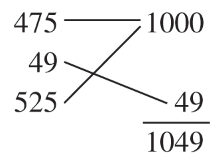
(c) $2003 + 3002 = 5005,$ which is a palindrome.

22. Answers may vary; possibilities include:

(a) $180 + 97 - 23 + 20 - 140 + 26$
$= 180 + 97 + 20 + 26 - 23 - 140$
$= (180 + 90 + 20 + 20 + 7 + 6)$
$- 23 - 140$
$= (310 + 13) - 23 - 140$
$= 323 - 23 - 140 = 300 - 140 = \mathbf{160}.$

(b) $87 - 42 + 70 - 38 + 43$
$= 87 + 70 + 43 - 42 - 38$
$= (80 + 70 + 40 + 7 + 3)$
$- 40 - 30 - 2 - 8$
$= (190 + 10) - 70 - 10$
$= 200 - 80 = \mathbf{120}.$

23. **(a)** Answers vary; possibilities include:
Compatible numbers



(b) Answers vary; possibilities include:
Breaking up and bridging.
$375 - 76 = 375 - 75 - 1$
$= 300 - 1 = 299.$

(c) Answers vary; possibilities include:
Compatible numbers
$230 + 60 + 70 + 44 + 6 =$
$(230 + 70) + (60 + 40) + (4 + 6) =$
$300 + 100 + 10 =$
410

24. **(a)** $567 + 38$:

(***i***) $567 + 30 = 597$ (add 567 to the tens value of 38).

(***ii***) $597 + 8 = \mathbf{605}$ (add 597 to the units value of 38).

(b) $418 + 215$:

(***i***) $418 + 200 = 618$ (add 418 to the hundreds value of 215).

(***ii***) $618 + 10 = 628$ (add 618 to the tens value of 215).

(***iii***) $628 + 5 = 633$ (add 628 to the units value of 215).

25. **(a)** $85 - 49 \Rightarrow (85 + 1) - (49 + 1) =$ $86 - 50 = \mathbf{36}.$

(b) $87 + 33 \Rightarrow (87 + 3) + (33 - 3) =$ $90 + 30 = \mathbf{120}.$

(c) $143 - 97 = (143 + 3) - (97 + 3) =$ $146 - 100 = \mathbf{46}.$

(d) $58 + 39 \Rightarrow (58 + 2) + (39 - 2) =$ $60 + 37 = \mathbf{97}.$

26. **(a)** $28 + 2 = 30; 30 + 20 = 50; 50 + 3 = 53.$ Then $2 + 20 + 3 = \mathbf{25}$.

(b) $47 + 3 = 50; 50 + 10 = 60; 60 + 3 = 63.$ Then $3 + 10 + 3 = \mathbf{16}$.

27. **(a)** $5\underline{2}80$: The number is between 5200 and 5300; The midpoint is 5250; The number is greater than the midpoint; So it rounds up to **5300**.

(b) $\underline{1}15{,}234$: The number is between 100,000 and 200,000; The midpoint is 150,000; The number is less than the midpoint; So it rounds down to **100,000**.

(c) $1\underline{1}5{,}234$: The number is between 110,000 and 120,000; The midpoint is 115,000; The number is greater than the midpoint; So it rounds up to **120,000**.

(d) $23\underline{2}5$: The number is between 2320 and 2330; The midpoint is 2325; When the number is at the midpoint it is conventional to round up (to **2330**).

28. Answers may vary:

(a) $878 + 2340 \approx 900 + 2300 = 3200$

(b) $25{,}201 - 19{,}987 \approx 25{,}000 - 20{,}000 = \mathbf{5000}$.

(c) $2215 + 3023 + 5967 + 975 \approx 2000 + 3000 + 6000 + 1000 = \mathbf{12{,}000}$.

29. **(a)** $2215 + 3023 + 5987 + 975$:

(***i***) $2 + 3 + 5 + 0 = 10$ (add front-end digits);

(***ii***) $10{,}000$ (place value);

(***iii***) $215 + 23 + 987 + 975 \approx 200 + 0 + 1000 + 1000 = 2200$ (adjust);

(***iv***) $10{,}000 + 2200 = \mathbf{12{,}200}$ (adjusted estimate).

(b) $234 + 478 + 987 + 319 + 469$:

(***i***) $2 + 4 + 9 + 3 + 4 = 22$ (add front-end digits)

(***ii***) 2200 (place value);

(***iii***) $34 + 78 + 87 + 19 + 69 \approx 30 + 80 + 90 + 20 + 70 = 290$ (adjust);

(***iv***) $2200 + 290 = \mathbf{2490}$ (adjusted estimate).

30. **(a)** (***i***) **No**. The numbers are not clustered.

(***ii***) **Yes**. The numbers are clustered around 500.

(b) Estimate may vary:

(***i***) Front-end: $0 + 1 + 0 + 2 = 3$; place value 3000; adjust by $500 + 500 + 100 + 400 = 1500$; estimate $3000 + 1500 = \mathbf{4500}$.

Grouping to nice numbers: $474 + 1467 \approx 2000$; $64 + 2445 \approx 2500$; sum $\approx 2000 + 2500 = \mathbf{4500}$.

Rounding: Sum $\approx 500 + 1500 + 100 + 2400 = \mathbf{4500}$.

(***ii***) Front-end: $4 + 4 + 5 + 5 + 5 = 23$; place value 2300; adjust by $80 + 80 + 30 + 0 + 30 = 220$; estimate $2300 + 220 = \mathbf{2520}$.

Grouping to nice numbers: $483 + 475 \approx 1000$; $530 + 528 \approx 1000$; $503 \approx 500$; sum $\approx 1000 + 1000 + 500 = \mathbf{2500}$.

Rounding: sum $\approx 500 + 500 + 500 + 500 + 500 = \mathbf{2500}$.

31. **(a)** The range is $100 + 600 = 700$ to $200 + 700 = 900$. Then $700 < (145 + 678) < 900$.

(b) The range is $200 + 0 = 200$ to $300 + 100 = 400$. Then $200 < (278 + 36) < 400$.

32. Answers may vary; e.g., $3300 - 100 - 300 - 400 - 500 = \mathbf{2000}$. The estimate is **high** because the amounts were rounded to the hundreds place. \$8 was taken away from the check amounts while \$13 was added to the \$3287.

33. **(a)** **False**. Sweden would be estimated to be about 44,000 square miles larger than Finland.

(b) **False**. Twice Norway's size would be about 250,000 square miles.

(c) **False**. France would be estimated to be about 85,000 square miles larger than Norway.

(d) **True**. About 195,000 square miles to about 174,000 square miles.

34. The clustering strategy yields $6 \cdot$ (about 70,000) $\approx$ $\mathbf{420,000}$ in attendance.

Assessment 3-4B

1. **(a)** In the units column, $1\underline{3} - 9 = 4$ (trade from the tens column).

In the tens column, $7 - 5 = 2$, (1 has been traded from $\underline{8}$).

In the hundreds column, $3 - 1 = \underline{2}$.

$$\begin{array}{rccc} & 3 & \underline{\mathbf{8}} & \underline{\mathbf{3}} \\ - & 1 & 5 & 9 \\ \hline & \underline{\mathbf{2}} & 2 & 4 \end{array}$$

(b) In the units column, 10 has been traded from the tens column.

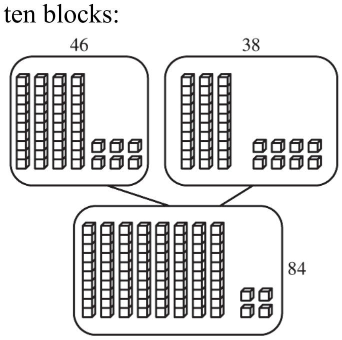
In the tens column, $8 - 0 = 8$ but 1 has been traded from $\underline{9}$.

In the hundreds column, $12 - 3 = 9$ (trade 1000 from the thousands column).

In the thousands column, $12 - 8 = 4$, (1 has been traded from $\underline{3}$).

$$\begin{array}{rcccc} & 1\,\underline{\mathbf{3}} & \underline{\mathbf{2}} & \underline{\mathbf{9}} & 6 \\ - & 8 & 3 & 0 & 9 \\ \hline & 4 & 9 & 8 & 7 \end{array}$$

2. Base ten blocks:



3. If whole numbers are used:

(a)
$$\begin{array}{rccc} & \boxed{8} & \boxed{7} & \boxed{6} \\ - & \boxed{2} & \boxed{3} & \boxed{5} \\ \hline & 6 & 4 & 1 \end{array}$$

(b)
$$\begin{array}{rccc} & \boxed{6} & \boxed{2} & \boxed{3} \\ - & \boxed{5} & \boxed{8} & \boxed{7} \\ \hline & & 3 & 6 \end{array}$$

4. 4 and 5 can be used in the units column to sum to 9. 8 and 4 can then be used in the tens column; they sum to 12 with the 1 regrouped to the hundreds column. The two 3's can be used in the hundreds column to sum, with the regrouped 1, to 7. The only remaining numbers, to be placed in the thousands column, are 5 and 6.

$$\begin{array}{rcccc} & 5 & 3 & 8 & 4 \\ + & 6 & 3 & 4 & 5 \\ \hline & 11 & 7 & 2 & 9 \end{array}$$

5. **(a)**
$$\begin{array}{r} 86 \\ -\ 38 \\ \hline \end{array} \Rightarrow \begin{array}{r} 86+2 \\ -\ 38+2 \\ \hline \end{array} \Rightarrow \begin{array}{r} 88 \\ -\ 40 \\ \hline \mathbf{48} \end{array}$$

(b)
$$\begin{array}{r} 582 \\ -\ 44 \\ \hline \end{array} \Rightarrow \begin{array}{r} 582+6 \\ -\ 44+6 \\ \hline \end{array} \Rightarrow \begin{array}{r} 588 \\ -\ 50 \\ \hline \mathbf{538} \end{array}$$

6. **(a)**
$$\begin{array}{rcccc} & & 9 & 8 & 7 \\ + & & 3 & 5 & 6 \\ \hline & & & 1 & 3 \\ & & 1 & 3 & \\ & 1 & 2 & & \\ \hline & 1 & 3 & 4 & 3 \end{array}$$

(b)
$$\begin{array}{rccc} & 4 & 1 & 5 \\ + & & 7 & 9 \\ \hline & & 1 & 4 \\ & 4 & 8 & \\ \hline & 4 & 9 & 4 \end{array}$$

7. Answers may vary, for example:

(a) A tens digit was not regrouped as 1 ten when the sum of the units digits was more than 9.

(b) Partial sums are not in the correct place value position.

(c) The units minuend is subtracted from the units subtrahend.

(d) One tens value should have been traded from the 5 in the minuend's ten position.

8. 75 minutes + 18 minutes + 45 seconds + 30 seconds = 93 minutes + 75 seconds = 93 minutes + 1 minute + 15 seconds = 94 minutes + 15 seconds = 1 hour + 34 minutes + 15 seconds. George's meal required **1 hour 34 minutes 15 seconds cooking time**.
He must start at 2 : 25 : 45 pm.

9. Step 1 → Expanded form;
Step 2 → Commutative and associative properties of addition;
Step 3 → Distributive property of multiplication over addition;
Step 4 → Closure property of addition, one digit addition facts;
Step 5 → Expanded form condensed.

10. **(a)**
$$\begin{aligned} 46 + 32 &= (4 \cdot 10 + 6) + (3 \cdot 10 + 2) \\ &= (4 \cdot 10 + 3 \cdot 10) + (6 + 2) \\ &= (4 + 3) \cdot 10 + (6 + 2) \\ &= 7 \cdot 10 + 8 \\ &= \mathbf{78} \end{aligned}$$

(b)
$$\begin{aligned} 3214 + 783 &= (3 \cdot 10^3 + 2 \cdot 10^2 + 1 \cdot 10 + 4) \\ &\quad + (7 \cdot 10^2 + 8 \cdot 10 + 3) \\ &= (3 \cdot 10^3) + (2 + 7) \cdot 10^2 \\ &\quad + (1 + 8) \cdot 10 + (4 + 3) \\ &= 3 \cdot 10^3 + 9 \cdot 10^2 + 9 \cdot 10 + 7 \\ &= \mathbf{3997} \end{aligned}$$

11. **(a)**
$$\begin{array}{ccccc} & 2 & 3 & 4 & 5 \\ + & 8 & 8 & 8 & 8 \\ \hline & \boxed{1/0} & \boxed{1/1} & \boxed{1/2} & \boxed{1/3} \\ 1 & 1 & 2 & 3 & 3 \end{array}$$

(b)
$$\begin{array}{ccccc} & 8 & 7 & 1 & 3 \\ + & 4 & 2 & 1 & 4 \\ \hline & \boxed{1/2} & \boxed{0/9} & \boxed{0/2} & \boxed{0/7} \\ 1 & 2 & 9 & 2 & 7 \end{array}$$

12. Remember that in base five the number 11, for example, means $1 \cdot 5 + 1$, whereas in base ten it means $1 \cdot 10 + 1$.

(a)
$$\begin{array}{cccl} & 3 & 13 & \\ & \cancel{4} & \cancel{3} & _{five} \\ - & 2 & 4 & _{five} \\ \hline & 1 & 4 & _{five} \end{array}$$

(b)
$$\begin{array}{ccccl} & 1 & 1 & & \\ & 1 & 4 & 3 & _{five} \\ + & & 2 & 3 & _{five} \\ \hline & 2 & 2 & 1 & _{five} \end{array}$$

(c)
$$\begin{array}{cccl} & 2 & 12 & \\ & \cancel{3} & \cancel{2} & _{five} \\ - & 2 & 3 & _{five} \\ \hline & & 4 & _{five} \end{array}$$

(d)
$$\begin{array}{ccccl} & 1 & 1 & & \\ & 2 & 3 & 2 & _{five} \\ + & & 4 & 3 & _{five} \\ \hline & 3 & 3 & 0 & _{five} \end{array}$$

(e)
$$\begin{array}{ccccl} 1 & 1 & & & \\ & 1 & 1 & 0 & _{two} \\ + & 1 & 1 & 1 & _{two} \\ \hline 1 & 1 & 0 & 1 & _{two} \end{array}$$

(f)
$$\begin{array}{cccccl} & 1 & & & & \\ & \cancel{10} & 10 & & & \\ \cancel{1} & \cancel{0} & \cancel{0} & 0 & 1 & _{two} \\ - & & 1 & 0 & 1 & _{two} \\ \hline & 1 & 1 & 0 & 0 & _{two} \end{array}$$

13. For an example of use of the table, moving down the rows in the + column to 3 and then across the row to the column headed by 4 will give the sum of $3 + 4$, or 11_{six}.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	11
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

$$\begin{array}{crrrl} & 1 & 12 & 1 & \\ & \cancel{2} & \cancel{3} & \cancel{1} & _{six} \\ - & 1 & 4 & 4 & _{six} \\ \hline & & 4 & 3 & _{six} \end{array}$$

Check:

$$\begin{array}{crrrl} & 1 & 1 & & \\ & 1 & 4 & 4 & _{six} \\ + & & 4 & 3 & _{six} \\ \hline & 2 & 3 & 1 & _{six} \end{array}$$

$$\begin{array}{crrrl} & 2 & 13 & 12 & \\ & \cancel{3} & \cancel{4} & \cancel{2} & _{six} \\ - & 1 & 4 & 4 & _{six} \\ \hline & 1 & 5 & 4 & _{six} \end{array}$$

Check:

$$\begin{array}{crrrl} & 1 & 1 & & \\ & 1 & 5 & 4 & _{six} \\ + & 1 & 4 & 4 & _{six} \\ \hline & 3 & 4 & 2 & _{six} \end{array}$$

14. **(a)** 2 cups = 1 pint and 2 pints = 1 quart. This is essentially operating in base two:

$$\begin{array}{crlrlrl} & 1 & & 1 & & & \\ & 1 & \text{quart} & 1 & \text{pint} & 1 & \text{cup} \\ + & & & 1 & \text{pint} & 1 & \text{cup} \\ \hline & 2 & \text{quarts} & 1 & \text{pint} & 0 & \text{cup} \end{array}$$

(b) Again in base two; 2 pints must be traded from 1 quart in order to subtract 1 pint:

$$\begin{array}{crlrlrl} & & & 2 & & & \\ & \cancel{1} & \text{quart} & 0 & \text{pint} & 1 & \text{cup} \\ - & & & 1 & \text{pint} & 1 & \text{cup} \\ \hline & & & 1 & \text{pint} & 0 & \text{cup} \end{array}$$

(c) 4 cups = 1 quart and 4 quarts = 1 gallon. This is essentially operating in base four:

$$\begin{array}{crlrlrl} & & & 6 & & & \\ & 0 & & 2 & & 5 & \\ & \cancel{1} & \text{gallon} & \cancel{3} & \text{quarts} & \cancel{1} & \text{cup} \\ - & & & 4 & \text{quarts} & 2 & \text{cups} \\ \hline & & & 2 & \text{quarts} & 3 & \text{cups} \end{array}$$

(or 2 quarts, 1 pint, 1 cup)

15. **(a)** Each row, column, and diagonal sums to **34**.

(b) The sum is **34**.

(c) The sum is **34**.

(d) **Yes**. All rows, columns, and diagonals still contain four numbers; adding 11 to each adds 44 to all sums and they are still equal (to 78).

(e) **Yes**. This subtracts 44 from all sums and they are still equal (to $^{-}10$).

16. **(a)** Use the scratch algorithm. Whenever a sum is 10 or more, scratch a line through the last digit added and write the number of units next to the scratched digit; count the number of scratches in each column and add to the column to the left.

$$\begin{array}{crrrr} & & 1 & 2 & \\ & & 5 & 3 & 7 \\ & & 3 & 1 & \cancel{8}_5 \\ + & {}^{1}2 & \cancel{3}_2 & \cancel{4}_0 & \cancel{5}_0 \\ \hline & 3 & 2 & 0 & 0 \end{array}$$

(b) A base six addition table might be helpful (see problem 13):

$$\begin{array}{crrrl} & & {}^{2}\cancel{4}_0 & 1 & _{six} \\ & & 3 & 2 & _{six} \\ & & 2 & 2 & _{six} \\ & & \cancel{4}_3 & \cancel{3}_2 & _{six} \\ & & 2 & 2 & _{six} \\ + & & \cancel{5}_4 & \cancel{4}_2 & _{six} \\ \hline & 3 & 4 & 2 & _{six} \end{array}$$

17. There is **no numeral 6 in base six**. $23_{six} + 43_{six} = 110_{six}$.

18. **(a)** Subtract 213_{five} from 342_{five}.

$$\begin{array}{crrrl} & 3 & 4 & 2 & _{five} \\ - & 2 & 1 & 3 & _{five} \\ \hline & \mathbf{1} & \mathbf{2} & \mathbf{4} & _{\mathbf{five}} \end{array}$$

(b) Subtract 1011_{two} from 1101_{two}.

$$\begin{array}{crrrrl} & 1 & 1 & 0 & 1 & _{two} \\ - & 1 & 0 & 1 & 1 & _{two} \\ \hline & & & \mathbf{1} & \mathbf{0} & _{two} \end{array}$$

(c) Subtract 9 from $E08_{twelve}$:

$$\begin{array}{rrrl} E & 0 & 8 & _{twelve} \\ - & & 9 & _{twelve} \\ \hline \boldsymbol{T} & \boldsymbol{E} & \boldsymbol{E} & _{\mathbf{twelve}} \end{array}$$

(d) Subtract 100_{two} from $10{,}000_{two}$.

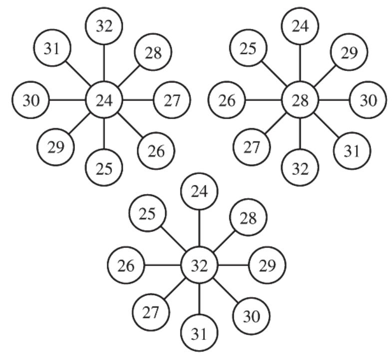
$$\begin{array}{rrrrrl} 1 & 0 & 0 & 0 & 0 & _{two} \\ - & & 1 & 0 & 0 & _{two} \\ \hline & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & _{\mathbf{two}} \end{array}$$

19.
- The Hawks scored 15 points in the first quarter, so 15 goes in that block.
- The Hawks were behind by 5 points at the end of the first quarter, so 20 goes to the Elks in the first.
- The Elks scored 5 more points in the second quarter than in the first, so 25 goes in that block.
- The Hawks scored 7 more points than the Elks in the second quarter, so 32 goes to the Hawks in the second.
- The Hawks scored twice as many points in the third quarter as the Elks did in the first, so 40 goes in that block.
- The Hawks scored 120 points in the game, so $120 - (15 + 32 + 40) = 33$ goes to the Hawks in the fourth quarter.
- The Elks outscored the Hawks by 6 points in the fourth quarter, so 39 goes in that block.
- The Elks scored as many points in the third quarter as the Hawks did in the first two; $15 + 32 = 47$ goes in that block.
- The Elks scored a total of $20 + 25 + 47 + 39 = 131$ in the game. Thus:

TEAMS	**QUARTERS 1**	**2**	**3**	**4**	**FINAL SCORE**
Hawks	**15**	**32**	**40**	**33**	**120**
Elks	**20**	**25**	**47**	**39**	**131**

20. Answers may vary

(a)



(b) **Three**. Let the sum of all three terms in a line be s. There are four lines in the completed shape, so adding them all is $4s$. This is the same, however, as adding all the terms together (i.e., $24, 25, \ldots, 32$) with one number used four times. Call the number in the middle $\boldsymbol{x}$. So $24 + 25 + \cdots + 32 + 3x = 4s$, where x is already used once in the sum of all the numbers 24 to 32. This reduces to $\frac{9 \cdot 56}{2} = 252$, making the equation $252 + 3x = 4s$. Solving for x yields: $3x = 4s - 252 = 4(s - 63)$. Thus x must be a multiple of 4, the only options for which are 24, 28, and 32.

21. The greatest addend possible to make 87 + 78 and answer with a 2 in the hundreds place would be 99. The largest 2-digit number is 99. Add 87 + 78 + 99 = 264 which is a number with a 2 in the hundreds place.

22. Answers may vary; possibilities include:

(a) $160 + 92 - 32 + 40 - 18$
$= 160 + 92 + 40 - 32 - 18$
$= (160 + 90 + 40 + 2) - 32 - 18$
$= (290 + 2) - 32 - 18$
$= 292 - 32 - 18 = 292 - 50 = \mathbf{242}$.

(b) $36 + 97 - 80 + 44$
$= 36 + 97 + 44 - 80$
$= (30 + 90 + 40 + 6 + 7 + 4) - 80$
$= (160 + 17) - 80$
$= 177 - 80 = \mathbf{97}$.

23. Step 1: **Associative** property of addition.
Step 2: **Commutative** property of addition.
Step 3: **Associative** property of addition.
Step 4: **Closure property of whole-number addition**.

24. **(a)** $997-32$:

(***i***) $997-30=967$ (subtract the tens value of 32 from 997).

(***ii***) $967-2=$ **965** (subtract the units value of 32 from 967).

(b) $560+136$:

(i) $560+100=660$ (add 560 to the hundreds value of 136).

(ii) $660+30=690$ (add 660 to the tens value of 136).

(iii) $690+6=696$ (add 690 to the untis value of 136).

25. **(a)** $75-38 \Rightarrow (75+2)-(38+2)=77-40=$ **37**.

(b) $57+35 \Rightarrow (57+3)+(35-3)=60+32=$ **92**.

(c) $137-29 \Rightarrow (137+1)-(29+1)=138-30=$ **108**.

(d) $78+49 \Rightarrow (78+2)+(49-2)=80+47=$ **127**.

26. **(a)** $63+7=70$; $70+4=74$. Then $7+4=$ **11**.

(b) $57+3=60$; $60+10=70$; $70+3=73$. Then $3+10+3=$ **16**.

27. **(a)** $3\underline{5}87$: The number is between 3500 and 3600;
The midpoint is 3550;
The number is greater than the midpoint;
So it rounds up to **3600**.

(b) $\underline{1}48{,}213$: The number is between 100,000 and 200,000;
The midpoint is 150,000;
The number is less than the midpoint;
So it rounds down to **100,000**.

(c) $2\underline{3}{,}785$: The number is between 23,000 and 24,000;
The midpoint is 23,500;
The number is greater than the midpoint;
So it rounds up to **24,000**.

(d) $23\underline{5}7$: The number is between 2350 and 2360;
The midpoint is 2355;
The number is greater than the midpoint;
So it rounds up to **2360**.

28. Estimates may vary.

(a) $937+28 \approx 940+30=970$

(b) $32{,}285-18{,}988 \approx 32{,}000-19{,}000=$ **13,000**.

(c) $3215+3789+5987 \approx 3000+4000+6000=$ **13,000**.

29. Estimates may vary.

(a) $2345+5250+4210+910$:

(***i***) $2+5+4+0=11$ (add front-end digits);

(***ii***) $11{,}000$ (place value);

(***iii***) $345+250+210+910 \approx 350+250+200+900=1700$ (adjust);

(***iv***) $11{,}000+1700=$ **12,700** (adjusted estimate).

(b) $345+518+655+270$:

(***i***) $3+5+6+2=16$ (add front-end digits);

(***ii***) 1600 (place value);

(***iii***) $50+20+60+70=200$ (adjust);

(***iv***) $1600+200=1800$ (adjusted estimate).

30. **(a)** (***i***) **No**. The numbers are not clustered.

(***ii***) **Yes**. The numbers are clustered around 2000.

(b) Estimates may vary:

(***i***) Front-end: $0+2+0+3=5$; place value 5000; adjust by $300+300+100+500=1200$; estimate $5000+1200=$ **6200**.

Grouping to nice numbers:
$318 + 2314 \approx 2600$; $57 + 3489 \approx 3600$;
sum $\approx 2600 + 3600 =$ **6200**.

Rounding: sum $\approx 300 + 2300 + 100 + 3500 =$ **6200**.

(*ii*) Front-end:
$2 + 1 + 2 + 2 + 1 = 8$; place value 8000; adjust by $400 + 1000 + 0 + 100 + 900 = 2400$; estimate $8000 + 2400 =$ **10,400**.

Grouping to nice numbers:
$2350 + 1987 \approx 4000$; $2036 + 2103 \approx 4000$; $1890 \approx 2000$; sum $\approx 4000 + 4000 + 2000 =$ **10,000**.

Rounding: sum $\approx 2000 + 2000 + 2000 + 2000 + 2000 =$ **10,000**.

31. Estimates may vary:

(a) The range is $100 + 700 = 800$ to $200 + 800 = 1000$. Then $800 < (123 + 780) < 1000$.

(b) The range is $400 + 200 = 600$ to $500 + 300 = 800$. Then $600 < (482 + 246) < 800$.

32. Answers vary. For example

$\$65 + \$190 + \$45 + \$212 + \$420 \approx$

$\$70 + \$200 + \$50 + \$200 + \$400 = \$920$

$\$1237 \approx \$1200$

$\$1200 - \$920 \approx \$1200 - \$900 = \$300$

The remaining balance is about \$300.

33. **(a)** **Yes**. Rounding, Josh plans to write checks for about \$40, \$30, \$60, and \$250, or about \$380. Since the check amounts were rounded up, he will have enough.

(b) **Yes**. (assuming a non-negative beginning balance). $\$981 + \$1140$ is greater than $\$900 + \$1100 = \$2000$.

(c) **Alberto**. He received 10 more votes than Juan from the first district, but only 1 less from the second.

34. Answers may vary. Possibilities include:

<u>Rounding</u>

$$\begin{array}{r} 71,000 \\ 65,000 \\ 68,000 \\ 73,000 \\ 85,000 \\ +\ 70,000 \\ \hline 432,000 \end{array}$$

<u>Trading off</u>

Observe that the high attendance on Friday distributed among the other day's attendance for approximately 72,000 visitors per day.
$72,000 \times 6 =$ **432,000**.

Mathematical Connections 3-4: Review Problems

27. **1410**. $\boldsymbol{M} = 1000$, $\boldsymbol{CD} = 500 - 100$, and $\boldsymbol{X} = 10$.

28. **(a)** $E0T_{twelve} = 11 \cdot 12^2 + 0 \cdot 12^1 + 10 \cdot 12^0 =$ **1594**.

(b) $1011_{two} = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 =$ **11**.

(c) $43_{five} = 4 \cdot 5^1 + 3 \cdot 5^0 =$ **23**.

29. **No**. For example, $2 + 3 = 5$, which is not an element of the set $\{1, 2, 3\}$.

30. For example, $(2 + 3) + 4 = 2 + (3 + 4)$.

31. **(a)** $20 - x = x$
$\Rightarrow 20 - x + x = x + x$
$\Rightarrow 20 = 2x$
$\Rightarrow 10 = x \Rightarrow \boldsymbol{x = 10}$.

(b) $20 - x - 6 = 0$
$\Rightarrow 20 - 6 = x$
$\Rightarrow 14 = x$
$\Rightarrow \boldsymbol{x = 14}$.

(c) $x + 4 = 3 + x + 1$
$\Rightarrow x + 4 = x + 4$
$\Rightarrow x = x$.
All whole numbers **solve the equation.**

Assessment 3-5A: Multiplication and Division Algorithms, Mental Computation, and Estimation

1. **(a)** There are $7 + 12 = 19$ factors of 5. Thus $5^7 \cdot 5^{12} = 5^{7+12} = \mathbf{5^{19}}$.

   (b) $6^{10} \cdot 6^2 \cdot 6^3 = 6^{10+2+3} = \mathbf{6^{15}}$.

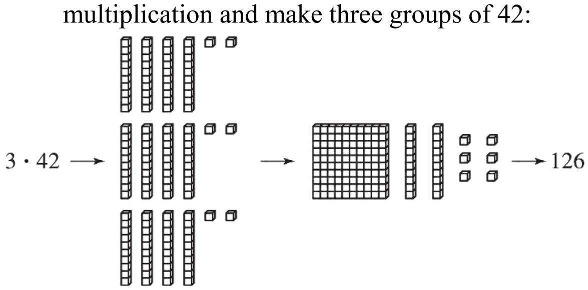
   (c) $10^{296} \cdot 10^{17} = 10^{296+17} = \mathbf{10^{313}}$.

   (d) $$\begin{aligned} 2^7 \cdot 10^5 \cdot 5^7 &= 2^7 \cdot 5^7 \cdot 10^5 \\ &= (2 \cdot 5)^7 \cdot 10^5 \\ &= 10^7 \cdot 10^5 \\ &= 10^{7+5} = \mathbf{10^{12}}. \end{aligned}$$

2. **(a)** $\mathbf{2^{100}}$ is greater. $2^{80} + 2^{80} = 2^{80}(1 + 1) = 2^{80} \cdot 2 = 2^{81} < 2^{100}$.

   (b) $\mathbf{2^{102}}$ is greatest. $2^{102} = 2^2 \cdot 2^{100} > 3 \cdot 2^{100} > 2^{101} = 2 \cdot 2^{100}$.

3. We can use the repeated addition model of multiplication and make three groups of 42:



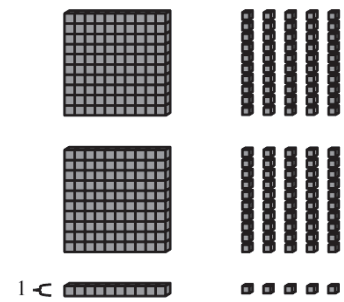
4. **(a)** The following partial products are obtained through the distributive property of multiplication over addition. The parenthetical computations below illustrate the corresponding area notions from the model:

$$\begin{array}{rrrl} & 2 & 2 & \\ \times & 1 & 3 & \\ \hline & & 6 & (3 \times 2) \\ & 6 & 0 & (3 \times 20) \\ & 2 & 0 & (10 \times 2) \\ 2 & 0 & 0 & (10 \times 20) \\ \hline 2 & 8 & 6 & \end{array}$$

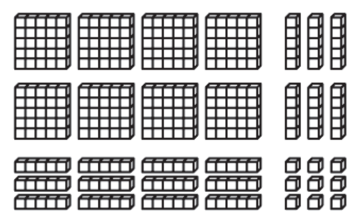
   (b)

$$\begin{array}{rrrl} & 1 & 5 & \\ \times & 2 & 1 & \\ \hline & & 5 & (1 \times 5) \\ & 1 & 0 & (1 \times 10) \\ 1 & 0 & 0 & (20 \times 5) \\ 2 & 0 & 0 & (20 \times 10) \\ \hline 3 & 1 & 5 & \end{array}$$

   Which is illustrated by:



   (c)



   To find $43_{five} \cdot 23_{five}$ count the number of flats, longs, and ones. There are $4 \cdot 2$ flats, $2 \cdot 3 + 3 \cdot 4$ longs, and $3 \cdot 3$ units. 5 units $= 1$ long, 5 longs $= 1$ flat, and 5 flats $= 1$ block, thus $43_{five} \cdot 23_{five} = 2$ blocks, 1 flat, 4 longs, and 4 units, or $\mathbf{2144_{five}}$.

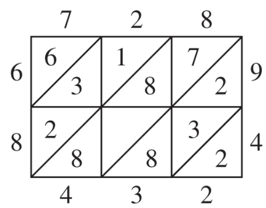
5. **(a)** Start with multiplication of 4_6 by 3. The _ must be 2, since 1 must be regrouped from $3 \cdot 6$ and only $3 \cdot 2 + 1 = 7$. Then $3 \cdot 4 = 12$ for 2 in the _ of the first partial product. Similar reasoning gives:

$$\begin{array}{rrrrrr} & & 4 & \underline{2} & 6 & \\ & \times & 7 & 8 & 3 & \\ \hline & 1 & \underline{2} & 7 & 8 & \\ & 3 & 4 & 0 & 8 & \\ \underline{2} & 9 & 8 & 2 & & \\ \hline 3 & 3 & 3 & 5 & \underline{5} & 8 \end{array}$$

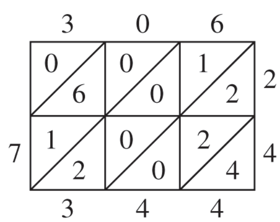
(b) The _ in the multiplier must be 4, since only $4 \cdot 7$ gives 8 in the units place of the second partial product. Similar reasoning gives:

$$\begin{array}{rrrrrr} & & & 3 & 2 & 7 \\ & & \times & 9 & \underline{\mathbf{4}} & 1 \\ \hline & & & 3 & 2 & 7 \\ & 1 & \underline{\mathbf{3}} & 0 & 8 & \\ \underline{\mathbf{2}} & 9 & \underline{\mathbf{4}} & 3 & & \\ \hline 3 & 0 & \underline{7} & \underline{7} & 0 & 7 \end{array}$$

6. (a) $728 \times 94 = 68{,}432$:



(b) $306 \times 24 = 7344$:



7. (a) Annexation can be interpreted as append. In base two, two is 10_{two}. To illustrate the property consider a three digit number in base two: $abc_{two} \cdot 10_{two} = (a \cdot 10^2_{two} + b \cdot 10_{two} + c \cdot 1_{two}) \cdot 10_{two} = a \cdot 10^3_{two} + b \cdot 10^2_{two} + c \cdot 10_{two} + 0 \cdot 1_{two} = abc0_{two}$. So multiplying abc_{two} by 10_{two} "annexed" the numeral abc_{two} with a 0 in the "ones" place.

(b) In base two, 4 can be expressed as $10_{two} \cdot 10_{two}$. Thus, given what we learned in (b), multiplying by 4 in base two "annexes" a base two numeral by 00 by annexing the original number by
0 twice.

(c)
$$\begin{aligned} 110_{two} \cdot 11_{two} &= 110_{two}(10_{two} + 1_{two}) \\ &= 110_{two} \cdot 10_{two} + 110_{two} \cdot 1_{two} \\ &= 1100_{two} + 110_{two} \\ &= \mathbf{10010_{two}}. \end{aligned}$$

8. (a) Annexation can be interpreted as append. In base two, two is 10_{two}. To illustrate the property consider a three digit number in base two: $abc_{two} \cdot 10_{two} = (a \cdot 10^2_{two} + b \cdot 10_{two} + c \cdot 1_{two}) \cdot 10_{two} = a \cdot 10^3_{two} + b \cdot 10^2_{two} + c \cdot 10_{two} + 0 \cdot 1_{two} = abc0_{two}$. So multiplying abc_{two} by 10_{two} "annexed" the numeral abc_{two} with a 0 in the "ones" place.

(b) In base two, 4 can be expressed as $10_{two} \cdot 10_{two}$. Thus, given what we learned in (b), multiplying by 4 in base two "annexes" a base two numeral by 00 by annexing the original number by
0 twice.

(c)
$$\begin{aligned} 110_{two} \cdot 11_{two} &= 110_{two}(10_{two} + 1_{two}) \\ &= 110_{two} \cdot 10_{two} + 110_{two} \cdot 1_{two} \\ &= 1100_{two} + 110_{two} \\ &= \mathbf{10010_{two}}. \end{aligned}$$

9.

		Halves		Doubles	
	→	17	×	$\boxed{63}$	
Halve 17		8		126	Double 63
Halve 8		4		252	Double 126
Halve 4		2		504	Double 252
Halve 2	→	1		$\boxed{1008}$	Double 504

The numbers in the doubles column paired with odd numbers in the halves column are 63 and 1008; and $63 + 1008 = 17 \times 63 = \mathbf{1071}$.

10. (a) 3 hrs skiing × 444 calories/hr = **1332** calories.

(b) Jane: 2 hrs × 462 calories/hr = 924 calories.

Carolyn: 3 hrs × 198 calories/hr = 594 calories.

Thus Jane burned $924 - 594 = \mathbf{330}$ more.

(c) Lyle: 3 hrs × 708 calories/hr = 2124 calories.

Maurice: 5 hrs × 444 calories/hr = 2220 calories.

Thus Maurice burned $2220 - 2124 = \mathbf{96}$ more.

11. 2 hrs $\times$ 666 calories/hr $=$ 1332 calories/day swimming.

1500 $-$ 1332 $=$ 168 calories/day increased intake.

168 $\times$ 14 $=$ 2352 excess calories consumed.

2352 $<$ 3500, so he gained **less** than 1 pound.

12. Assuming the price is \$30 per \$1000 per year, there are $\$50,000 \div \$1,000 = 50$ installments of \$30. Each quarter, the cost is $50 \cdot \$30 \div 4 =$ **\$375**.

13. **(a)** Repeated subtraction

$$\begin{array}{r|l} 8 & 623 \\ & \underline{560} \quad 70 \text{ eights} \\ & 63 \\ & \underline{56} \quad \underline{7} \text{ eights} \\ & 7 \quad 77 \text{ reminder } 7 \end{array}$$

Standard algorithm

$$\begin{array}{r|l} & 77 \quad \text{remainder } 7 \\ \hline 8 & 623 \\ & \underline{560} \\ & 63 \\ & \underline{56} \\ & 7 \end{array}$$

(b) Repeated subtraction

$$\begin{array}{r|l} 36 & 298 \\ & \underline{288} \quad \underline{8} \text{ 36's} \\ & 10 \quad 8 \text{ remainder } 10 \end{array}$$

Standard algorithm

$$\begin{array}{r|l} & 8 \quad \text{remainder } 10 \\ \hline 36 & 298 \\ & \underline{288} \\ & 10 \end{array}$$

(c) Repeated subtraction

$$\begin{array}{r|l} 391 & 4001 \\ & \underline{3910} \quad \underline{10} \text{ 391's} \\ & 91 \quad 10 \text{ remainder } 91 \end{array}$$

Standard algorithm

$$\begin{array}{r|l} & 10 \quad \text{reminder } 91 \\ \hline 391 & 4001 \\ & \underline{391} \\ & 91 \end{array}$$

14. **(a)** The divisor will be the least digit value and the dividend will be arranged so that the greatest is placed for the greatest place value. $\mathbf{3\overline{)754}}$

(b) The thinking in (a) is reversed. $\mathbf{7\overline{)345}}$

15. Reverse the operation: $300 \div 10 = 30$ and $30 \div 10 =$ **3**.

16. **(a)** Answers may vary. One such is $36 \cdot 84 = 3024$ and $63 \cdot 48 = 3024$.

(b) Let the digits be a, b, c and d. Then $(10a + b) \cdot (10c + d) = (10b + a) \cdot (10d + c) \Rightarrow 100ac + 10bc + 10ad + bd = 100bd + 10bc + 10ad + ac \Rightarrow 99ac = 99bd \Rightarrow ac = bd$.

So if $a \cdot c = b \cdot d$ then the products will always be the same when the digits are reversed (e.g., in part (a) above, $a \cdot c = b \cdot d \Rightarrow 3 \cdot 8 = 6 \cdot 4$).

17. Let p be the number of pennies in box 3. Then $3p$ is the number of pennies in box 1 and $2(3p) = 6p$ is the number of pennies in box 2. Thus $3p + 6p + p = 4520 \Rightarrow 10p = 4520 \Rightarrow p = 452$.

So box $1 = 3(452) =$ **1356** pennies;
box $2 = 6(452) =$ **2712** pennies;
box $3 =$ **452** pennies.

18. 36 apples $\times$ 50 boxes $=$ 1800 apples, or 600 3-apple bags. She had $18 \div 3 = 6$ bags left over, so she sold $600 - 6 = 594$ bags at \$1 per bag. Thus her profit was $\$594 - \$452 =$ **\$142**.

19. **(a)** 5 was multiplied by 6 to obtain 30. The 3 was regrouped, then 3 was multiplied by 2 to obtain 6. The regrouping was added to obtain 9 which was recorded in the tens place.

(b) When 1 was brought down the quotient of 0 was not recorded.

20. $56 \cdot 10$

$= (5 \cdot 10 + 6) \cdot 10$	expanded form.
$= (5 \cdot 10) \cdot 10 + 6 \cdot 10$	distributive property.
$= 5(10 \cdot 10) + 6 \cdot 10$	associative property.
$= 5 \cdot 10^2 + 6 \cdot 10$	definition of a^n.
$= 5 \cdot 10^2 + 6 \cdot 10 + 0 \cdot 1$	additive identity.
$=$ **560**	place value.

21. **(a)** Finding a number that leaves a remainder of 3 upon division by 4 is equivalent to thinking of a number multiplied by four and then add 3. For example 15 is a number that leaves a remainder of 3 upon division by 4:

$15 \div 4 = 3$ with a remainder of 3 because

$15 = 4 \bullet 3 + 3 \Rightarrow 15 = 12 + 3 \Rightarrow 15 = 15.$

In general, $4n + 3$ is all whole numbers that leave a remainder of 3 when divided by 4. In set builder notation $\{4n+3 \mid n \in W\}$.

(b) $4n+3$

$4(0)+3=3$

$4(1)+3=7$

$4(2)+3=11$

$4(3)+3=15$

$4(4)+3=19$

etc.

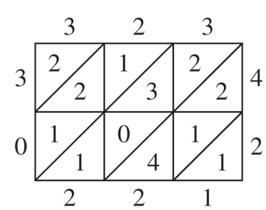
3, 7, 11, 15, 19, …

(c) Arithmetic because each subsequent term is the result of adding 4 to the previous term.

22. **(a)** **Base nine**. In the units column $3 + 8 = 12$ with the two brought down and the 1 regrouped, so $(3 + 8)_{nine} = 12_{nine}$.

(b) **Base six**. $2 \cdot 3$ produced 0 in the units column, so $(2 \cdot 3)_{six} = 10_{six}$.

23. $323_{five} \cdot 42_{five} = \mathbf{30221_{five}}$:



24. $32_a = 23_b \Rightarrow 3a + 2 = 2b + 3 \Rightarrow b = \frac{3a-1}{2}$. The smallest value of $a(a > 1)$ for b to be whole is $a = 3 \Rightarrow a = 3$ and $b = 4$. But 32_{three} is not possible because there is no numeral 3 in base three. The smallest possible solution is thus $a = 5$ and $b = 7$, or $32_{five} = 23_{seven}$.

25. **(a)** The greatest product requires the largest multiplicands which can be formed using the four numbers: $8 \times 763 > 7 \times 863$ because $8 \times 700 = 7 \times 800$ but $8 \times 63 > 7 \times 63$. Thus:

$$\begin{array}{cccc} & \boxed{7} & \boxed{6} & \boxed{3} \\ & \times & & \boxed{8} \\ \hline 6 & 1 & 0 & 4 \end{array}$$

(b) The least product requires the smallest multiplicands which can be formed using the four numbers; $3 \times 678 < 6 \times 378$ because $3 \times 600 = 6 \times 300$ but $3 \times 78 < 6 \times 78$. Thus:

$$\begin{array}{cccc} & \boxed{6} & \boxed{7} & \boxed{8} \\ & \times & & \boxed{3} \\ \hline 2 & 0 & 3 & 4 \end{array}$$

26. In each case it may be helpful to generate a multiplication table in the appropriate base; e.g. in base 5:

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

(a)

$$\begin{array}{cccl} & 1 & & \\ & 3 & 2 & five \\ \times & & 4 & five \\ \hline 2 & 3 & 3 & five \end{array}$$

(b)

$$\begin{array}{r@{}l} & 4_{five} \text{ remainder } 1_{five} \\ 4_{five} & \overline{)\,3\ 2}_{five} \\ & \underline{3\ 1} \\ & 1 \end{array}$$

(c)

$$\begin{array}{r} 4\ 3_{six} \\ \times\ 2\ 3_{six} \\ \hline 2\ 1\ 3 \\ 1\ 3\ 0 \\ \hline 1\ 5\ 1\ 3_{six} \end{array}$$

(d)

$$\begin{array}{r@{}l} & 3\ 1_{five} \\ 3_{five} & \overline{)\,1\ 4\ 3}_{five} \\ & \underline{1\ 4} \\ & 0\ 3 \\ & \underline{0\ 3} \\ & 0 \end{array}$$

(e)

$$\begin{array}{r@{}l} & 1\ 1\ 0_{two} \\ 11_{two} & \overline{)\,1\ 0\ 0\ 1\ 0}_{two} \\ & \underline{1\ 1} \\ & 1\ 1\ 0 \\ & \underline{1\ 1\ 0} \\ & 0 \end{array}$$

(f)

$$\begin{array}{r} 1\ 0\ 1\ 1\ 0_{two} \\ \times\ \qquad 1\ 0\ 1_{two} \\ \hline 1\ 0\ 1\ 1\ 0 \\ 1\ 0\ 1\ 1\ 0 \\ \hline 1\ 1\ 0\ 1\ 1\ 1\ 0_{two} \end{array}$$

27. 8 hours $\times$ 62 mph $= (8 \cdot 60) + (8 \cdot 2) = 480 + 16 =$ **496 miles** (i.e., the front-end multiplying method).

28. Estimates may vary. The range would be from $30 \cdot 20 = 600$ to $40 \cdot 30 = 1200$. Rounding, $38 \approx 35$ or 40 and $23 \approx 20$ or 25, thus about $35 \cdot 20 = 700$ seats, or $40 \cdot 25 = 1000$ seats. 700 would be low (rounded down) and 1000 would be high (rounded up).

29. Answers may vary:

(a) **Different**. One factor is the same in each and the other is 4 times larger.

(b) **Same**. 22 was divided by 2 to obtain 11 while 32 was multiplied by 2 to obtain 64. The result is to multiply the original computation by $\frac{2}{2} = 1$ which does not change it.

(c) **Same**. 13 was multiplied by 3 and 33 was divided by 3, thus the original computation was multiplied by 1.

30. Note that in parts (b) through (d), a larger divisor produces a lower quotient when the dividend stays the same and a larger dividend produces a higher quotient when the divisor stays the same.

(a) **High**. $299 \cdot 300 < 300 \cdot 300$.

(b) **Low**. $6001 \div 299 > 6000 \div 300$.

(c) **Low**. $6000 \div 299 > 6000 \div 300$.

(d) **Low**. $10 \cdot 99 = 990 < 999 \Rightarrow 999 \div 99 > 10$.

31. Answers may vary. One strategy to find $(n5)^2$ would be to write $n(n+1)$ and append 25 (because 5^2 is always 25); e.g., $65^2 = (6 \cdot 7 = 42)$ and append $25 = 4225$ and $75^2 = (7 \cdot 8 = 56)$ and append $25 = 5625$.

32. Answers vary. If the estimated product is 42,000 and 42 = 6 • 7 so 42,000 = 600 • 70, then an estimate of $42{,}000 \approx 612 \bullet 73$.

Assessment 3-5B

1. **(a)** There are $8 + 4 = 12$ factors of 3. Thus $3^8 \cdot 3^4 = 3^{8+4} = \mathbf{3^{12}}$.

(b) $5^2 \cdot 5^4 \cdot 5^2 = 5^{2+4+2} = \mathbf{5^8}$.

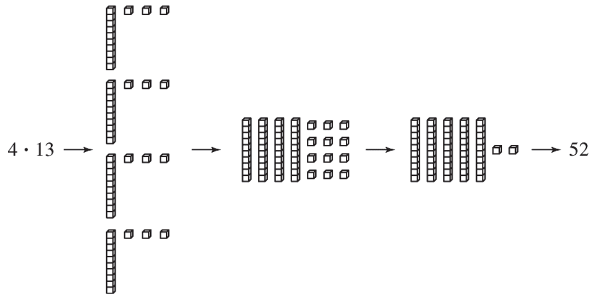
(c) $6^2 \cdot 2^2 \cdot 3^2 = 6^2 \cdot (2 \cdot 3)^2 = 6^2 \cdot 6^2 = \mathbf{6^4} = (2 \cdot 3)^4$.

(c) $4^8 \cdot 8^4 \cdot 32^5 = 2^{16} \cdot 2^{12} \cdot 2^{25} = \mathbf{2^{53}}$.

2. **(a)** $2^{20} + 2^{20} = 2^{20}(1+1) = 2^{20} \cdot 2 = 2^{21}$, so **they are equal**.

(b) $9 \cdot 3^{30} = 3^2 \cdot 3^{30} = 3^{32}$, so $\mathbf{3^{31} < 9 \cdot 3^{30} < 3^{33}}$.

3. We can use the repeated addition model of multiplication and make four groups of 13:



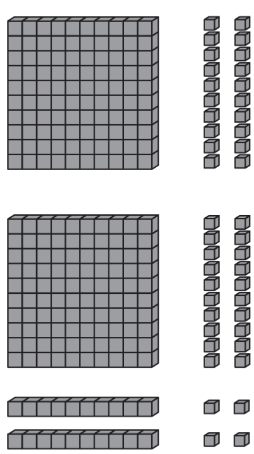
4. **(a)** The following partial products are obtained through the distributive property of multiplication over addition, as shown below:

$$\begin{array}{rl} 13 & \\ \times\ 12 & \\ \hline 6 & (2\times 3) \\ 20 & (2\times 10) \\ 30 & (10\times 3) \\ 100 & (10\times 10) \\ \hline 156 & \end{array}$$

(b)

$$\begin{array}{rl} 12 & \\ \times\ 22 & \\ \hline 4 & (2\times 2) \\ 20 & (2\times 10) \\ 40 & (20\times 2) \\ 200 & (20\times 10) \\ \hline 264 & \end{array}$$

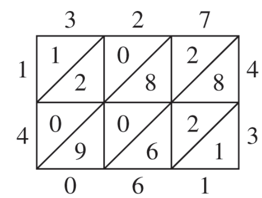
Which is illustrated by:



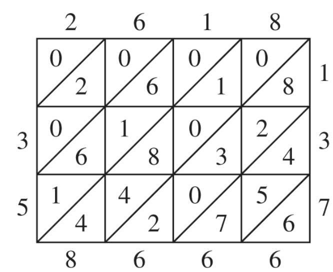
5. Start by multiplication of 4_4 by 7. The _ must be 8 since the 2 must be regrouped from $7\cdot 4$ and only $7\cdot 8+2$ yields an 8 in the tens column. Now that the factors are 484 and 327, compute the partial products in the usual way to solve for the other missing terms.

$$\begin{array}{r} 4\,\underline{\mathbf{8}}\,4 \\ \times\ 3\,2\,7 \\ \hline 3\,\underline{\mathbf{3}}\,8\,8 \\ 9\,6\,8 \\ \underline{\mathbf{1}}\,4\,5\,2 \\ \hline 1\,5\,8\,2\,\underline{\mathbf{6}}\,8 \end{array}$$

6. **(a)** $327\times 43 = 14061$:



(b) $2618\times 137 = 358,666$:



7. **(a)** Answers may vary. Assume the daily averages for taking a shower, toilet flushing, washing of hands and face, drinking, and brushing teeth, and assume three dish washes and two cookings (have cold cereal for breakfast). Then $75 +$
$22 + 7 + 1 + 1 + (3\cdot 30) + (2\cdot 18) =$
232 liters.

(b) About 310,000,000 people $\times$ 200 liters per day
$=$ **about 62,000,000,000 liters** per day.

8. (a) Use a three-digit number in base five, abc_{five}.

$abc_{five} \cdot 10_{five} = (a \cdot 10^2 + b \cdot 10^1 + c \cdot 10^0)_{five} \cdot 10_{five} = (a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1)_{five} = \boldsymbol{abc0_{five}}$.

(b) Multiplying by 100_{five} is the same as multiplying twice by 10_{five} (i.e., $10_{five} \cdot 10_{five} = 100_{five}$). Since a zero is annexed each time a number is multiplied by 10, multiplication by 100 will result in annexation of two zeros (this is true for multiplying by 100 in any base, not just base five).

(c) $14_{five} \cdot 23_{five} = (10 + 4)_{five} \cdot 23_{five} = (10 \cdot 23 + 4 \cdot 23)_{five} = (230 + 202)_{five} = \boldsymbol{432_{five}}$.

Observe that in base five notation

$$\begin{aligned} 4_{five} \cdot 23_{five} &= [4 \cdot (2 \cdot 10 + 3)]_{five} \\ &= (130 + 22)_{five} = 202_{five}. \end{aligned}$$

9.

		Halves		Doubles	
	$\rightarrow$	31	$\times$	$\boxed{69}$	
Halve 31	$\rightarrow$	15		$\boxed{138}$	Double 69
Halve 15	$\rightarrow$	7		$\boxed{276}$	Double 138
Halve 7	$\rightarrow$	3		$\boxed{552}$	Double 276
Halve 3	$\rightarrow$	1		$\boxed{1104}$	Double 552

$$\begin{array}{r} 69 \\ 138 \\ 276 \\ 552 \\ 1104 \\ \hline \mathbf{2139} \end{array}$$

2139 which is the product of 31×69.

10. (a) 444 calories per hour $\cdot$ 4 hours = **1776 calories**.

(b) Jane burned $462 \cdot 3 = 1386$ calories and Carolyn burned $198 \cdot 4 = 792$ calories. **Jane** burned 594 more calories.

(c) Lyle burned $708 \times 4 = 2832$ calories and Maurice burned $444 \times 5 = 2220$ calories. **Lyle** burned 612 more calories.

11. Each day Glenn burned $666 \times 3 = 1998$ calories. He burned more calories working out than he consumed in additional calories. Assuming that the three hour swims were not part of his usual daily routine, he **did not gain weight**.

12. \$24/thousand $\times$ 30 thousands = \$720 annual premiums. \$720 $\div$ 12 months = **\$60**/month in installment payments.

13. (a) Repeated subtraction

$$\begin{array}{r|l l} 7 & 392 & \\ & \underline{350} & 50 \text{ sevens} \\ & 42 & \\ & \underline{42} & \underline{6} \text{ sevens} \\ & 0 & 56 \text{ reminder } 0 \end{array}$$

Standard algorithm

$$\begin{array}{r} 56 \text{ remainder } 0 \\ 7\overline{)392} \\ \underline{35} \\ 42 \\ \underline{42} \\ 0 \end{array}$$

(b) Repeated subtraction

$$\begin{array}{r|l l} 37 & 925 & \\ & \underline{740} & 20 \text{ 37's} \\ & 185 & \\ & \underline{185} & \underline{5} \text{ 37's} \\ & 0 & 25 \text{ remainder } 0 \end{array}$$

Standard algorithm

$$\begin{array}{r} 25 \text{ remainder } 0 \\ 37\overline{)925} \\ \underline{74} \\ 185 \\ \underline{185} \\ 0 \end{array}$$

(c) Repeated subtraction

$$\begin{array}{r|l l} 423 & 5002 & \\ & \underline{4653} & \underline{11} \text{ 423's} \\ & 349 & 11 \text{ remainder } 349 \end{array}$$

Standard algorithm

$$\begin{array}{r|l} & 1\ 1 \quad \text{remainder } 349 \\ 4\ 2\ 3 & \overline{5\ 0\ 0\ 2} \\ & \underline{4\ 2\ 3} \\ & 7\ 7\ 2 \\ & \underline{4\ 2\ 3} \\ & 3\ 4\ 9 \end{array}$$

14. **(a)** The greatest quotient follows from dividing the smallest number possible into the largest number possible $(876 \div 3 = 292)$:

$$\boxed{3}\ \overline{)\ \boxed{8}\ \boxed{7}\ \boxed{6}}$$

(b) The least quotient follows from dividing the largest number possible into the smallest number possible
$(367 \div 8 = 45$ remainder $7)$:

$$\boxed{8}\ \overline{)\ \boxed{3}\ \boxed{6}\ \boxed{7}}$$

15. $5 \cdot x = 250 \Rightarrow x = 50$, the original number. The answer should have been $50 \div 5 = \mathbf{10}$.

16. Because sum of two numbers less than 10,000 is less than 20,000, ***M* must be 1**. To produce an exchange (carry), *S* must be 8 or 9. Since $S + M$ is 9 or 10, *O* must be 0 or 1. But *M* is 1, so ***O* is 0**. At this point we have

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & 1 & 0 & R & E \\ \hline & 1 & 0 & N & E & Y \end{array}$$

and we know that *S* is 8 or 9.

If there is an exchange from column three to four, then $E = 9$ (since $E + 0 = E$) and then $N = 0$. But $O = 0$. So there is no exchange from column three to four and thus, $\boldsymbol{S = 9}$.

If there is no exchange from column two to column three, then $E = N$, which is not possible. There is a carry and thus $N = E + 1$.

We have
$$\begin{array}{cccccc} & & 9 & E & E+1 & D \\ + & & 1 & 0 & R & E \\ \hline & 1 & 0 & E+1 & E & Y \end{array}$$

If there were no exchange from column 1, then *R* is 9. But S is 9. So $\boldsymbol{R = 8}$.

Similar reasoning results in
$\boldsymbol{E = 5, N = 6, D = 7,\ \& \ Y = 2}$

$$\begin{array}{ccccc} & & 1 & 1 & \\ & 9 & 5 & 6 & 7 \\ & 1 & 0 & 8 & 5 \\ \hline 1 & 0 & 6 & 5 & 2 \end{array}$$

17. Let x represent the number of dimes in the second box. Then $4x + x + 3(4x) = 340$. Solving we find that $x = 20$ dimes. Converting to a dollar value, the second box contains $20 \times \$0.10 = \$2$. Thus the boxes contain **8, 2, and 24 dollars** respectively.

18. $\$5340 - (12 \text{ months} \times \$95 \text{ per month}) = \$4200$ saved in the final two years.
$\$4200 \div 24 \text{ months} =$ **$175** per month.

19. **(a)** Regrouping was not used. Place value was not observed.

(b) 4 was multiplied by 6 to obtain 24. The 2 was regrouped; then 6 and 3 were added instead of multiplied and the 2 was added to obtain 11.

20.
$$\begin{aligned} 35 \cdot 100 &= (3 \cdot 10 + 5) \cdot 100 && \text{expanded form} \\ &= (3 \cdot 10 + 5) \cdot 10^2 && \text{definition of } a^n \\ &= (3 \cdot 10) \cdot 10^2 + 5 \cdot 10^2 && \text{distributive property} \\ &= 3(10 \cdot 10^2) + 5 \cdot 10^2 && \text{associative property of multiplication} \\ &= 3 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 0 \cdot 1 && \text{rules of exponents multiplication by zero property} \\ &= 3500 && \text{place value} \end{aligned}$$

21. **(a)** Let n be a whole number. Then $\boldsymbol{n = 4q + 1}$, $q \in W$, will produce all the whole numbers with remainder of 1 when divided by 4.

(b)

q	n
0	1
1	5
2	9

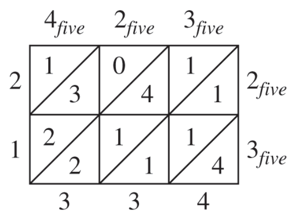
etc

(c) This is an **arithmetic sequence** with $a_1 = 1$ and $d = 4$.

22. **(a)** **Base four**. In the units column it was necessary to regroup to subtract; since $6 - 3 = 3$, a 4 must have been regrouped. Then $(4 + 2)_{ten} = 12_{four}$ and $(12 - 3)_{four} = 3_{four}$.

(b) **Any base $\geq$ two** would yield this quotient.

23. $423_{five} \cdot 23_{five} = 21334_{five}$:



24. Form the equation $4a + 1 = b + 4 \Rightarrow 4a - 3 = b$. When $a = 2, b = 5$, which is a solution to the equation but not a solution to the problem since $a = 2$ cannot be the base for 41_a. Continuing, the first useable value for a is 5, which leads to a base of 17 for b. Thus the solution is $\boldsymbol{a = 5, b = 17}$, or $41_{five} = 14_{seventeen}$.

25. **(a)** The greatest product requires the largest multiplicands which can be formed using the five numbers; $83 \times 762 > 73 \times 862$ because $80 \times 700 = 70 \times 800$ but $83 \times 62 > 73 \times 62$:

$$\begin{array}{r} \boxed{7}\ \boxed{6}\ \boxed{2} \\ \times\ \boxed{8}\ \boxed{3} \\ \hline 6\ 3\ 2\ 4\ 6 \end{array}$$

(b) The least product requires the smallest multiplicands which can be formed using the five numbers; $26 \times 378 < 36 \times 278$ because $20 \times 300 = 30 \times 200$: but $26 \times 78 < 36 \times 78$

$$\begin{array}{r} \boxed{3}\ \boxed{7}\ \boxed{8} \\ \times\ \boxed{2}\ \boxed{6} \\ \hline 9\ 8\ 2\ 8 \end{array}$$

26. In each case it may be helpful to generate a multiplication table in the appropriate base; e.g. in base 5:

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

(a)
$$\begin{array}{rl} {}^{1}\ \ \ \ & \\ 4\ 2 & _{five} \\ \times\ \ \ 3 & _{five} \\ \hline 2\ 3\ 1 & _{five} \end{array}$$

(b)
$$\begin{array}{r} 3_{five} \\ 4_{five} \overline{\smash{\big)}\,2\ 2}_{five} \\ 2\ 2 \\ \hline 0 \end{array}$$

(c)
$$\begin{array}{rl} 3\ 2 & _{five} \\ \times\ 4\ 2 & _{five} \\ \hline 1\ 1\ 4 & \\ 2\ 3\ 3\ \ \ & \\ \hline 2\ 4\ 4\ 4 & _{five} \end{array}$$

(d)
$$\begin{array}{r} 3\ 1_{five} \\ 2\ 3_{five} \overline{\smash{\big)}\,1\ 3\ 1\ 3}_{five} \\ 1\ 2\ 4\ \ \ \\ \hline 2\ 3 \\ 2\ 3 \\ \hline 0 \end{array}$$

(e)
$$\begin{array}{rl} 1\ 0\ 1 & _{two} \\ \times\ 1\ 0\ 1 & _{two} \\ \hline 1\ 0\ 1 & \\ 1\ 0\ 1\ \ \ \ \ & \\ \hline 1\ 1\ 0\ 0\ 1 & _{two} \end{array}$$

(f)

$$\begin{array}{r} 1\ 1_{\ two} \\ 11_{two}\overline{)1\ 0\ 0\ 1}_{\ two} \\ \underline{0\ 1\ 1} \\ 1\ 1 \\ \underline{1\ 1} \\ 0 \end{array}$$

27. 6 hours $\times$ 65 mph $= (6 \cdot 60) + (6 \cdot 5) = 360 + 30 =$ **390 miles** (i.e., the front-end multiplying method).

28. About **21,000** calories. 3540 calories per pound is about 3500 calories, and $6 \cdot 3500 = 3 \cdot (2 \cdot 3500) = 3 \cdot (7000) = 21{,}000$ calories. (Actual is 21,240.)

29. Answers may vary:

(a) **Different**. The estimates are quite different. The second number (22) is $\frac{1}{4}$ the first number (88).

(b) **Same**. 93 was divided by 3 to obtain 31 and 15 was multiplied by 3 to obtain 45.

(c) **Different**. $20 \cdot 17 = 17 \cdot 20$; $12 < 17$ and $18 < 20$.

30. Note that in parts (a) through (d), a larger divisor produces a lower quotient when the dividend stays the same and a larger dividend produces a higher quotient when the divisor stays the same.

(a) **High**. $398 \cdot 500 < 400 \cdot 500$.

(b) **Low**. $8001 \div 398 > 8001 \div 400 > 8000 \div 400$.

(c) **Low**. $10{,}000 \div 999 > 10{,}000 \div 1000$.

(d) **High**. $1999 \div 201 < 2000 \div 200$.

31. Answers may vary. One strategy to mentally find $ab \cdot 99$, where ab represents the number $10a + b$, would be to write $(ab - 1)$ as the first two digits and then append the digits obtained from $100 - ab$. E.g., to find $12 \cdot 99$ the first two digits are $12 - 1 = 11$ and the second two digits are $100 - 12 = 88$, or $12 \cdot 99 = 1188$.

Another method is to show $ab \cdot 99$ as $ab(100 - 1) = 100ab - ab$.

32. Answers vary. If the estimated product is 2400 and $24 = 3 \cdot 8$ so $2400 = 30 \cdot 80$, then an estimate of $2400 \approx 34 \cdot 82$.

Mathematical Connections 3-5: Review Problems

22. Answers may vary; for example, $3 + 0 = 3 = 0 + 3$.

23. **(a)** $ax + bx + 2x = \mathbf{(a + b + 2)x}$.

(b) $3(a + b) + x(a + b) = \mathbf{(3 + x)(a + b)}$.

24. 59,260 miles $-$ 52,281 $=$ **6979 miles** traveled.

25. **(a)** $36 \div 4 = 9 \Rightarrow \mathbf{36 = 4 \cdot 9}$.

(b) $112 \div 2 = x \Rightarrow \mathbf{112 = 2x}$.

(c) $48 \div x = 6 \Rightarrow \mathbf{48 = x6}$.

(d) $x \div 7 = 17 \Rightarrow \mathbf{x = 7 \cdot 17}$.

Chapter 3 Review

1. **(a)** Tens.

(b) Thousands.

(c) Hundreds.

2. **(a)** $\overline{\text{CD}}\text{XLIV} = 1000 \cdot \text{CD} + \text{XLIV} = 1000 \cdot 400 + 40 + 4 =$ **400,044**.

(b) $432_{five} = 4 \cdot 5^2 + 3 \cdot 5 + 2 = 100 + 15 + 2 =$ **117**.

(c) $ET0_{twelve} = 11 \cdot 12^2 + 10 \cdot 12 + 0 = 1584 + 120 =$ **1704**.

(d) $1011_{two} = 1 \cdot 2^3 + 1 \cdot 2 + 1 = 8 + 2 + 1 =$ **11**.

(e) $4136_{seven} = 4 \cdot 7^3 + 1 \cdot 7^2 + 3 \cdot 7 + 6 = 1372 + 49 + 21 + 6 =$ **1448**.

3. **(a)** CM $\rightarrow$ 900; XC $\rightarrow$ 90; IX $\rightarrow$ 9, or 999 $=$ **CMXCIX**.

(b) Eight groups of 10 and 6 units of one, or ∩∩∩∩∩∩∩∩||||||.

(c) Six units of 20 and three units of 1, or .

(d) 2 groups of 125 + 3 groups of 25 + 4 groups of 5 + 1, or $\mathbf{2341_{five}}$.

(e) 1 group of 16 + 1 group of 8 + 0 groups of 4 + 1 group of 2 + 1, or $\mathbf{11011_{two}}$.

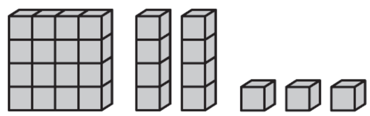
4. **(a)** $3^{4+7+6} = \mathbf{3^{17}}$.

(b) $2^{10+11} = \mathbf{2^{21}}$.

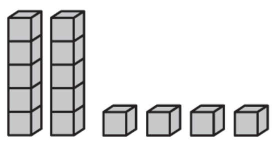
5. $1000_{three} + 200_{three} + 100_{three} + 20_{three} = 1000_{three} + 1000_{three} + 20_{three} = \mathbf{2020_{three}}$.

6. $51_{ten} = (1 \cdot 10^3)_{three} + (2 \cdot 10^2)_{three} + (2 \cdot 10)_{three} + (0 \cdot 1)_{three} \Rightarrow$ **<u>1</u> block, <u>2</u> flats, <u>2</u> longs, <u>0</u> units**. So the fewest blocks needed is **5.**

7. **(a)** 123_{four}:



(b) 24_{five}:



8. **(a)** $40{,}000{,}000{,}000 = 4 \cdot 10^{10}$, so the place value of 4 is $\mathbf{10^{10}}$.

(b) A number in base five having ten digits is $m \cdot 5^9 + n \cdot 5^8 + \cdots$. The place value of the second digit is therefore $\mathbf{5^8}$.

(c) 30 zeros and a 1 in base two represents a number $n = 1 \cdot 2^{31} + 0 \cdot 2^{30} + \cdots + 1$. The place value of the lead digit is therefore $\mathbf{2^{31}}$.

9. **(a)** $10^{10} + 23 = 1 \cdot 10^{10} + 0 \cdot 10^9 + \cdots + 2 \cdot 10 + 3$, or $\mathbf{10{,}000{,}000{,}023}$.

(b) $2^{10} + 1 = 1 \cdot 2^{10} + 0 \cdot 2^9 + \cdots + 1$, or $\mathbf{10{,}000{,}000{,}001_{two}}$.

(c) $5^{10} + 1 = 1 \cdot 5^{10} + 0 \cdot 5^9 + \cdots + 1$, or $\mathbf{10{,}000{,}000{,}001_{five}}$.

(d)
$$\begin{array}{rccccccccccc} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & & & & & & & & & & & 1 \\ \hline & & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \end{array}$$

(e) 10 must be borrowed from each 0 digit in the subtrahend, starting with the rightmost, just as with base ten.
$$\begin{array}{rcccccccccccl} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & _{two} \\ - & & & & & & & & & & & 1 & \\ \hline & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & _{two} \end{array}$$

(f)
$$\begin{array}{rccccccl} & 1 & 0 & 0 & 0 & 0 & 0 & _{twelve} \\ - & & & & & & 1 & \\ \hline & & E & E & E & E & E & _{twelve} \end{array}$$

(g)
$$\begin{aligned} & 7 \cdot 5^6 + 3 \cdot 5^4 + 11 \\ & = (5+2) \cdot 5^6 + 0 \cdot 5^5 + 3 \cdot 5^4 \\ & \quad + 0 \cdot 5^3 + 0 \cdot 5^2 + 10 + 1 \\ & = 5 \cdot 5^6 + 2 \cdot 5^6 + 0 \cdot 5^5 + 3 \cdot 5^4 \\ & \quad + 0 \cdot 5^3 + 0 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0 \\ & = 1 \cdot 5^7 + 2 \cdot 5^6 + 0 \cdot 5^5 + 3 \cdot 5^4 \\ & \quad + 0 \cdot 5^3 + 0 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0 \\ & = 12030021 \end{aligned}$$

10. Answers may vary. One example would be selling pencils by units, dozens, and gross; i.e., base twelve.

11. **(a)** The Egyptian system had seven symbols. It was a tally system and a grouping system, and it used the additive property. It did not have a symbol for zero which was not important inasmuch as they did not use place value.

(b) The Babylonian system used only two symbols. It was a place value system (base 60) and was additive within the positions. It lacked a symbol for zero until around 300 B.C.

(c) The Roman system used seven symbols. It was additive, subtractive, and multiplicative. It did not have a symbol for zero.

(d) The Hindu-Arabic system uses ten symbols. It uses place value involving base ten and has a symbol for zero.

12. **(a)** There is 1 group of $125(5^3)$ in 128, with remainder 3.

There are 0 groups of 25 in 3, with remainder 0.

There are 0 groups of 5 in 3, with remainder 0.

There are 3 groups of 1 in 3, with remainder 0.

Thus $128 = \mathbf{1003_{five}}$.

(b) There is 1 group of $128(2^7)$ in 128, with remainder 0.

There are 0 groups of 64 in 0, with remainder 0.

There are 0 groups of 32 in 0, with remainder 0.

There are 0 groups of 16 in 0, with remainder 0.

There are 0 groups of 8 in 0, with remainder 0.

There are 0 groups of 4 in 0, with remainder 0.

There are 0 groups of 2 in 0, with remainder 0.
There are 0 groups of 1 in 0, with remainder 0.

Thus $128 = \mathbf{10000000}_{\boldsymbol{two}}$.

(c) There are $10(T)$ groups of 12 in 128, with remainder 8. There are 8 groups of 1 in 8, with remainder 0.

Thus $128 = \boldsymbol{T8}_{\boldsymbol{twelve}}$.

13. Place value is determined by powers of each base.

(a) $2^{10} + 2^3 = \mathbf{10000001000}_{\boldsymbol{two}}$.

(b) $11 \cdot 12^5 + 10 \cdot 12^3 + 20 = 11 \cdot 12^5 + 10 \cdot 12^3 + 1 \cdot 12 + 8 = \boldsymbol{E0T018}_{\boldsymbol{twelve}}$.

14. $1 \cdot b^2 + 2b + 3 = 83 \Rightarrow b^2 + 2b - 80 = 0 \Rightarrow (b-8)(b+10) = 0 \Rightarrow b = 8$ or $b = -10$. Since the base must be positive, $\boldsymbol{b = 8}$.

15. (a) **Distributive** property of multiplication over addition.

(b) **Commutative** property of addition.

(c) **Identity** property of multiplication for whole numbers.

(d) **Distributive** property of multiplication over addition.

(e) **Commutative** property of multiplication.

(f) **Associative** property of multiplication.

16. (a) $13 = 3 + 10$. Since 10 is a natural number, $3 < 13$.

(b) $12 = 3 + 9$. Since 3 is a natural number, $12 > 9$.

17. (a) $4 \cdot \boxed{\mathbf{10} \le \boldsymbol{whole\ number} \le \mathbf{15}} - 37 < 27$.

(b) $398 = \boxed{\mathbf{10}} \cdot 37 + 28$.

(c) $\boxed{\boldsymbol{n}} \cdot (3+4) = \boxed{\boldsymbol{n}} \cdot 3 + \boxed{\boldsymbol{n}} \cdot 4$, where $n \in W$ is any whole number.

(d) $42 - \boxed{\boldsymbol{Any\ whole\ number} \le \mathbf{26}} \ge 16$.

18. (a) $3a + 7a + 5a = (3 + 7 + 5)a = \mathbf{15}\boldsymbol{a}$.

(b) $3x^2 + 7x^2 - 5x^2 = (3 + 7 - 5)x^2 = \mathbf{5}\boldsymbol{x}^{\mathbf{2}}$.

(c) $x(a + b + y) = \boldsymbol{xa + xb + xy}$.

(d) $(x+5)3 + (x+5)y = \mathbf{3}\boldsymbol{x} + \mathbf{15} + \boldsymbol{xy} + \mathbf{5}\boldsymbol{y}$

or

$(x+5)3 + (x+5)y = \boldsymbol{(x+5)(3+y)}$.

(e) $3x^2 + x = x(3x+1)$.

(f) $2x^5 + x^3 = x^3(2x^2 + 1)$.

19. 60 people $\times$ 8 ounces $=$ 480 ounces required.
$480 \div 12$ ounces per can $=$ **40** twelve-ounce cans.

20. 2 slacks $\times$ 3 blouses $\times$ 2 sweaters $=$ **12** outfits.

21. Work backward from 93 using inverse operations:
Subtract 89, giving 4;
Add 20, giving 24;
Divide by 12, giving 2; then
Multiply by 13, giving **26** as the original number.

22. \$80 per person $\times$ 80 people $=$ \$6400. The **\$6000** package is less expensive.

23. 30 hours per week $\times$ \$5 per hour $+$ 8 hours overtime $\times$ \$8 per overtime hour $=$ **\$214**.

24. Let q be the amount from the first question. Then winnings are $q + 2q + 4q + \cdots$ which is a geometric sequence with $a_1 = q, r = 2,$ and $n = 5$. $6400 = q(2)^{5-1} \Rightarrow 16q = 6400 \Rightarrow q = \mathbf{\$400}$.

25. (a) Let n be the original number. Then

$$\frac{2[2(n+17) - 4] + 20}{4} - 20 = \frac{4(n+17) - 8 + 20}{4} - 20 = \frac{4n + 80}{4} - 20 = n + 20 - 20 = \boldsymbol{n}.$$

(b) Answers may vary. For example, if n is the original number:

$4(n+18) - 7 = 4n + 65$.

Then two more steps might be:
$4n + 65 - 65$ (subtract 65);
$\frac{4n}{4}$ (divide by 4).

(c) Answers may vary; use the techniques of parts (a) and (b).

26. Scratch:

$$\begin{array}{rrrr} & {\scriptstyle 1} & & \\ & 3 & 1 & 6 \\ & \cancel{7}_1 & 1 & 2 \\ + & & \cancel{9}_1 & 1 \\ \hline 1 & 1 & 1 & 9 \end{array}$$

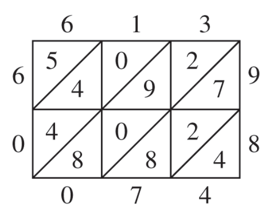
Traditional:

$$\begin{array}{rrrr} & {\scriptstyle 1} & & \\ & 3 & 1 & 6 \\ & 7 & 1 & 2 \\ + & & 9 & 1 \\ \hline 1 & 1 & 1 & 9 \end{array}$$

27. Traditional:

$$\begin{array}{rrrrr} & & 6 & 1 & 3 \\ & \times & & 9 & 8 \\ \hline & 4 & 9 & 0 & 4 \\ 5 & 5 & 1 & 7 & \\ \hline 6 & 0 & 0 & 7 & 4 \end{array}$$

Lattice:



28. **(a)** Repeated subtraction:

$$\begin{array}{r|rrrrl} 9\ 1\ 2 & 4 & 8 & 0 & 3 & \\ & \underline{4} & \underline{5} & \underline{6} & \underline{0} & \underline{5\text{-}912\text{'s}} \\ & & 2 & 4 & 3 & 5\text{-}912\text{'s} \Rightarrow 5 \text{ r } 243 \end{array}$$

Traditional:

$$\begin{array}{r|rrrr} & & & & 5 \quad \Rightarrow \quad 5 \text{ r } 243 \\ 9\ 1\ 2 & 4 & 8 & 0 & 3 \\ & 4 & 5 & 6 & 0 \\ \hline & & 2 & 4 & 3 \end{array}$$

(b) Repeated subtraction:

$$\begin{array}{r|rrrrl} 1\ 1 & 1 & 0 & 1 & 1 & \\ & & \underline{9} & \underline{9} & \underline{0} & 90\text{-}11\text{'s} \\ & & & 2 & 1 & \\ & & & \underline{1} & \underline{1} & \underline{1 - 11} \\ & & & 1 & 0 & 91\text{-}11\text{'s} \Rightarrow \quad 91 \text{ r } 10 \end{array}$$

Traditional:

$$\begin{array}{r|rrrr} & & & 9 & 1 \quad \Rightarrow \quad 91 \text{ r } 10 \\ 1\ 1 & 1 & 0 & 1 & 1 \\ & & \underline{9} & \underline{9} & \underline{0} \\ & & & 2 & 1 \\ & & & \underline{1} & \underline{1} \\ & & & 1 & 0 \end{array}$$

(c) Repeated subtraction:

$$\begin{array}{r|rrrrl} 2\ 3_{five} & 3 & 3 & 1 & 2_{five} & \\ & \underline{2} & \underline{3} & \underline{0} & \underline{0} & (100\text{-}23\text{'s})_{five} \\ & 1 & 0 & 1 & 2 & \\ & \underline{1} & \underline{0} & \underline{1} & \underline{0} & \underline{(20\text{-}23\text{'s})_{five}} \\ & & & & 2_{five} & (120\text{-}23\text{'s})_{five} \end{array}$$

$\Rightarrow 120_{five}$ remainder 2_{five}

Traditional:

$$\begin{array}{r|rrrr} & & 1 & 2 & 0_{five} \\ 2\ 3_{five} & 3 & 3 & 1 & 2_{five} \\ & \underline{2} & \underline{3} & & \\ & 1 & 0 & 1 & \\ & \underline{1} & \underline{0} & \underline{1} & \\ & & & 0 & 2_{five} \end{array}$$

$\Rightarrow 120_{five}$ remainder 2_{five}

(d) Repeated subtraction:

$$\begin{array}{r|rrrrl} 1\ 1_{two} & 1 & 0 & 1 & 1_{two} & \\ & & \underline{1} & \underline{1} & \underline{0} & \\ & & 1 & 0 & 1 & (10\text{-}11\text{'s})_{two} \\ & & & \underline{1} & \underline{1} & \underline{(1\text{-}11)_{two}} \\ & & & 1 & 0_{two} & (11\text{-}11\text{'s})_{two} \end{array}$$

$\Rightarrow 11_{two}$ remainder 10_{two}

(d) Traditional:

$$\begin{array}{r|l} & 1\ 1_{two} \\ \cline{2-2} 1\ 1_{two} & 1\ 0\ 1\ 1_{two} \\ & \underline{1\ 1} \\ & 1\ 0\ \ 1 \\ & \underline{1\ \ 1} \\ & 1\ \ 0\ two \end{array}$$

$\Rightarrow 11_{two}$ remainder 10_{two}

29. **(a)** If $4803 \div 912 = 5$ remainder 243
Then $912 \cdot 5 + 243 = 4803$.

(b) If $1011 \div 11 = 91$ remainder 10 Then $11 \cdot 91 + 10 = 1011$.

(c) If $(3312 \div 23)_{five} = (120 \text{ remainder } 2)_{five}$
Then $(23 \cdot 120)_{five} + 2_{five} = 3312_{five}$.

(d) If $(1011 \div 11)_{two} = (11 \text{ remainder } 10)_{two}$
Then $(11 \cdot 11)_{two} + 10_{two} = 1011_{two}$.

30. **(a)** $19 \cdot 5 \cdot 194 \cdot 2 = (19 \cdot 194) \cdot 10 = \mathbf{36{,}860}$.

(b) $379 \cdot 4 \cdot 193 \cdot 25 = (379 \cdot 193) \cdot 100 = \mathbf{7{,}314{,}700}$.

(c) $8 \cdot 481 \cdot 73 \cdot 125 = (481 \cdot 73) \cdot 1000 = \mathbf{35{,}113{,}000}$.

(d) $374 \cdot 200 \cdot 893 \cdot 50 = (374 \cdot 893) \cdot 10{,}000 = \mathbf{3{,}339{,}820{,}000}$.

31. $(\$320 \times 6 \text{ mos}) + (\$410 \times 6 \text{ mos}) = \mathbf{\$4380}$.

32. $15{,}600$ cans per hour $\div$ 24 cans per case $=$ 650 cases per hour. Then 650×4 hours $=$ **2600 cases**.

33. There are 8 groups of 3 in 24 (or two dozen) apples. $8 \times 69¢$ per group $= 552¢$ on sale;
$32¢$ each $\times$ 24 apples $= 768¢$ regular price.
$768¢ - 552¢ =$ **216¢ saved** (or \$2.16).

34. Let b be the number of bicycles and t be the number of tricycles. Then

$$\begin{array}{lr} 2b + 3t = 126 \text{ wheels} & (i) \\ 2b + 2t = 108 \text{ pedals} & (ii) \end{array}$$

Subtracting (ii) from $(i) \Rightarrow t =$ **18 tricycles**.

Substituting $t = 18$ into $(i) \Rightarrow b =$ **36 bicycles**.

35. **(a)**
$$\begin{array}{rrrl} {\scriptstyle 1} & {\scriptstyle 1} & & \\ 1 & 2 & 3 & {}_{five} \\ + & 3 & 4 & {}_{five} \\ \hline \mathbf{2} & \mathbf{1} & \mathbf{2} & {}_{five} \end{array}$$

(b)
$$\begin{array}{rrrrl} & {\scriptstyle 10} & {\scriptstyle 0} & {\scriptstyle 10} & \\ \cancel{1} & 0 & \cancel{1} & 0 & {}_{two} \\ - & 1 & 0 & 1 & {}_{two} \\ \hline & \mathbf{1} & \mathbf{0} & \mathbf{1} & {}_{\mathbf{two}} \end{array}$$

(c)
$$\begin{array}{rrrrl} & & 2 & 3 & {}_{five} \\ & \times & 3 & 4 & {}_{five} \\ \hline & 2 & 0 & 2 & \\ 1 & 2 & 4 & & \\ \hline \mathbf{1} & \mathbf{4} & \mathbf{4} & \mathbf{2} & {}_{\mathbf{five}} \end{array}$$

(d)
$$\begin{array}{rrrrrrl} & & 1 & 0 & 0 & 1 & {}_{two} \\ & & \times & 1 & 0 & 1 & {}_{two} \\ \hline & & 1 & 0 & 0 & 1 & \\ & 0 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 1 & & & \\ \hline \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & {}_{\mathbf{two}} \end{array}$$

36.
$$\begin{aligned} 44_{five} \cdot 34_{five} &= (4 \cdot 10_{five} + 4) \cdot 34_{five} \\ &= 4 \cdot 34_{five} \cdot 10_{five} + 4 \cdot 34_{five} \\ &= 3010_{five} + 301_{five} \\ &= \mathbf{3311_{five}} \end{aligned}$$

37.
$$\begin{array}{r|ll} 4_{five} & 434_{five} & \\ \cline{2-2} & -\,400_{five} & (100 \cdot 4)_{five} \\ \cline{2-2} & 34_{five} & \\ & -\,31_{five} & (4 \cdot 4)_{five} \\ \cline{2-2} & 3_{five} & \end{array}$$

Thus, $\mathbf{434_{five} = 104_{five} \cdot 4_{five} + 3_{five}}$

38. Answers may vary; for example:

(a) $(26 + 24) + (37 - 7) = 50 + 30 = \mathbf{80}$.

(b) $(7 \cdot 9) \cdot (4 \cdot 25) = 63 \cdot 100 = \mathbf{6300}$.

39. Methods may vary; for example:

(a) $63 \cdot 7 = (7 \cdot 60) + (7 \cdot 3) = 420 + 21 =$ **441** (front-end multiplying).

(b) $85 - 49 = (85 + 1) - (49 + 1) = 86 - 50 =$ **36** (trading off).

(c) $(18 \cdot 5) \cdot 2 = 18 \cdot (5 \cdot 2) = 18 \cdot 10 =$ **180** (using compatible numbers).

(d) $2436 \div 6 = (2400 \div 6) + (36 \div 6) = 400 + 6 =$ **406** (breaking up the dividend).

40. Answers may vary; for example:

(a) Front-end: $5 + 3 + 2 + 4 + 9 = 23$; place value 2300; adjustments $40 + 100 + 60 + 0 + 100 = 300$; adjusted sum $= 2300 + 300 = 2600$.

(b) Rounding: $500 + 400 + 300 + 400 + 1000 = 2600$.

In this case, both estimates give the same result (the actual sum is 2602, so both are reasonable).

41. The addends cluster around 2400, so one would estimate the sum to be $4 \cdot 2400 =$ **9600**.

42. **(a)** $999 \cdot 47 + 47 = 47(999 + 1) = 47 \cdot 1000 = \mathbf{47{,}000}$.

(b) $43 \cdot 59 + 41 \cdot 43 = 43(59 + 41) = 43 \cdot 100 = \mathbf{4300}$.

(c) $1003 \cdot 79 - 3 \cdot 79 = 79(1003 - 3) = 79 \cdot 1000 = \mathbf{79{,}000}$.

(d) $1001 \cdot 113 - 113 = 113(1001 - 1) = 113 \cdot 1000 = \mathbf{113{,}000}$.

(e) $101 \cdot 35 = (100 + 1) \cdot 35 = 35 \cdot 100 + 35 \cdot 1 = 3500 + 35 = \mathbf{3535}$.

(f) $98 \cdot 35 = (100 - 2) \cdot 35 = 35 \cdot 100 - 35 \cdot 2 = 3500 - 70 = \mathbf{3430}$.

43. **(a)**

$$\begin{array}{rcccccccc} & 3x^3 & + & 4x^2 & + & 7x & + & 8 \\ + & & & 5x^2 & + & 2x & + & 1 \\ \hline & 3x^3 & + & 9x^2 & + & 9x & + & 9 \end{array}$$

(b) Answers may vary. For example:

$$\begin{array}{rcccccccc} & 3 \cdot 10^3 & + & 5 \cdot 10^2 & + & 7 \cdot 10 & + & 8 \\ - & & & 4 \cdot 10^2 & + & 2 \cdot 10 & + & 1 \\ \hline & 3 \cdot 10^3 & + & 1 \cdot 10^2 & + & 5 \cdot 10 & + & 7 \end{array}$$

Is equivalent (when $x = 10$) to:

$$\begin{array}{rcccccccc} & 3x^3 & + & 5x^2 & + & 7x & + & 8 \\ - & & & 4x^2 & + & 2x & + & 1 \\ \hline & 3x^3 & + & x^2 & + & 5x & + & 7 \end{array}$$

(c) Answers may vary. For example:

$25 \cdot 10^2 = (2 \cdot 10 + 5) \cdot 10^2 = 2 \cdot 10^3 + 5 \cdot 10^2 + 0 \cdot 10 + 0 = 2500$.

Is equivalent to:

$(2x + 5)x^2 = 2x^3 + 5x^2$.

44.

Suppose $a, b \in B$. Then $a = 5j$ and $b = 5k$, where $j, k \in W$. Therefore $a + b = 5j + 5k = 5(j + k)$, where $(j + k) \in W$. Therefore $a + b \in B$ and B is closed under addition.

45. After 0 seconds 1 person knew (the principal).

After 30 seconds the principal had called 1 person (2 members notified).

After 60 seconds the principal and one member called 2 other members (4 members notified).

After 90 seconds the one member and the 2 new members called another 3 members (7 members notified).

Continuing this pattern:

After 120 seconds 12 members notified.

After 150 seconds 20 members notified.

After 180 seconds 33 members notified.

After 210 seconds 54 members notified.

After 240 seconds 88 members notified.

It took 240 seconds = 4 minutes to notify all 85 members.

46. She started with 124 cookies and there were 7 left so the children took 124 – 7= 117. A number between 10 and 30 that evenly divides into 117 is 13. $117 \div 13 = 9$. There were 13 children who each took 9 cookies.

47. In the list of digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 there are three digits that read the same upside down: 0, 1, and 8. A 6 reads as 9 upside down and 9 reads as 6 upside down. So the five digits that are possible are 0, 1, 6, 8, and 9.

The difference of the first digit in the upside down number (call this number B) and the first number in the real license plate number (call this number A) is 7. Using the list of possible digits only

8 – 1 = 7 works.

This gives us:

$$\begin{array}{l} 8\_,\_\_1(\text{B}) \\ -\underline{1\_,\_\_8}(\text{A}) \\ =78,633 \end{array}$$

In order for the last digit in B (1) minus the last digit in A (8) to be 1, a ten was added to the 1 to make 11 units to subtract 8 and get 3. In order for the difference between the second to last digit in B and the second last digit in A to be 3 and using the 3 left over digits 0, 6, 9 one can only use 0 and 6 because using 9-6 would not allow for the ten to be added to the 1in B.

This gives us:

$$\begin{array}{l} 89,\_01\ (\text{B}) \\ -\underline{10,\_68}\ (\text{A}) \\ =78,633 \end{array}$$

The only digit not used in the top is 6. Placing the numbers into the five digit numbers:

$$\begin{array}{l} 89,601\ (\text{B}) \\ -\underline{10,968}\ (\text{A})\ . \\ =78,633 \end{array}$$

The real license plate is 10,968.

CHAPTER 4

NUMBER THEORY

Assessment 4-1A: Divisibility

1. **Odd. For example: 2 times 2 plus 1 is 5 which is odd.**

2. **(a)** **False; there is no value $c \in W$ such that $24c = 5$.**
   b) True; $30 \div 10 = 3$.
   (c) False; there is no value $c \in W$ such that $8c = 324$.
   (d) False; there is no value $c \in W$ such that $0c = 24$.

3. Use these tests for each number:

(***i***) $2|n$ if the units digit is divisible by 2.

(***ii***) $3|n$ if the sum of the digits is divisible by 3.

(***iii***) $4|n$ if the last two digits are divisible by 4.

(***iv***) $5|n$ if the units digit is 0 or 5.

(***v***) $6|n$ if $2|n$ and $3|n$.

(***vi***) $8|n$ if the last three digits are divisible by 8.

(***vii***) $9|n$ if the sum of the digits is divisible by 9.

(***viii***) $10|n$ if the units digit is 0.

(***ix***) $11|n$ if the sum of the digits in places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11.

		2	3	4	5	6	8	9	10	11
(a)	4,201,012	Y	N	Y	N	N	N	N	N	N
(b)	1573	N	N	N	N	N	N	N	N	Y
(c)	15,810	Y	Y	N	Y	Y	N	N	Y	N

4. **(a)** **Yes**. The question is really, "does 9|1379?" Using the divisibility test for 9, $9 \nmid (1 + 3 + 7 + 9)$ so $9 \nmid 1379$ and there will be a remainder; i.e., a group of less than 9 players.
   (b) **No**. The question is really, "does 11|1354?" Using the divisibility test for 11, $11 \nmid [(1 + 5) - (3 + 4)]$ so $11 \nmid 1354$ and there will be a remainder; i.e. trees left over.
   (c) **Yes**. The question is really, "does 8|1216?" Using the divisibility test for 8, $8|216$ so there will be no remainder.

5. **(a)** **Any digit from 0 to 9**. The units digit of $1{,}427{,}4\square 2$ is divisible by 2.
   (b) **1, 4, or 7**. $3|(1 + 4 + 2 + 7 + 4 + \square + 2)$ $\Rightarrow 3|(20 + \square) \Rightarrow \square = 1, 4, \text{or } 7$.
   (c) **1, 3, 5, 7, or 9**. $4|\square 2 \Rightarrow \square = 1, 3, 5, 7, \text{or } 9$.
   (d) **7**. $9|(1 + 4 + 2 + 7 + 4 + \square + 2) \Rightarrow$ $9|(20 + \square) \Rightarrow \square = 7$.

6. **(a)** $3|74\underline{7}$. $3|(7 + 4 + 7 = 18)$. The _ could be filled by 1, 4, or 7, but 7 is the greatest.
   (b) $9|83\underline{7}45$. $9|(8 + 3 + 7 + 4 + 5 = 27)$. Only 7 makes them viable.
   (c) $11|6\underline{6}55$. $11|(6 + 5) - (6 + 5)$. 6_55 in this case must result in the sum of the digits in places that are even powers of 10 being equal to the sum of the digits in places that are odd powers of 10.
   (d) $5|136\underline{5}$. The _ could be filled in with 0 or 5, but 5 is the greatest.
   (e) $6|2\underline{8}44$. $2|$ 2_44 and $3|(2 + \_ + 4 + 4)$ $\Rightarrow 3|(10 + \_)$ so _ could be 2, 5, 8, but 8 is the greatest.
   (f) $8|38\underline{6}4$. The _ could be filled in with 2 or 6, but 6 is the greatest.

7. **(a)** $7|280$ because $280 = 7 \cdot 40$ (definition of division).
   (b) $19|(3800 + 19)|$ because $19|3800$ and $19\ |19$. [Theorem 4-2(a)].
   (c) $15|(2^4 \cdot 3^5 \cdot 5)$ because $2^4 \cdot 3^5 \cdot 5 = (3 \cdot 5) \cdot 2^4 \cdot 3^4$ (definition of division).
   (d) $19 \nmid (3800 + 37)$ because $19|3800$ but $19 \nmid 37$ [Theorem 4-2(b)].

8. **(a)** **True**. Break into parts of compatible number: $390026 = 390000 + 26 = 13(30000) + 13(2) = 13(30002)$.

(b) **True**. By Theorem 4-2(b), 13 | 260000 and $13 \nmid 33$. Thus $13 \nmid (260000 + 33)$.

(c) **False.** Because 17 | 34,000 and 17 ∤ 15; 17 does not divide the sum.

(d) **True**. Because 17 | 34,000 and 17 | 15; 17 divides the sum.

(e) **False.** Because 19 | 19,000 and 19 ∤ 31; 19 does not divide the sum.

(f) **True**. $93^{11} = 93 \cdot 93^{10} = 31 \cdot 3 \cdot 93^{10} = 31(3 \cdot 93^{10})$. Therefore, since $3 \cdot 93^{10} \in W$, there exist a $g \in W$ such that $93^{11} = 31(g)$.

9. **(a)** **True** by Theorem 4-1. For any whole numbers a and d, if $d|a$ and n is any whole number, then $d|na.$. So if 7 | 21, then 7 | (21•21) or 7 | 21^2.

(b) **False**. Let $a = 4, b = 2,$ and $c = 3$. Two divides 4 $(b|a)$ and $(2 + 3)$ does not divide $(4 + 3), (b + c) \nmid (a + c)$.

(c) **True**. Because $b|a$, there is a whole number c such that $a = bc$. Then, $a^3 = b^3c^3 = (bc^3)b^2$, therefore $b^2|a^3$.

(d) **True**. Because $b|a$, there exist $c \in W$ such that $a = bc$. Thus, $a + b = bc + b = b(c + 1)$. Since $c + 1 \in W$, b divides $a + b$, written $b|(a + b)$.

10. **(a)** A number divisible by 2 and 3 but not by 5 has to end on 2, 4, 6, or 8 and the sum of its digits has to be divisible by 3. For example 6.

(b) A number divisible by 2 and 4 but not by 8 has to end on 0, 2, 4, 6, or 8 and the last two digits are divisible by 4. However, the last three digits cannot be divisible by 8. For example 12.

(c) A number divisible by 5 and t10 has to end in 0 but if it ends in 0, then it is divisible by 2. Such a number does not exist.

11. **19¢.** $209 = 11 \cdot 19$ (both prime numbers). For whole cent pricing 1¢, 11¢, 19¢, and 209¢ are possibilities, but the problem statement is for pencils (plural) and the pencils must cost more than 12¢.

12. **(a)** **True**. For example 3 | (1 · 2 · 3). In general for three consecutive numbers n, $n + 1$, $n + 2$ every other number is even, at least one (or two) of the three numbers n, $n + 1$, $n + 2$ will be even. Since every third number is divisible by three, exactly one of n, $n + 1$, $n + 2$ will have a factor of three. So in $n \cdot (n+1) \cdot (n+2)$ there will be at least one factor of two and exactly one factor of three.

(b) **False.** For example $4 \nmid (5 \cdot 6)$ the product of two consecutive numbers is divisible by four if one of the numbers is divisible by 4.

13. There are 11 ways: 99911, 99731, 99551, 99533, 97751, 97733, 97553, 95555, 77771, 77753, 77555.

14. **(a)** **True**. Let $n = 3(a \cdot 10^k + b \cdot 10^{k-1} + \cdots + x \cdot 10^0)$, where $3a$, $3b$, $\ldots 3x$ are all single digits. Then every digit is divisible by 3, and the number itself is divisible by 3.

(b) **False**. Twelve is divisible by 3 but its digits $\{1, 2\}$ are not divisible by 3.

(c) **False**. This is an "if and only if" statement. For it to be true, both (a) and (b) must be true. Since (b) is false, this statement is false.

15. **(a)**

$$\begin{array}{r} 2414271 \\ 3\overline{)7242815} \\ \underline{-6000000} \\ 1242815 \\ \underline{-1200000} \\ 42815 \\ \underline{-30000} \\ 12815 \\ \underline{-12000} \\ 815 \\ \underline{-600} \\ 215 \\ \underline{-210} \\ 5 \\ \underline{-3} \\ \mathbf{2} \end{array}$$

$7 + 2 + 4 + 2 + 8 + 1 + 5 = 29$

$$\begin{array}{r} 9 \\ 3\overline{)\;29} \\ \underline{-27} \\ \mathbf{2} \end{array}$$

(b) $7242815 = 7 \cdot 1{,}000{,}000 + 2 \cdot 100{,}000 + 4 \cdot 10{,}000 + 2 \cdot 1{,}000 + 8 \cdot 100 + 1 \cdot 10 + 5$

$= 7(999{,}999 + 1) + 2(99{,}999 + 1) + 4(9{,}999 + 1) + 2(999 + 1) + 8(99 + 1) + 1(9 + 1) + 5$

$= 7 \cdot 999{,}999 + 2 \cdot 99{,}999 + 4 \cdot 9{,}999 + 2 \cdot 999 + 8 \cdot 99 + 1 \cdot 9 + 7 + 2 + 4 + 2 + 8 + 1 + 5$

Because a 3 can be factored out of the first six terms, which is equivalent to dividing by 3, applying the division algorithm to the sum of the digits at the end will yield the reminders.

(c) **Yes.** The key idea in (b) was that each place value greater than $1(10^{\circ})$ can be written as the digit times 9…9 plus 1, for example $d \cdot (9{,}999 + 1) = d \cdot 10^4$. Thus, any number can be rewritten as 9 times some whole number plus the sum of the digits.

16. 1, 3, 5, 9, and 15 divide n. 1 divides every number.

Because $n = 45 \cdot d$, $d \in W$, then $n = (3 \cdot 15)\ d$ which implies n is divisible by 3 and 15. And, $n = (5 \cdot 9)d$ which implies n is divisible by 5 and 9.

17. **16|*n* if 16|(last four digits of *n*)**. This continues the pattern of divisibility by 2, 4, and 8.

18. The term "casting out nines" implies that any 9 or sum of digits equaling 9 in n may be "cast out." The remaining digit is the remainder when n is divided by 9. Then the remainder when n is divided by 9 is the same as the remainder when the sum of the digits of n is divided by 9 ("casting out" as needed).

(a) (*i*) $12{,}343 + 4546 + 56 = 16{,}945$.

(*ii*) $12343 = 9 \cdot 1371 + 4$

$4546 = 9 \cdot 505 + 1$

$56 = 9 \cdot 6 + 2$

and $4 + 1 + 2 = 7$.

(*iii*) $16945 = 9 \cdot 1882 + 7$.

(b) (*i*) $987 + 456 + 8765 = 10{,}208$.

(*ii*) $987 = 9 \cdot 109 + 6$

$456 = 9 \cdot 50 + 6$

$8765 = 9 \cdot 973 + 8$

and $6 + 6 + 8 = 20$; $20 = 9 \cdot 2 + 2$.

(*iii*) $10208 = 9 \cdot 1134 + 2$.

(c) $1003 - 46 = 957.$ 1003 has a remainder of 4 when divided by 9; 46 has a remainder of 1 when divided by 9; $4 - 1 = 3$ has a remainder of 3 when divided by 9. $9 + 5 + 7 = 21$ with a remainder of 3 when divided by 9.

(d) $345 \cdot 56 = 19{,}320.$ 345 has a remainder of 3 when divided by 9; 56 has a remainder of 2 when divided by 9; $3 \cdot 2 = 6$ has a remainder of 6 when divided by 9. $1 + 9 + 3 + 2 + 0 = 15$ has a remainder of 6 when divided by 9.

(e) Answers may vary. The division may not have a whole number quotient, in which case the test fails.

19. **(a)** **False**. The palindrome 12121 is not divisible by 11.

(b) **True.** Let a, b, and c be digits. All six digit palindromes can be written in the form *abccba*. The sum of the digits in places with even power is $b + c + a$. The sum of the digits in places with odd powers is $a + c + b$. This difference is zero, which is divisible by 11.

Alternative: *abccba*

$$= a \cdot 10^5 + b \cdot 10^4 + c \cdot 10^3 + c \cdot 10^2 + b \cdot 10 + a \cdot 1$$
$$= (a \cdot 10^5 + a) + (b \cdot 10^4 + b \cdot 10) + (c \cdot 10^3 + c \cdot 10^2)$$
$$= a \cdot 100{,}001 + b \cdot 10{,}010 + c \cdot 1{,}100$$
$$= 11(a \cdot 9091 + b \cdot 910 + c \cdot 100).$$

20. Any five-digit number may be written as $a \cdot 10^4 + b \cdot 10^3 + c \cdot 10^2 + d \cdot 10 + e = a(9999 + 1) + b(999 + 1) + c(99 + 1) + d(9 + 1) + e = (9999a + 999b + 99c + 9d) + (a + b + c + d + e)$.

The first group is divisible by 9 [Theorem 4-2(a)]. Then the five-digit number is divisible by 9 if and only if the second group is divisible by 9 [Theorem 4-2(b)]; i.e., if the sum of the digits is divisible by 9.

Assessment 4-1B

1. Odd. For example: 3 times 2 plus 1 is 7 which is odd.

2. **(a)** **True**. 5 is a factor of 20.

(b) **False**. There is no value $c \in W$ such that $20c = 10$.

(c) **True**. $20 \div 5 = 4$.

(d) **False**. There is no value $c \in W$ such that $30c = 6$.

3. Use these tests for each number:

(***i***) $2|n$ if the units digit is divisible by 2.

(***ii***) $3|n$ if the sum of the digits is divisible by 3.

(***iii***) $4|n$ if the last two digits are divisible by 4.

(***iv***) $5|n$ if the units digit is 0 or 5.

(***v***) $6|n$ if $2|n$ and $3|n$.

(***vi***) $8|n$ if the last three digits are divisible by 8.

(***vii***) $9|n$ if the sum of the digits is divisible by 9.

(***viii***) $10|n$ if the units digit is 0.

(***ix***) $11|n$ if the sum of the digits in places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11.

		2	3	4	5	6	8	9	10	11
(a)	7163	N	N	N	N	N	N	N	N	N
(b)	34,200	Y	Y	Y	Y	Y	Y	Y	Y	N
(c)	199,990	Y	N	N	Y	N	N	N	Y	N

4. **(a)** **Yes**. 9 | 1,111,500 because 9 | (1 + 1 + 1 + 1 + 5 + 0 + 0), 9 | (9).

(b) **Yes**. 4 | 548 because 4 | 48.

(c) **No**. $12 \nmid 26{,}600$.

5. **(a)** Since $1{,}182{,}1\square 6$ will always be even, **any digit** 0-9 can be placed in the square.

(b) Represent the digit to be place in the square by x.
$1 + 1 + 8 + 2 + 1 + x + 6 = 19 + x$.
$19 + x$ is divisible by 3 if and only if $x =$ **2, 5, or 8.**

(c) For $1{,}182{,}1\ \square 6$ to be divisible by 4, $\square 6$ must be divisible by 4. This happens if and only if **1, 3, 5, 7, or 9** are placed in the square.

(d) Considering 5(b), $19 + x$ must be divisible by 9. Therefore, $x = \mathbf{8}$.

6. **(a)** **True.** Because $5 \mid 2 \cdot 3 \cdot 5 \cdot 7$ because $5 \mid 5$.

(b) False. $5 \mid 2 \cdot 3 \cdot 5 \cdot 7$, but $5 \nmid 1$, so $5 \nmid [(2 \cdot 3 \cdot 5 \cdot 7) + 1]$.

(c) **True**. $6 \mid (2 \cdot 3) \cdot 2^2 \cdot 3^2 \cdot 17^4$ because $6 \mid (2 \cdot 3)$.

(d) True. $7 \mid 4200$ but $7 \nmid 22$ so $7 \nmid 4222$.

7. **(a)** $26|(13^4 \cdot 100)$ because $13^4 \cdot 100 = 13 \cdot 2 \cdot 13^3 \cdot 50 = \mathbf{26(13^3 \cdot 50)}$ [Theorem 4-1].

(b) $13 \nmid (2^4 \cdot 5^3 \cdot 26 + 1)$ because $\mathbf{13|(2^4 \cdot 5^3 \cdot 26)}$ but $\mathbf{13 \nmid 1}$ [Theorem 4-2 (b)].

(c) $2^4 \nmid (2 \cdot 4 \cdot 6 \cdot 8 \cdot 17^{10} + 1)$ because $2^4|(1 \cdot 2 \cdot 3 \cdot 4) \cdot 2^4 \cdot 17^{10}$ but $\mathbf{2^4 \nmid 1}$ [Theorem 4-2(b)].

(d) $2^4|(10^4 + 6^4)$ because $\mathbf{2^4|10^4 = 2^4 \cdot 5^4}$ and $\mathbf{2^4|6^4 = 2^4 \cdot 3^4}$ [Theorem 4-2 (a)].

8. **(a)** **False.** Because 12 | 24,000 and $12 \nmid 13$; 12 does not divide the sum.

(b) **True**. Because 12 | 24,000 and 12 | 36; 12 divides the sum.

(c) **False**. Because 19 | 38,000 and 19 37; 19 does not divide the sum.

(d) **True**. Because 19 | 3,800,000 and $19 \nmid 18$; 19 does not divide the sum.

(e) **False.** Because 23 | 23,000 and 23 | 23; 23 divides the sum.

(f) **True**. Because 23 | 46; $23 \mid 46 \cdot 46^{10}$.

9. **(a)** **True** by Theorem 4-2(b). If $d|a$ and $d \nmid b$, then $d \nmid (a + b)$.

(b) **False.** 3 | (50 • 30) because 3 | 30 so 3 will divide any multiple of 30 by Theorem 4-1. if $d|a$ and n is any whole number, then $d|na$..

(c) **True** by Theorem 4-2(b). Consider $a + b$ a single whole number.

(d) **True** by Theorem 4-1.

10. **(a)** **No.** 12 | 24,000 and $12 \nmid 13$, so $12 \nmid 24{,}013$.

(b) **Yes**. 12 | 24,000 and 12 | 36, so 12 | 24,036.

(c) **Yes**. 17 | 17,000 and 17 | 34, so 17 | 17,034.

(d) **Yes**. 3 is a factor of $2 \cdot 3 \cdot 5 \cdot 7$.

(e) **No**. $6 \mid (2 \cdot 3 \cdot 5 \cdot 7)$ and $6 \nmid 1$, so $6 \nmid (2 \cdot 3 \cdot 5 \cdot 7) + 1$.

11. The number of notepads sold must be between 16 (if the price was close to \$2) and 31 (if the price was close to \$l) and 17|3145. **17 notepads** at \$1.85 each totals \$31.45.

12. (a) True. For example 3 | (1+2+3). In general 3 | $(x + (x+1) + (x + 2))$ ⇨ 3 | $(3x + 3)$ because 3 | $3x$ and 3 | 3.

(b) False. For example $4 \nmid (3 + 4 + 5 + 6)$. In general $4 \nmid (x + (x+1) + (x + 2) + (x+3))$ ⇨ 4 ⇨ $(4x + 6)$ because $4 \nmid 6$.

13. (a) **1, 2, 4, 5, 8, and 11**. None are the result of combinations divisible by 3 or 7.

(b) **1 touchdown and 11 field goals or 4 touchdowns and 4 field goals**. Consider 0 touchdowns: $3 \nmid 40$; impossible. Consider 2 touchdowns: $3 \nmid (40 - 14)$; impossible. Other situations follow.

(c) **5 Field goals**. 6 touchdowns ·7 points each $= 42$ points; $57 - 42 = 15$; and $15 \div 3 = 5$ field goals.

14. (a) **True**. Any number is divisible by 6 if and only if it is divisible by 2 and by 3.

(b) **True**. $6 = 2 \cdot 3$.

(c) **False** E.g., 20. The difference between (c) and (b) is that 2 and 3 have no common factors, while 2 and 4 do.

(d) **True**. $8 = 2 \cdot 4$.

(e) **False.** Consider the number 4. 2|4 and 4|4, but $8 \nmid 4$. Alternative: 2|12 and 4|12, but $8 \nmid 12$.

15.

	n	Remainder when n is divided by 9	Sum of the digits of n	Remainder when the sum of the digits of n is divided by 9
(a)	31	4	4	4
(b)	143	8	8	8
(c)	345	3	12	3
(d)	2987	8	26	8
(e)	7652	2	20	2

(f) The remainder when n is divided by 9 seems to be equal to the remainder when the sum of the digits of n is divided by 9.

16. If $28|n$, then $n = 28k$, where k is a whole number. Thus, $n = 2^2 \cdot 7 \cdot k$. This shows that 1, 2, 2^2(=4), $2 \cdot 7$ (=14), and 7 also divide n.

17. **25|*n* if 25|(last two digits of *n*)**; i.e., if n ends in 00, 25, 50, or 75.

18. (a) The sum of the digits in $99 + 28$ is $9 + 9 + 2 + 8$, which leaves remainder 1 upon division by 9. However, 227 leaves the same remainder as $2 + 2 + 7$, i.e., 2 upon division by 9.

(b) 11,199 – 21 leaves remainder 3 – 3 or 0 upon division by 9 (using the casting out technique). However, 11,168 has remainder 8, when divided by 9.

(c) $99 \cdot 26$ divided by 9 has remainder 0, while 2575 divided by 9 has remainder 1.

19. (a) (***i***) $11|[(4 + 5) - (5 + 4)] \Rightarrow$ **11|4554**.

(***ii***) $11|[(9 + 3) - (3 + 9)] \Rightarrow$ **11|9339**.

(***iii***) $11|[(2 + 0) - (0 + 2)] \Rightarrow$ **11|2002**.

(***iv***) $11|[(2 + 2) - (2 + 2)] \Rightarrow$ **11|2222**.

(b) **Yes**. By definition, all four-digit palindromes are of the form $a \cdot 10^3 + b \cdot 10^2 + b \cdot 10 + a$ for whole numbers a and b (where a and b may be equal). The divisibility test for 11 shows that n is divisible by 11 if $11|[(a + b) - (b + a)]$, or $11|0$.

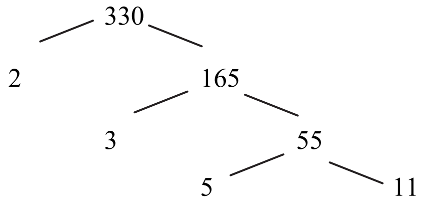
20. Let $abcd$ represent any four-digit number.

$$\begin{aligned} abcd &= a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d \\ &= a(999 + 1) + b(99 + 1) \\ &\quad + c(9 + 1) + \text{d} \\ &= a \cdot 999 + b \cdot 99 + c \cdot 9 \\ &\quad + a + b + c + d \\ &= 3(a \cdot 333 + b \cdot 33 + c \cdot 3) \\ &\quad + (a + b + c + d) \end{aligned}$$

Since 3 divides $a \cdot 999 + b \cdot 99 + c \cdot 9 + d$, if 3 divides $(a + b + c + d)$, then 3 divides $abcd$. If 3 divides $abcd$, since 3 divides $a \cdot 999 + b \cdot 99 + c \cdot 9$, then 3 must divide $(a + b + c + d)$.

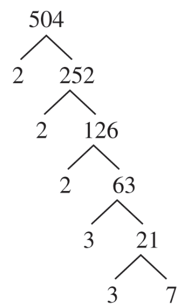
Assessment 4-2A: Prime and Composite Numbers

1. (a)



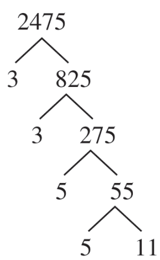
(b) $2 \cdot 3 \cdot 5 \cdot 11 = 330$

2. (a)



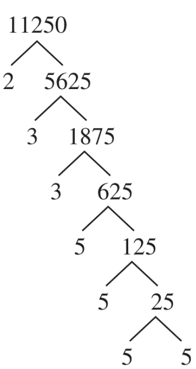
$504 = 2^3 \cdot 3^2 \cdot 7$

(b)



$2475 = 3^2 \cdot 5^2 \cdot 11$

(c)



$11250 = 2 \cdot 3^2 \cdot 5^4$

3. (a) $1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2$
$= \mathbf{2^7 \cdot 3^4 \cdot 5 \cdot 7}$.

(b) $(2^2 \cdot 5^2) \cdot (13) \cdot (7^2)^{10} =$
$\mathbf{2^2 \cdot 5^2 \cdot 7^{20} \cdot 13}$.

(c) 251 is prime, so its factorization is **251**.

(d) $100^{10} = (2^2 \cdot 5^2)^{10} = 2^{20} \cdot 5^{20}$.

4. 23. $23^2 < 769$, but $29^2 > 769$ (23 is prime and 29 is the next largest prime).

5. (a) **Prime**. Fails divisibility test for primes up to 7; no need to test for primes >7 because $11^2 > 103$.

(b) **Not prime**. $7 \cdot 17 = 119$.

(c) **Prime**. Fails divisibility test for primes up to 5.

(d) **Not prime.** $101 \cdot 3 = 303$.

(e) **Prime.** Fails divisibility test for primes up to 19; no need to test for primes >19.

(f) **Prime.** Fails divisibility test for primes up to 7.

(g) **Prime.** $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$, which fails divisibility test for primes up to 13.

(h) **Not prime**. $2 \cdot 3 \cdot 5 \cdot 7 + 11 = 221 = 13 \cdot 17$.

6. The three smallest primes are 2, 3, and 5, so the least number divisible by three different primes is $2 \cdot 3 \cdot 5 = \mathbf{30}$.

7. Since 2 is the least prime, 2^k for some k must be our answer. If $k = 4$, then $\mathbf{2^4 = 16}$ has factors 1, 2, 4, 8, and 16. But this is a two digit number. However, $\mathbf{5^4 = 625}$ has five factors 1, 5, 25, 125, and 625.

8. (a) The Fundamental Theorem of Arithmetic tells us that each composite number, n, can be written as a product of primes in one and only one way. Since $2|n$ and $3|n$, we know that 2 and 3 must both occur in ns prime decomposition. Therefore, $6|n$.

(b) **True.** If $a \mid n$ and $b \mid n$ then there exist natural numbers k and l such that $n = ak$ and $n = bl$. Thus, $n^2 = ak \cdot bl = ab \cdot kl$.

9. (a) $36^{10} \cdot 49^{20} \cdot 6^{15}$
$= (2^2 \cdot 3^2)^{10} \cdot (7^2)^{20} \cdot (2 \cdot 3)^{15}$
$= 2^{20} \cdot 3^{20} \cdot 7^{40} \cdot 2^{15} \cdot 3^{15}$
$= \mathbf{2^{35} \cdot 3^{35} \cdot 7^{40}}$.

(b) $100^{60} \cdot 300^{40} = (2^2 \cdot 5^2)^{60} \cdot (2^2 \cdot 3 \cdot 5^2)^{40}$
$= 2^{120} \cdot 5^{120} \cdot 2^{80} \cdot 3^{40} \cdot 5^{80}$
$= \mathbf{2^{200} \cdot 3^{40} \cdot 5^{200}}$.

(c) $(2 \cdot 3^4 \cdot 5^{110} \cdot 7) + (2^2 \cdot 3^4 \cdot 5^{110})$
$= 2 \cdot 3^4 \cdot 5^{110} \cdot (7 + 2)$
$= 2 \cdot 3^4 \cdot 5^{110} \cdot 3^2$
$= \mathbf{2 \cdot 3^6 \cdot 5^{110}}$.

(d) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 + 1 =$ **2311**, which is prime.

10. **(*i*)** $2^3 \cdot 3^2 \cdot 25^3$ is not a prime factorization because 25 is not prime.

(*ii*) $25^3 = (5^2)^3 = 5^6$, so **prime factorization** is $2^3 \cdot 3^2 \cdot 5^6$.

11. **No**. $8^z = (2^3)^z = 2^{3z}$. In other words, 8^z will always have a unique prime factorization that contains only 2*s*. On the other hand $3^x \cdot 5^y$ will always have a unique prime factorization that contains only 3*s* and 5*s*.

12. $32\,n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$
$= 2^5 \cdot 2 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$
$= 32 \cdot \mathbf{2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6} \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11$

The Fundamental Theorem of Arithmetic tells us that

$m = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6 \cdot (3^2 \cdot 5^3 \cdot 7^2 \cdot 11)$.

Since $3^2 \cdot 5^3 \cdot 7^2 \cdot 11$ is a whole number, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11^6$ divides m.

13. **Yes**. $7^5 \cdot 11^3 = 7(7^4 \cdot 11^3)$.

14. **(a)** **1 · 48; 2 · 24; 3 · 16; or 4 · 12**, all pairs of divisors of 48.

(b) One**. 47 is prime, so the only possibility would be** 1×47.

15. **(a)** **1, 2, 3, 5, 6, 10, 15, or 30** rows. The prime factorization of 30 is $2 \cdot 3 \cdot 5$, so there are $(1 + 1) \cdot (1 + 1) \cdot (1 + 1) = 8$ divisors.

(b) **1, 2, 4, 7, 14, or 28** rows. The prime factorization 28 is $2^2 \cdot 7$, so there are $(2 + 1) \cdot (1 + 1) = 6$ divisors.

(c) **1 or 23** rows. 23 is prime, so there are only two divisors.

(d) **1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, or 120** rows. The prime factorization of 120 is $2^3 \cdot 3 \cdot 5$, so there are $(3 + 1) \cdot (1 + 1) \cdot (1 + 1) = 16$ divisors.

16. There are 70 days in 10 weeks. Briah swims on day 1 and then every other day (1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69). Emma swims on day 1 and then every three days (1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, 55, 58, 61, 64, 67, 70). They swim together 12 times (1, 7, 13, 19, 25, 31, 37, 43, 49, 55, 61, 67).

17. **91 eggs.** Let n be the number of eggs in the basket. Then $n - 1$ is a multiple of 3 and of 5. Hence, $n - 1 = 3 \cdot 5k$ for some whole number k. Thus, $n = 15k + 1$. We know that $n \leq 100$ and that $7 \mid n$. We substitute $k = 1, 2, 3, \ldots$ to obtain the following values for n that are less than or equal to 100: 16, 31, 46, 61, 76, 91. Among these values, only 91 is divisible by 7. Hence, $n =$ **91**.

18. The least number of coins the pirates stole is a number that is a multiple of 15 and has a remainder of 3 when divided by 17 and a remainder of 10 when divided by 16. On a spreadsheet we can create a list of multiples of 15 (15, 30, 45, 60, 75, …) and a list of multiples of 16 plus 10 (26, 42, 58, 74, 90, …) and a list of multiples of 17 plus 3 (20, 37, 54, 71, 88, …) to find out that the least number of coins is **3930**.

19. Consider the two-digit primes: 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. The ones with the tens digit greater than the units digit are 31, 41, 43, 53, 61, 71, 73, 83, and 97. The sum of the two digits is a two-digit prime so we only have 83 left. The last three digits are all and different: 1, 3, 5, 7, 9. The sum of the three digits is palindromic. Only 1 + 3 + 7 = 11 works. The sum of the first and third digit is one-half the sum of the first and second. The only order possible is 173 because $1+7=\frac{(1+3)}{2}$. So the license plate is 83-173.

20. There is no analytic method; one must work through the list of primes. The twin primes less than 200 are: 3 and 5; 5 and 7; 11 and 13; 17 and 19; 29 and 31; 41 and 43; 59 and 61; 71 and 73; 101 and 103; 107 and 109; 137 and 139; 149 and 151; 179 and **181; 191 and 193; and 197 and 199.**

21. **(a) (i)** 1 + 2 + 3 + 4 + 6 = 16, so 12 is not perfect.

(ii) 1 + 2 + 4 + 7 + 14 = 28, so 28 is perfect.

(iii) 1 + 5 + 7 = 13, so 35 is not perfect.

(b) Answers vary. For example 496 is perfect.

Assessment 4-2B

1. **(a)**

156
2 78
2 39
3 13

(b) $2 \cdot 2 \cdot 3 \cdot 13 = 156$

2. **(a)**

544
2 272
2 136
2 68
2 34
2 17

$544 = 2^5 \cdot 17$

(b)

3010
2 1505
5 301
7 43

$3010 = 2 \cdot 5 \cdot 7 \cdot 43$

(c)

9550
2 4775
5 955
5 191

$9550 = 2 \cdot 5^2 \cdot 191$

3. **(a)** $1001 = \mathbf{7 \cdot 11 \cdot 13}$.

(b) $1001^2 = (7 \cdot 11 \cdot 13)^2 = \mathbf{7^2 \cdot 11^2 \cdot 13^2}$.

(c) $999^{10} = (3^3 \cdot 37)^{10} = \mathbf{3^{30} \cdot 37^{10}}$.

(d) $111^{10} - 111^9 = 111^9(111 - 1) =$
$(3 \cdot 37)^9 \cdot (2 \cdot 5 \cdot 11) =$
$\mathbf{2 \cdot 3^9 \cdot 5 \cdot 11 \cdot 37^9}$.

4. 23. $23^2 < 671$, **but** $29^2 > 671$. **(23 is prime and 29 is the next largest prime.)**

5. **(a)** **Prime**. Fails divisibility test for primes less than 11; no need to test for primes >7 because $11^2 > 101$.

(b) **Prime**. Fails divisibility test for primes less than 11; no need to test for primes >7 because $11^2 > 113$.

(c) **Not prime**. 153=3(51) is divisible by 3.

(d) **Not prime**. $9 \cdot 21 = 189$.

(e) **Prime.** Fails divisibility test for primes less than or equal to 23; no need to test for primes >23 because $29^2 > 541$.

(f) **Prime.** Fails divisibility test for primes less than 23; no need to test for primes >23 because $29^2 > 601$.

(g) **Not prime.** 5 is a factor of both $2 \cdot 3 \cdot 5 \cdot 7$ and 5.

(h) **Not prime.** See (g).

6. $2 \cdot 3 \cdot 5 \cdot 7 = \mathbf{210}$.

7. Any number with an odd number of factors must be a perfect square; i.e., for exactly three factors the factors must be of the form: $1, p, p^2$, where p is a prime. Thus find the largest two-digit prime and square it, or $97^2 = \mathbf{9409}$, which has factors 1, 97, and 9409.

8. **(a)** Since $4|n$, 2^2 must be in the prime factorization of n. Since $9|n$, 3^2 must be in the prime factorization of n. Because 2^2 and 3^2 have no common factors, $2^2 \cdot 3^2 = 36$ must divide n.

(b) Part (a) has the key idea: that a and b have no common factors. This notion is after expressed as $GCD(a,b) = 1$.

9. **(a)** $16^4 \cdot 81^4 \cdot 6^6 = (2^4)^4 \cdot (3^4)^4 \cdot (2 \cdot 3)^6 = 2^{16} \cdot 3^{16} \cdot 2^6 \cdot 3^6 = \mathbf{2^{22} \cdot 3^{22}}$.

(b) $8^4 \cdot 32^5 = (2^3)^4 \cdot (2^5)^5 = 2^{12} \cdot 2^{25} = \mathbf{2^{37}}$.

(c) $2^2 \cdot 3^5 \cdot 7^{55} + 2^4 \cdot 3^4 \cdot 7^{55} = 2^2 \cdot 3^4 \cdot 7^{55}(1 \cdot 3 \cdot 1 + 2^2 \cdot 1 \cdot 1) = 2^2 \cdot 3^4 \cdot 7^{55}(7) = \mathbf{2^2 \cdot 3^4 \cdot 7^{56}}$.

(d) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 + 1 = 2731,$ which is prime.

10. **(*i*)** $2^2 \cdot 5^3 \cdot 9^2$ is not a prime factorization because 9 is not prime.

(*ii*) The prime factorization is $2^2 \cdot 5^3 \cdot (3^2)^2 = \mathbf{2^2 \cdot 5^3 \cdot 3^4}$.

11. **No**. 5^z has factors of only 5; there are no factors of 5 in 2^x or 3^y.

12. If $2n = 2^6 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7$ then $n = 2^5 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^7 = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) \cdot (2^4 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^6)$. Thus $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)$ is a factor of n.

13. **No**. No whole number times $3^2 \cdot 2^4 = 144$ will yield $3^3 \cdot 2^2 = 108$.

14. **(a)** The other arrays are 1 by 24, 2 by 12, 3 by 8, because 1 · 24 = 24, 2 · 12 = 24, and 3· 8 = 24.

(b) There would only be 1 array: 1 by 43 because 1 · 43 = 43 and there are no other factors of 43. 43 is prime.

15. **(a)** $15 = 3 \cdot 5$. Possible lengths and widths are 1 by 15, 3 by 5.

(b) $24 = 2^3 \cdot 3$. Possible lengths and widths are 1 by 24, 2 by 12, 3 by 8, 4 by 6.

(c) 17 is prime. There can only be a 1 by 17 rectangle.

(d) $200 = 2^3 \cdot 5^2$. Possible lengths and widths are 1 by 200, 2 by 100, 4 by 50, 5 by 40, 8 by 25, 10 by 20.

16. Barbara runs every other day. Starting on day 1 she runs on the following days: 1, 3, ,5 ,7 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29. She goes to dance class every fifth day. Starting on day 1 she goes to dance class on the following days: 1, 6, 11, 16, 21, 26. She does both on days 1, 11, and 21 or 3 days in a 30-day month.

17. If he has one card left over each time, we can list the multiples of 2, 3, 4, 5, and 6 and add 1. That gives us: 3,5,7,9, … and 4,7,10,13, … and 6, 11, 16, 21, … and 7, 13, 19, 25, … . We know we are looking for an odd number (because of the divisibility by 2 plus 1), less than 100 that ends on 1 (because of the divisibility by 5 plus 1). We check 91, 81, 71, and finally 61. Jose has 61 cards.

18. Working backward we know that 100 coins is after Isabelle divided the pile by 3 and took one part so 100 is 2 parts and the total was 150. That is 150 coins plus the 2 she gave her dad = 152. The 152 coins is after Noelle divided the pile by 3 and took one part so 152 is 2 parts and the total was 228. That is 228 coins plus the 2 she gave her dad = 230. The 230 coins is after Juliette divided the pile by 3 and took one part so 230 is 2 parts and the total was 345. That is 345 coins plus the 2 she gave her dad = 347. There were 347 coins.

19. We can assign a variable to each of the colors: r for red, g for green and, b for blue. The product of the number of red beads by the total number of red and green beads is a number exactly 120 greater than the number of blue beads. This translates into $r(r + g) = b + 120$. We also know that r, g, and b are all prime numbers. Using the guess and check method we find that there are 2 green beads, 11 red beads and 23 blue beads.

20. **(a)** **0, 4, 6, or 8.** All would cause divisibility 2 or 5 at some point; 2 or 5 would allow superprimes in the leftmost position only.

(b) **1 and 9.** 9 is composite; 1 is not prime by definition.

(c) **23, 29, 31, 37, 53, 59, 71, 73, and 79,** following the above restrictions.

(d) Answer may vary; e.g., 233, 239, or 373.

21. **(a)** **(*i*)** 1 + 2 + 3 + 4 + 6 = 16, so 12 is abundant.

(*ii*) 1 + 2 + 4 + 7 + 14 = 28, so 28 is perfect and neither abundant or deficient.

(***iii***) 1 + 5 + 7 = 13, so 35 is deficient.

(b) Answers vary. For example 14 is deficient and 18 is abundant.

Mathematical Connections 4-2: Review Problems

22. **(a)** **True**. $13|(13 \cdot 10)$.

(b) **True**. $11 \cdot 91 = 1001$.

(c) **True**. Theorem 4-2(d).

23. Use the standard divisibility tests for all except 7; for 7 just perform the division:

(a) Divisors are **2, 3, and 6**.

(b) Divisors are **2, 3, 5, 6, 9, and 10**.

24. If $15|n$ there exists a whole number a such that $15a = n$:

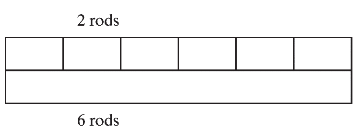
$(3 \cdot 5)a = n$

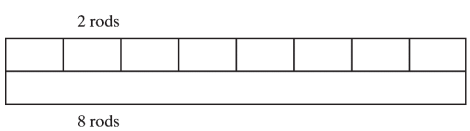
$3(5a) = n$

Since $5a$ is a whole number, $3|n$.

Assessment 4-3A: Greatest Common Divisor and Least Common Multiple

1. (***i***) The 2 rods can be used to build both the 6 rod and the 8 rod $\Rightarrow GCD(6,8) = \mathbf{2}$.





(***ii***) Four 6 rods (length 24) are the same length as three 8 rods (length 24) $\Rightarrow LCM(6,8) = 24$.

2. **(a)** (***i***) $D_{18} = \{1,2,3,6,9,18\}$;
$D_{12} = \{1,2,3,4,6,12\}$;
$\Rightarrow GCD(18,12) = 6$.

(***ii***) $M_{18} = \{18,36,54,72,90,...\}$;
$M_{12} = \{12,24,36,...\}$;
$\Rightarrow LCM(18,12) = 36$.

(b) (***i***) $D_{20} = \{1,2,4,5,10,20\}$;
$D_{36} = \{1,2,3,4,6,9,12,18,36\}$;
$\Rightarrow GCD(20,36) = 4$.

(***ii***) $M_{20} = \{20,40,60,80,100,...,180,...\}$;
$M_{36} = \{36,72,108,144,180,...\}$;
$\Rightarrow LCM(24,36) = 180$.

(c) (***i***) $D_8 = \{1,2,4,8\}$;
$D_{24} = \{1,2,3,4,6,8,12,24\}$;
$D_{64} = \{1,2,4,8,16,32,64\}$;
$\Rightarrow GCD(8,24,64) = 8$.

(***ii***) $M_8 = \{8,16,24,...,192,...\}$;
$M_{24} = \{24,48,72,...,192,...\}$;
$M_{64} = \{64,128,192,...\}$;
$\Rightarrow LCM(8,24,64) = 192$.

(d) (***i***) $D_7 = \{1,7\}$;
$D_9 = \{1,3,9\}$;
$\Rightarrow GCD(7,9) = \mathbf{1}$.

(***ii***) $M_7 = \{7,14,21,...,63,...\}$;
$M_9 = \{9,18,27,...,63,...\}$;
$\Rightarrow LCM(7,9) = \mathbf{63}$.

3. **(a)** $132 = 2 \cdot 2 \cdot 3 \cdot 11 = 2^2 \cdot 3 \cdot 11$;
$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7 =$
$2^3 \cdot 3^2 \cdot 7; \Rightarrow GCD(132,504) =$
$2^2 \cdot 3 = \mathbf{12}$. $LCM(132,504) =$
$2^3 \cdot 3^2 \cdot 7 \cdot 11 = \mathbf{5544}$.

(b) $65 = 5 \cdot 13$;
$1690 = 2 \cdot 5 \cdot 13 \cdot 13 = 2 \cdot 5 \cdot 13^2$;
$\Rightarrow GCD(65,1690) = 5 \cdot 13 = \mathbf{65}$.
$LCM(65,1690) = 2 \cdot 5 \cdot 13^2 = \mathbf{1690}$.

(c) $900 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = 2^2 \cdot 3^2 \cdot 5^2$;
$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^5 \cdot 3$;
$630 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 2 \cdot 3^2 \cdot 5 \cdot 7$;
$\Rightarrow GCD(900,96,630) = 2 \cdot 3 = \mathbf{6}$.
$LCM(900,96,630) = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
$= \mathbf{50400}$.

(d) $108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 = 2^2 \cdot 3^3$;
$360 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2^3 \cdot 3^2 \cdot 5$;
$\Rightarrow GCD(108,360) = 2^2 \cdot 3^2 = \mathbf{36}$.
$LCM(108,360) = 2^3 \cdot 3^3 \cdot 5 = \mathbf{1080}$.

4. $\rightarrow R$ symbolizes the reminder left after the indicated divisions.

(a) $GCD(2924,220) = GCD(220,64)$ since $2924 \div 220 \rightarrow R\ 64$;
$GCD(220,64) = GCD(64,28)$ since $220 \div 64 \rightarrow R\ 28$;
$GCD(64,28) = GCD(28,8)$ since $64 \div 28 \rightarrow R\ 8$;
$GCD(28,8) = GCD(8,4)$ since $28 \div 8 \rightarrow R\ 4$;
$GCD(8,4) = GCD(4,0)$ since $8 \div 4 \rightarrow R\ 0$;
$\Rightarrow GCD(2924,220) = \mathbf{4}$.

(b) $GCD(14595,10856) = GCD(10856,3739)$ since $14595 \div 10856 \rightarrow R\ 3739$;
$GCD(10856,3739) = GCD(3739,3378)$ since $10856 \div 3739 \rightarrow R\ 3378$;
$GCD(3739,3378) = GCD(3378,361)$ since $3739 \div 3378 \rightarrow R\ 361$;
$GCD(3378,361) = GCD(361,129)$ since $3378 \div 361 \rightarrow R\ 129$;
$GCD(361,129) = GCD(129,103)$ since $361 \div 129 \rightarrow R\ 103$;
$GCD(129,103) = GCD(103,26)$ since $129 \div 103 \rightarrow R\ 26$;
$GCD(103,26) = GCD(26,25)$ since $103 \div 26 \rightarrow R\ 25$;
$GCD(26,25) = GCD(25,1)$ since $26 \div 25 \rightarrow R\ 1$;
$GCD(25,1) = GCD(1,0)$ since $25 \div 1 \rightarrow R\ 0$;
$\Rightarrow GCD(14595,10856) = \mathbf{1}$.

5. (a) **(*i*)** Intersection of sets:
$M_{24} = \{24,48,72,96,...\}$;
$M_{36} = \{36,72,108,...\}$;
$\Rightarrow LCM(24,36) = \mathbf{72}$.

(*ii*) Prime factorization:
$24 = 2 \cdot 2 \cdot 2 \cdot 3 = 2^3 \cdot 3$;
$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$;
$\Rightarrow LCM(24,36) = 2^3 \cdot 3^2 = \mathbf{72}$.

(b) **(*i*)** Intersection of sets:
$M_{72} = \{72,144,216,...,1440,...\}$;
$M_{90} = \{90,180,270,...,1440,...\}$;
$M_{96} = \{96,192,288,...,1440,...\}$;
$\Rightarrow LCM(72,90,96) = \mathbf{1440}$.

(*ii*) Prime factorization:
$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2$;
$90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$;
$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
$= 2^5 \cdot 3; \Rightarrow LCM(72,90,96)$
$= 2^5 \cdot 3^2 \cdot 5 = \mathbf{1440}$.

(c) **(*i*)** Intersection of sets:
$M_{90} = \{90,180,270,...,630,...\}$;
$M_{105} = \{105,210,315,...,630,...\}$;
$M_{315} = \{315,630,945,...\}$;
$\Rightarrow LCM(90,105,315) = \mathbf{630}$.

(*ii*) Prime factorization:
$90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$;
$105 = 3 \cdot 5 \cdot 7$;
$315 = 3 \cdot 3 \cdot 5 \cdot 7 = 3^2 \cdot 5 \cdot 7$;
$\Rightarrow LCM(90,105,315) = 2 \cdot 3^2 \cdot 5 \cdot 7$
$= \mathbf{630}$

(d) $16^{100} = (4^2)^{100} = 4^{200}$ and
$25^{100} = (5^2)^{100} = 5^{200}$
$\Rightarrow LCM(16^{100},25^{100}) = 4^{\mathbf{200}} \cdot \mathbf{5}^{\mathbf{200}}$
$= (20)^{\mathbf{200}}$.

6. If the GCD is 17 then the product of the two numbers divided by the GCD will result in the LCM. $1734 \div 17 = 102$, so 102 is their LCM.

7. If the GCD is 19 we can find the other factors of 57 by dividing 57 by 19. $57 \div 19 = 3$. The other number is the LCM divided by 3: $228 \div 3 = 76$. The other number is 76.

8. (a) $LCM\,(a, b) = \boldsymbol{ab}$. a and b have no common factors.

(b) $GCD\,(a, a) = \boldsymbol{a}$ and $LCM(a, a) = \boldsymbol{a}$. a has all factors in common with a.

(c) $GCD\,(a^2, a) = \boldsymbol{a}$ and $LCM\,(a^2, a) = \boldsymbol{a^2}$.

(d) $GCD\,(a, b) = \boldsymbol{a}$ and $LCM\,(a, b) = \boldsymbol{b}$, since $a|b$.

9. (a) **True**. If both a and b are even, then $GCD\,(a, b) \geq 2$.

(b) **True**. $GCD\,(a, b) = 2$ implies that $2|a$ and $2|b$.

(c) **False**. GCD could be any larger multiple of 2; e.g., $GCD\,(8, 20) = 4$.

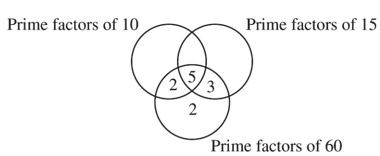
10. (a) $GCD\,(120, 75) = GCD\,(75, 45) = GCD\,(45, 30) = GCD(30, 15) = GCD(15, 0)$; $GCD(120, 75) = 15$.

$GCD\,(105, 15) = \text{GCD}\,(15, 0)$;
$GCD\,(105, 15) = 15$.
Thus $GCD\,(120, 75, 105) = \mathbf{15}$.

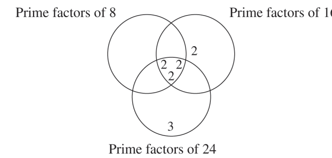
(b) $GCD\,(4618, 4619) = GCD(4618, 1) = 1$;
$GCD\,(34578, 1) = 1$;
Thus $GCD\,(34578, 4618, 4619) = \mathbf{1}$.

11. 2 is the only prime factor of 4 and $2 \nmid 97, 219, 988, 751$, so 1 is their only common divisor; i.e., they are relatively prime.

12. (a) The Venn diagram below shows 5 as the only prime factor common to 10, 15, and 60; 2 and 5 as common prime factors of 10 and 60; 3 and 5 as common prime factors of 15 and 60; 2 as the remaining prime factor of 60:



(b)



13. The prime factorization of 48 is $2^4 \cdot 3$. To find three pairs (a, b) such that their LCM is 48 is making sure that a and b have 2^4 and 3 in their LCM. For example: (1, 48), (2, 48), (3,48), etc.

14. This will be the set of all numbers between 1 and 49 that are relatively prime to 49. Since $49 = 7^2$, the set will contain all numbers between 1 and 49 **except** $1 \cdot 7, 2 \cdot 7, 3 \cdot 7, 4 \cdot 7, 5 \cdot 7, 6 \cdot 7$, and $7 \cdot 7$.

15. (a) The real question is “what is LCM (15, 40, 60)?” since this is when the alarms will coincide: $LCM\,(15, 40, 60) = 120$, or **120 minutes;** i.e., 2 hours later, at 8:00 A.M.

(b) **No**. This would be equivalent to changing locations of clocks A and B in the room.

16. The **780th “like”**. $LCM(12, 13, 20) = 780$.

17. There are 15 children. The candy will come out evenly for $LCM\,(15,12) = 60$; 60 candies $\div$ 12 candies per package $=$ **5 packages**.

18. The question is really, “What is LCM (12, 18, 16)?” since that is when the times will coincide $LCM\,(12, 18, 16) =$ **144 minutes**.

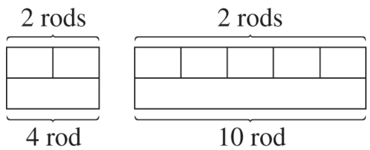
19. GCD (42, 54) = 6. She can make 7 bags of chocolate chip cookies and 9 bags of sugar cookies. Each bag has 6 cookies in it.

20. LCM (100,18) = 900 inches or 75 feet before points P and Q touch the sidewalk at the same time again.

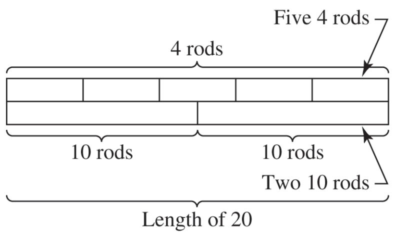
21. GCD(72, 42) = 6. The largest pieces she can cut are 6 yards long.

Assessment 4-3B

1. **(*i*)** $GCD\ (4, 10) = \mathbf{2}$**:**



(*ii*) $LCM\ (4, 10) = \mathbf{20}$:



2. **(a)** **(*i*)** $D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$\Rightarrow GCD(12,30) = \mathbf{6}.$

(*ii*) $M_{12} = \{12, 24, 36, 48, 60, 72, 84, \ldots\}$

$M_{30} = \{30, 60, 90, 120, 150, 180, \ldots\}$

$\Rightarrow LCM(12,30) = 60.$

(b) **(*i*)** $D_{18} = \{1, 2, 3, 6, 9, 18\}$

$D_{58} = \{1, 2, 29, 58\}$

$\Rightarrow GCD(18,58) = 2.$

(*ii*) $M_{18} = \{18, 36, 54, 72, 90, 108, \ldots, 522, \ldots\}$

$M_{58} = \{58, 116, 174, 232, 290, \ldots, 522, \ldots\}$

$\Rightarrow LCM(18, 58) = 522.$

(c) **(*i*)** $D_6 = \{1, 2, 3, 6\}$

$D_{18} = \{1, 2, 3, 6, 9, 18\}$

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$\Rightarrow GCD(6,18, 24) = \mathbf{6}.$

(*ii*) $M_6 = \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60, 66, 72, \ldots\}$

$M_{18} = \{18, 36, 54, 72, 90, 108, \ldots\}$

$M_{24} = \{24, 48, 72, 96, 120, 144, \ldots\}$

$\Rightarrow LCM(6,18, 24) = \mathbf{72}.$

(d) **(*i*)** $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$D_{13} = \{1, 13\}$

$\Rightarrow GCD(36,13) = \mathbf{1}.$

(*ii*) $M_{36} = \{36, 72, 108, \ldots, 468, \ldots\}$

$M_{13} = \{13, 26, 39, 52, 65, 78, \ldots, 468, \ldots\}$

$\Rightarrow LCM(36, 13) = 468.$

3. **(a)** $11 =$ prime number

$19 =$ prime number

$\Rightarrow GCD(11, 19) = \mathbf{1}.$

$LCM(11, 19) = 11 \cdot 19 = \mathbf{209}.$

(b) $140 = 2 \cdot 2 \cdot 5 \cdot 7 = 2^2 \cdot 5 \cdot 7$

$320 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^6 \cdot 5$

$\Rightarrow GCD(140, 320) = 2^2 \cdot 5 = \mathbf{20}$

$LCM(140, 320) = 2^6 \cdot 5 \cdot 7 = \mathbf{2240}.$

(c) $800 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 = 2^5 \cdot 5^2$

$75 = 3 \cdot 5 \cdot 5 = 3 \cdot 5^2$

$450 = 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3^2 \cdot 5^2$

$\Rightarrow GCD(800, 75, 450) = 5^2 = \mathbf{25}$

$LCM(800, 75, 450) =$

$2^5 \cdot 3^2 \cdot 5^2 = \mathbf{7200}.$

(d) $103 = 103$

$320 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^6 \cdot 5$

$\Rightarrow GCD(103, 320) = 1$

$LCM(103, 320) = 2^6 \cdot 5 \cdot 103 = 32{,}920.$

4. $\rightarrow R$ symbolizes the reminder left after the indicated divisions.

(a) $GCD(14560, 8250) = GCD(8250, 6310)$

since $14560 \div 8250 \rightarrow R\ 6310$

$GCD(8250, 6310) = GCD(6310, 1940)$

since $8250 \div 6310 \rightarrow R\ 1940$

$GCD(6310, 1940) = GCD(1940, 490)$ since

$6310 \div 1940 \rightarrow R\ 490$

$GCD(1940, 490) = GCD(490, 470)$ since

$1940 \div 490 \rightarrow R\ 470$

$GCD(490, 470) = GCD(470, 20)$ since

$490 \div 470 \rightarrow R\ 20$

$GCD(470, 20) = GCD(20, 10)$ since

$470 \div 20 \rightarrow R\ 10$

$GCD(20, 10) = GCD(10, 0)$ since

$20 \div 10 \rightarrow R\ 0$

$\Rightarrow GCD(14560, 8250) = \mathbf{10}.$

(b) $GCD(8424, 2520) = GCD(2520, 864)$ since $8424 \div 2520 \rightarrow R\ 864$

$GCD(2520, 864) = GCD(864, 792)$ since $2520 \div 864 \rightarrow R\ 792$

$GCD(864, 792) = GCD(792, 72)$ since $864 \div 792 \rightarrow R\ 72$

$GCD(792, 72) = GCD(72, 0)$ since $792 \div 72 \rightarrow R\ 0$

$\Rightarrow GCD(8424, 2520) = \mathbf{72}$.

5. **(a)** **(*i*)** Intersection of sets:

$M_{25} = \{25, 50, 75, 100, ..., 900, ...\}$

$M_{36} = \{36, 72, 108, 144, ..., 900, ...\}$

$\Rightarrow LCM(25, 36) = \mathbf{900}$.

(*ii*) Prime factorization:

$25 = 5 \cdot 5 = 5^2$

$36 = 2 \cdot 2 \cdot 3 \cdot 3 = 2^2 \cdot 3^2$

$\Rightarrow LCM(25, 36)$

$= 2^2 \cdot 3^2 \cdot 5^2 = \mathbf{900}$.

(b) Prime factorization:

$82 = 2 \cdot 41$

$90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$

$50 = 2 \cdot 5 \cdot 5 = 2 \cdot 5^2$

$\Rightarrow LCM(82, 90, 50)$

$= 2 \cdot 3^2 \cdot 5^2 \cdot 41 = \mathbf{18{,}450}$.

(c) Prime factorization:

$80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^4 \cdot 5$

$105 = 3 \cdot 5 \cdot 7$

$315 = 3 \cdot 3 \cdot 5 \cdot 7 = 3^2 \cdot 5 \cdot 7$

$\Rightarrow LCM(80, 105, 315)$

$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 = \mathbf{5040}$.

(d) Prime factorization:

$8^{100} = (2^3)^{100} = 2^{300}$

$50^{100} = (2 \cdot 5^2)^{100} = 2^{100} \cdot 5^{200}$

$\Rightarrow LCM(8^{100}, 50^{100}) = \mathbf{2^{300} \cdot 5^{200}}$.

6. If the GCD is 14 then the product of the two numbers divided by the GCD will result in the LCM. $5880 \div 14 = 420$, so 420 is their LCM.

7. If the GCD is 11 we can find the other factors of 66 by dividing 66 by 11. $66 \div 11 = 6$. The other number is the LCM divided by 6: $330 \div 6 = 55$. The other number is 55.

8. **(a)** $GCD\,(a, b) = \mathbf{1}$ and $LCM\,(a, b) = \boldsymbol{ab}$. a and b have no factors in common.

(b) $\boldsymbol{a|b}$. If $GCD(a, b) = a$, then a must divide both a and b.

(c) $\boldsymbol{b|a}$. If $LCM\,(a, b) = a$, then $b \cdot n = a$, (where n may be any integer), so $a \div b = n$, or $b|a$.

9. **(a)** **False**. For $a \neq b$, $LCM\,(a,b) > GCD\,(a,b)$.

(b) **True**. Theorem 4-19.

(c) **True**. If $GCD\,(a, b) >$ a, it could not divide a.

(d) **True**. If $LCM\,(a, b) <$ a, it could not be a multiple.

10. **(a)** $GCD\,(180, 240) = GCD\,(180, 60)$

$= GCD\,(60, 0) = 60$

$GCD\,(60, 306) = GCD\,(60, 6)$

$= GCD\,(6, 0) = 6$

$GCD\,(60, 6) = GCD\,(6, 0) = 6$

$\Rightarrow GCD\,(180, 240, 306) = \mathbf{6}$.

(b) $GCD\,(5284, 1250) = GCD\,(1250, 284)$

$= GCD\,(284, 114)$

$= GCD\,(114, 56)$

$= GCD\,(56, 2)$

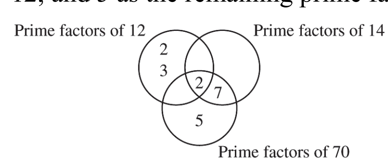
$= GCD\,(2, 0) = 2$

$GCD\,(2, 1280) = GCD\,(2, 0) = 2$.

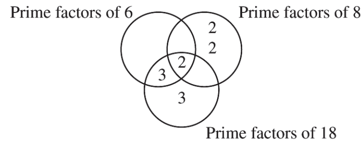
$\Rightarrow GCD\,(5284, 1250, 1280) = \mathbf{2}$.

11. 11 is its own only prime factor and $11 \nmid 181{,}345{,}913$ $[11 \nmid (1 + 1 + 4 + 9 + 3) - (8 + 3 + 5 + 1)]$, so 1 is their only common divisor; i.e., they are relatively prime.

12. **(a)** The Venn diagram below shows 2 as the only prime factor common to 12, 14, and 70; 2 as the only common prime factor of 12 and 70; 2 and 7 as common prime factors of 14 and 70; 3 as the remaining prime factor of 12; and 5 as the remaining prime factor of 70:



(b)



13. The prime factorization of 60 is $2^2 \cdot 3 \cdot 5$. To find three pairs (a, b) such that their LCM is 60 is making sure that a and b have 2^2, 3^2, and 5 in their LCM. For example: (1, 60), (2, 60), (3,60), etc.

14. GCD (25, x) $=$ 1 means that x can have no factors in common with 25, so x cannot have a factor of 5; the solution set is thus {1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24}.

15. The yellow light flashes 15 times per minute, or every 4 seconds (60 ÷ 15 = 4). The blue light flashes 6 times per minute, or every 10 seconds (60 ÷ 6 = 10). The red light flashes 12 times per minute or every 5 seconds (60 ÷ 12 = 5). The LCM (4,10,5) = 20. You must wait 20 seconds before the three lights flash on simultaneously again.

16. LCM (15,40) = 120. The 120th customer was the first to win both prizes.

17. GCD(245,238,84)= 7. Each bag will have 245 ÷ 7 = 35 white bouncy balls, 238 ÷ 7 = 34 yellow bouncy balls, and 84 ÷ 7 = 12 orange bouncy balls for a total of 35 + 34 + 12 = 81 bouncy balls.

18. $LCM\,(90, 75) = 450$ minutes, or $7\frac{1}{2}$ hours; 7:00 A.M. $+ 7\frac{1}{2}$ hours $=$ **2:30 P.M.**

19. $LCM(24, 45) = 360$; i.e., 360¢ worth of cookies was sold; 360¢ ÷ 24¢ per cookie $=$ **15 cookies.**

20. $LCM(40, 24, 60) = 120$. **Gear 1** must make 120 ÷ 40= **3 revolutions**; **gear 2** must make 120 ÷ 24 $=$ **5 revolutions**; and **gear 3** must make 120 ÷ 60 $=$ **2 revolutions** to line up the arrows.

21. *GCD* (300, 264) = 12. He will have 300 ÷ 12 = 25 piles of football cards and 264 ÷ 12 = 22 piles of baseball cards. Each pile will have 12 cards.

Mathematical Connections 4-3: Review Problems

16. **(a)** **83,151, 83,451 or 83751**. Let the number be $83a51$; then $8 + 3 + a + 5 + 1 = 17 + a$ must be a multiple of 3; if a is a single digit the only possibilities for $17 + a$ to be a multiple of 3 are 1, 4, or 7. The greatest digit is 7.

(b) **86,691.** Let the number be $8a691$; then $(1 + 6 + 8) - (a + 9) = 6 - a$ must be a multiple of 11; if a is a single digit the only possibility is $6 - a = 0 \Rightarrow a = 6$.

(c) **10,396**. $10{,}306 = 23 \cdot 448 + 2$ so $10{,}3\_6 > 23 \cdot 448$. The second factor must end in 2 to yield 6 in the units position, thus $23 \cdot 452 = 10{,}396$.

17. **(a)** $2^3 \cdot 3^7$

(b) $3^2 \cdot 5 \cdot 727$

(c) $2^{10} \cdot 3^9$

18. **No**. $3|(2 + 2 + 2 + 3)$ so $3|2223$; it is thus not prime.

19. Answers may vary; $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$ is one.

20. The question is really "What is the *LCM* (2, 4, 6, 8, 10)?". Their factors are $\{2, 2^2, \ 2 \cdot 3, \ 2^3, \ 2 \cdot 5\}$, thus $LCM\,(2, 4, 6, 8, 10) = 2^3 \cdot 3 \cdot 5 = 120$.

21. **53**. The next prime squared (59^2) is greater than 3359.

Chapter 4 Review

1. **(a)** **False**. $12 \nmid 4$ since $4 = 12x$ has no whole number solution for x.

(b) **False**. $0 \nmid 8$ since $8 = 0 \cdot x$ has no whole number solution for x.

(c) **True**. 0 divided by any natural number is 0.

(d) **False**. For example, 12. The test works only when the two numbers have no common factors and 4 and 6 have a common factor of 2.

(e) **False**. For example, 9 is not divisible by 12 but is divisible by 3.

2. **(a)** $m = 125{,}160$.

$2|m$ because $2|0$.

$3|m$ because $3|(1 + 2 + 5 + 1 + 6 + 0)$.

$4|m$ because $4|60$.

$5|m$ because $5|0$.

$6|m$ because $2|m$ and $3|m$.

$8|m$ because $8|160$.

$10|m$ because $10|0$.

(b) $m = 12{,}193$.

$2\nmid m$ because $2\nmid 3$.

$3\nmid m$ because $3\nmid(1+2+1+9+3)$.

$4\nmid m$ because $4\nmid 93$.

$5\nmid m$ because $5\nmid 3$.

$6\nmid m$ because $2\nmid m$ and $3\nmid m$.

$8\nmid m$ because $8\nmid 193$.

$9\nmid m$ because $9\nmid(1+2+1+9+3)$.

$11\nmid m$ because $11\nmid(1 + 1 + 3) - (2 + 9)$.

3. **(a)** Write the number as $87a4$.

$6|87a4$ if $2|87a4$ and $3|87a4$;
$2|4$ so $2|87a4$, thus
$6|87a4$ if
$3|87a4 \Rightarrow 3|(8 + 7 + a + 4 = 19 + a)$;

$3|(21, 24, \text{ or } 27)$ when $a =$ 2, 5, or 8;
so 6|**(87$\underline{2}$4, 87$\underline{5}$4, or 87$\underline{8}$4)**. The greatest digit is 8.

(b) Write the number as $4a856$.

$24|4a856$ if $3|4a856$ and $8|4a856$;

$8|856$ so $8|4a856$, thus

$24|4a856$ if

$3|(4 + a + 8 + 5 + 6) = 23 + a$;

$3|(24, 27, \text{ or } 30)$ when $a =$ 1, 4, or 7;

so 24|**(4$\underline{1}$856, 4$\underline{4}$856, or 4$\underline{7}$856)**. The greatest digit is 7.

(c) Write the number as $87ab4$.

$29|87{,}000$ so $29|ab4$.

The only whole number to return 4 in the units position when multiplied by 9 is 6. So the possibilities are $6 \cdot 29, 16 \cdot 29$, etc. until the product of $n \cdot 29 > 999$.

$6 \cdot 29 = 174; 16 \cdot 29 =$
$464; 26 \cdot 29 = 754;$
$36 \cdot 29 > 999$.

Thus $ab = 17, 46,$ or 75.

So 29|**(87$\underline{17}$4, 87$\underline{46}$4, or 87$\underline{75}$4)**. The greatest digits are 75.

4. **(a)** The student's claim is true; examples may vary. For example.,

(i) $5|(3 + 4 + 5 + 6 + 7)$.

(ii) $5|(5 + 6 + 7 + 8 + 9)$.

(b) Let n be a whole number; then
$n + (n + 1) + (n + 2) + (n + 3) +$
$(n + 4) = 5n + 10 = 5(n + 2)$.
Thus the sum is divisible by 5.

5. ***(i)*** The number must be **divisible by 3 and 8**.

(ii) $3|4152$ and $8|4152$ so $24|4152$.

6. ***(i)*** Answers may vary; e.g., 16.

(ii) To obtain k divisors raise any prime to the $(k - 1)$st power; e.g., to obtain 5 divisors take $2^{5-1} = 16$; divisors are $2^0, 2^1, 2^2, 2^3$, and 2^4.

7. $144 = (2^2 \cdot 3)^2 = 2^4 \cdot 3^2$. There are thus $(4 + 1) \cdot (2 + 1) = 15$ divisors: **1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 36, 48, 72 and 144**.

8.
$$\begin{aligned} n &= a \cdot 10^2 + b \cdot 10 + c \\ &= a(99 + 1) + b(9 + 1) + c \\ &= 99a + 9b + (a + b + c). \end{aligned}$$
Since $9|99a$ and $9|9b$ then
$9 \mid [99a + 9b + (a + b + c)]$
if and only if $9|(a + b + c)$.

9. 1009 is prime, so $17 \nmid 1009$ but $17|17$.
$17 \nmid (1009 + 17)$ by Theorem 4-2(b).

10. **(a)** **Composite**. $3 \mid 147$.

(b) **Prime**. Primes through 19 do not divide 373; $23^2 > 373$.

11. **(a)** $7 \cdot 11 \cdot 13 \cdot 17 + 17 = 17(7 \cdot 11 \cdot 13 + 1)$, so 17 is a factor.

(b) $10! + k$ can be factored as in part (a), depending on the value of $k (2 \le k \le 10)$, thus composite.

12. First show that among any three consecutive odd whole numbers there is always one divisible by 3. Suppose that the first whole number in the triplet is not divisible by 3. By the Division Algorithm that whole number can be written in the form $3n + 1$ or $3n + 2$ for some whole number n.

Then the three consecutive odd whole numbers are $(3n+1, 3n+3, 3n+5)$ or $(3n+2, 3n+4, 3n+6)$.

In the first triplet $3|3n+3$; in the second $3|3n+6$. If the first whole number is greater than 3 and not divisible by 3 then the second or third must be divisible by 3 and so cannot be prime.

13. **(a)**

$$\begin{array}{r|l} 3 & 1\ 1\ 1 \\ \hline & \ \ 3\ 7 \end{array} \Rightarrow 111 = 3 \cdot 37.$$

(b) $\Rightarrow 144 = 2^4 \cdot 3^2$.

$$\begin{array}{r|l} 2 & 144 \\ \hline 2 & 72 \\ \hline 2 & 36 \\ \hline 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

(c)

$$\begin{array}{r|l} 2 & 188 \\ \hline 2 & 94 \\ \hline & 47 \end{array} \Rightarrow 188 = 2^2 \cdot 47.$$

(d)

$$\begin{array}{r|l} 2 & 520 \\ \hline 2 & 260 \\ \hline 2 & 130 \\ \hline 5 & 65 \\ \hline & 13 \end{array} \Rightarrow 520 = 2^3 \cdot 5 \cdot 13.$$

14. **(a)** $10^{10} = (2 \cdot 5)^{10} = 2^{10} \cdot 5^{10}$.

(b) $89^4 = 89^4$.

(c) $8^3 \cdot 6^4 \cdot 13^2 = (2^3)^3 \cdot (2 \cdot 3)^4 \cdot 13^2 =$
$2^9 \cdot 2^4 \cdot 3^4 \cdot 13^2 = 2^{13} \cdot 3^4 \cdot 13^2$

(d) $2^3 \cdot 3^2 + 2^4 \cdot 3^3 \cdot 7 =$
$2^3 \cdot 3^2(1 + 2 \cdot 3 \cdot 7) =$
$\mathbf{2^3 \cdot 3^2 \cdot 43}$.

(e) $2^4 \cdot 3 \cdot 5^7 + 2^4 \cdot 5^6 =$
$2^4 \cdot 5^6(3 \cdot 5 + 1) = 2^4 \cdot 5^6 \cdot 16 =$
$2^4 \cdot 5^6 \cdot 2^4 = \mathbf{2^8 \cdot 5^6}$.

15. Product (all whole numbers ≤ 10) $=$
$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 =$
$1 \cdot 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5)$.

Thus LCM (all whole numbers ≤ 10) $=$
$1 \cdot 2^3 \cdot 3^2 \cdot 5 \cdot 7 = \mathbf{2520}$.

16. **(a)** $24 = 2^3 \cdot 3$ and $52 = 2^2 \cdot 13$. Thus $GCD(24, 52) = 2^2 = \mathbf{4}$.

(b) $GCD(5767, 4453) = GCD(4453, 1314)$
$= GCD(1314, 511) = GCD(511, 292)$
$= GCD(292, 219) = GCD(219, 73)$
$= GCD(73, 0) \Rightarrow GCD(5767, 4453) = \mathbf{73}$.

17. **(a)** $LCM(2^3 \cdot 5^2 \cdot 7^3, 2 \cdot 5^3 \cdot 7^2 \cdot 13,$
$2^4 \cdot 5 \cdot 7^4 \cdot 29) =$
$\mathbf{2^4 \cdot 5^3 \cdot 7^4 \cdot 13 \cdot 29}$.

(b) $GCD(277, 278) = \mathrm{GCD}(277, 1) = 1$

Therefore no common factors in 277 and 278.
Thus $LCM(277, 278) \cdot 1 = 277 \cdot 278$
$= 77{,}006$.

18. **No**. *LCM* and *GCD* are the same if the number are equal.

19. $LCM(a, b, c) = LCM(m, c)$, where $m = LCM(a, b)$. $LCM(a, b) = \frac{ab}{GCD(a,b)}$ and $LCM(m, c) = \frac{mc}{GCD(m,c)}$. Each of the above *GCD*s can be found using the Euclidean algorithm.

20. If $GCD(a, b) = 1 \Rightarrow \mathbf{LCM(a, b) = ab}$.
$GCD(a, b) \cdot LCM(a, b) = ab$; since $GCD(a, b) = 1$ then $LCM(a, b) = ab/1 = ab$.

21. **31¢**. The price must divide 3193¢; $31|3100$ and $31|93$ thus $31|3193$.

22. $9869 = 71 \cdot 139$ (both prime) implies **71 lattes** at \$1.39 each. She could not have sold 139 lattes at \$0.71 each, since she never sells for less than \$1.

23. LCM (45,30) = 90 minutes. 8:00 A.M. + 90 minutes = **9:30 A.M.**

24. The question is really "What is GCD (120, 144)?" since what is needed is the greatest number which will divide both 120 and 144: GCD (120, 144) = **24 coins**.

25. The runners will be at the starting place at the same time at LCM (3, 5) = **15 minutes**.

26. GCD(60,24) = 12. There will be 60 ÷12=5 bags of oranges and 24 ÷ 12 = 2 bags of apples. Each bag will have 12 pieces of fruit.

27. One revolution of gear 2, rotates 28 teeth. One revolution of gear 1, rotates 48 teeth. For each of the gears, the arrows are back in their original positions every multiple of 28 teeth for gear 2 and 48 teeth for gear 1. Thus, the arrows realign when LCM (48, 28) = 336 teeth have rotated. So gear 2 must make $336 \div 28 =$ **12 revolutions**.

CHAPTER 5

INTEGERS

Assessment 5-1A: Addition and Subtraction of Integers

1. For every integer a, there exists a unique integer ^-a, such that $a + {}^-a = 0 = {}^-a + a$. ^-a is called the additive inverse of a.

   (a) The unique integer $\mathbf{^-2}$ is the additive inverse of 2 because $2 + {}^-2 = 0$.

   (b) The additive inverse of ^-a can be written as $^-(^-a) = a$, or the number of the opposite sign. Thus the additive inverse of $^-6$ is $^-(^-6) = 6$.

   (c) The additive inverse of m is $\boldsymbol{^-m}$.

   (d) The additive inverse of 0 is **0** because $0 + 0 = 0$.

   (e) The additive inverse of ^-m is $^-(^-m) = \boldsymbol{m}$.

   (f) The additive inverse of $(a + b)$ is $^-(a + b) = \boldsymbol{^-a + {}^-b}$.

2. **(a)** $^-(^-2) = \mathbf{2}$ is the additive inverse of $^-2$.

   (b) $^-(^-m) = \boldsymbol{m}$.

   (c) $^-0 = \mathbf{0}$ since $^-0 + 0 = 0$.

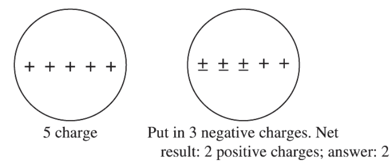
3. **(a)** Absolute value is the distance on a number line between the origin (0) and a specified number. Distance on the number line between 0 and $^-5$ is 5 units, so $|^-5| = \mathbf{5}$.

   (b) Distance on the number line between 0 and 10 is 10 units, so $|10| = \mathbf{10}$.

   (c) $^-|^-5|$ means the additive inverse of the absolute value of $^-5$, so $^-|^-5| = {}^-(5) = \mathbf{^-5}$.

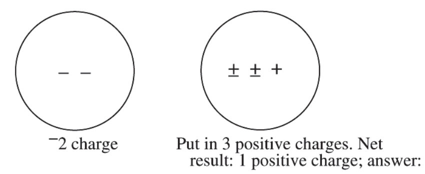
   (d) $^-|5| = {}^-(5) = \mathbf{^-5}$.

4. **(a)** Charged-field model:

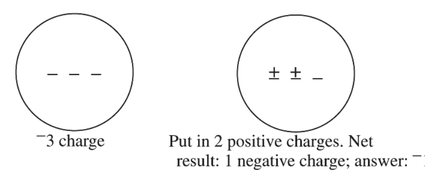


   For the chip model, draw black chips to represent positive numbers and red chips to represent negative numbers. Each red chip neutralizes one black chip.

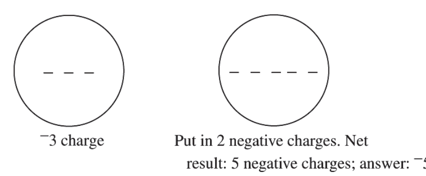
   (b) Charged-field model:



   (c) Charged-field model:

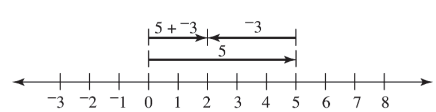


   (d) Charged-field model:

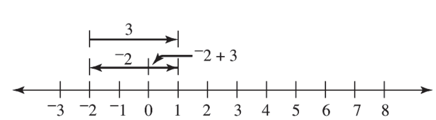


5. Movement on the number line in addition is always to the right for a positive number and to the left for a negative number.

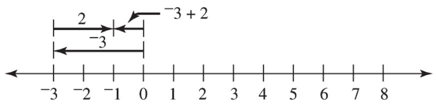
   (a)



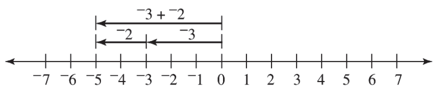
   (b)



(c)



(d)



6. (a) $7+{}^-13={}^-(|{}^-13|-|7|)$

$={}^-(13-7)$

$={}^-(6)$

$={}^-6$

(b) ${}^-7+{}^-13={}^-(|{}^-7|+|{}^-13|)$

$={}^-(7+13)$

$={}^-(20)$

$={}^-20$

7. Using the theorem that $a-b=a+{}^-b$:

(a) $3-{}^-2=3+{}^-({}^-2)=3+2=\mathbf{5}$.

(b) ${}^-3-2={}^-3+{}^-2=\mathbf{{}^-5}$.

(c) ${}^-3-{}^-2={}^-3+{}^-({}^-2)={}^-3+2=\mathbf{{}^-1}$.

8. (a) $3-({}^-2)=n$ if and only if $3={}^-2+n$. Thus $\boldsymbol{n}=\mathbf{5}$.

(b) ${}^-3-2=n$ if and only if ${}^-3=2+n$. Thus $\boldsymbol{n}=\mathbf{{}^-5}$.

(c) ${}^-3-{}^-2=n$ if and only if ${}^-3={}^-2+n$. Thus $\boldsymbol{n}=\mathbf{{}^-1}$.

9. (a) A drop in stock value is negative; an increase is positive, so a drop of 17 points followed by a gain of 20 points is represented by ${}^-17+20$, or a **net drop of 3 points.**

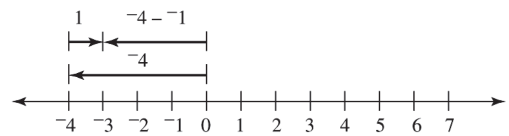
(b) ${}^-10°\text{C}+8°\text{C}=\mathbf{{}^-2°C}$.

(c) $35,000$ feet $+{}^-1000$ feet $=34,000$ *feet*.

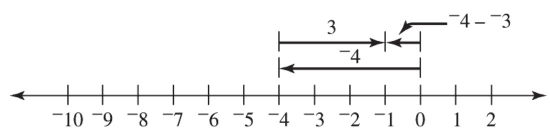
10. (a) $({}^-45)+({}^-55)+({}^-165)+({}^-35)+({}^-100)$ $+75+25+400={}^-400+500$, or **$100**.

(b) $300 (beginning balance) + $100 (in transactions) = **$400**.

11. (a) In the number-line model of ${}^-4-{}^-1$, ${}^-4$ is depicted as a line that starts at 0 and moves 4 units to the left to ${}^-4$; the operation of *subtraction* now changes the direction of ${}^-1$ from moving left (as ${}^-1$ would during addition) to moving *right* 1 unit, ending at ${}^-3$. Thus ${}^-4-{}^-1=\mathbf{{}^-3}$:



(b) ${}^-4$ is depicted as a line that starts at 0 and moves 4 units to the left; *subtraction* of ${}^-3$ is a line that moves 3 units to the *right*, ending at ${}^-1$. Thus ${}^-4-{}^-3=\mathbf{{}^-1}$:



12. (a) Starting with subtraction that is already known: ${}^-4-2={}^-6$

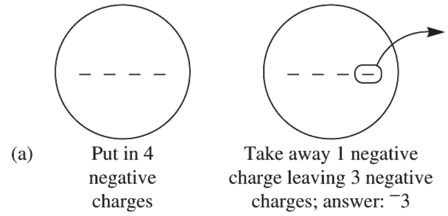
Continuing the pattern:

$${}^-4-1={}^-5$$
$${}^-4-0={}^-4$$
$${}^-4-{}^-1={}^-3.$$

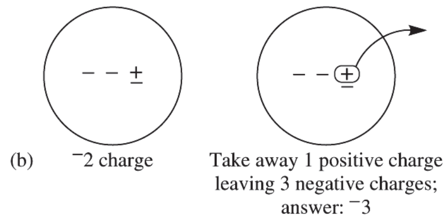
(b) Starting with subtraction that is already known:

$$2-1=1$$
$$1-1=0$$
$$0-1={}^-1$$
$${}^-1-1={}^-2$$
$${}^-2-1={}^-3.$$

13. (a)



(b)



14. (a) $^-2 + (3 - 10) = {^-2} + {^-7} = \mathbf{^-9}$.

(b) $[8 - (^-5)] - 10 = [8 + {^-(^-5)}] - 10 =$
$[8 + 5] - 10 = 13 - 10 = \mathbf{3}$.

(c) $(^-2 - 7) + 10 = (^-2 + {^-7}) + 10 =$
$(^-9) + 10 = \mathbf{1}$.

15. (a) (*i*) $55 - 60$.

(*ii*) $55 + (^-60)$.

(*iii*) $T = 55 + (^-60) = \mathbf{^-5°F}$.

(b) (*i*) $200 - 220$.

(*ii*) $200 + (^-220)$.

(*iii*) Balance $= \$200 + {^-\$220} = \mathbf{^-\$20}$.

16. (a) $3 - (2 - 4x) = 3 + {^-(2 + {^-4x})}$
$= 3 + {^-(2)} + {^-(^-4x)}$
$= 3 + {^-2} + 4x = \mathbf{1 + 4x}$.

(b) $x - (^-x - y) = x + {^-(^-x + {^-y})}$
$= x + {^-(^-x)} + {^-(^-y)}$
$= x + x + y = \mathbf{2x + y}$.

17. It can be shown that for all integers a, b, and c;

$a - b + c = a + {^-b} + c$ Theorem 5 - 4
$= a + {^-(b + {^-c})}$ Theorem 5 – 3 and Theorem 5 – 4
$= a - (b - c)$ Theorem 5 – 4

thus the original equation holds true for all integers.

18. (a) $W \cup I = \boldsymbol{I}$ $(W \subset I)$.

(b) $W \cap I = \boldsymbol{W}$ $(W \subset I)$.

(c) $I^+ \cup I^- = \boldsymbol{I} - \{\mathbf{0}\}$. 0 is neither positive nor negative, so it must be removed from the sets of integers.

(d) $W - I = \boldsymbol{\emptyset}$. $W \subset I$, so there are no elements in W that are not in I.

19. Answers may vary. The key idea for the approach below is to make the sum across all rows, columns, and diagonals equal to zero.

$^-3$	4	$^-1$
2	0	$^-2$
1	$^-4$	3

20. If $y = {^-x} - 1$:

(a) $y = {^-(^-1)} - 1 = 1 + {^-1} = \mathbf{0}$,
when $x = {^-1}$

(b) $y = {^-(100)} + {^-1} = {^-100} + {^-1} = \mathbf{^-101}$,
when $x = 100$

(c) $y = {^-(^-2)} - 1 = 2 + {^-1} = \mathbf{1}$,
when $x = {^-2}$

(d) $y = {^-(^-a)} - 1 = a - \mathbf{1}$,
when $x = {^-a}$

(e) $3 = {^-x} - 1 \Rightarrow 3 + 1 = {^-x} - 1 + 1$
$\Rightarrow 4 = {^-x} \Rightarrow x = \mathbf{^-4}$.

21. (a) The common difference is 1. The n^{th} term is $n + (^-41)$. Set the n^{th} term equal to the last term and solve for n.

$n + (^-41) = 40 \Rightarrow n = 81$. There are 81 terms in the arithmetic sequence.

(b) The common ratio is 2. The n^{th} term is $2^{(n+4)}$. Set the n^{th} term equal to the last term and solve for n.

$$2^{n+4} = 4096 \Rightarrow 2^{n+4} = 2^{12} \Rightarrow n + 4 = 12 \Rightarrow n = 8$$

There are 8 terms in the geometric sequence.

22. The sum of the first 100 negative integers = $^{-}1 + {}^{-}2 + {}^{-}3 + \ldots + {}^{-}98 + {}^{-}99 + {}^{-}100$. Using the Gauss method from Chapter 1:
$\dfrac{({}^{-}1 + {}^{-}100)100}{2} = {}^{-}5050$. The sum is $^{-}5050$.

23. The sum of $^{-}40 + {}^{-}39 + {}^{-}38 + \ldots + 38 + 39 + 40$
Using the Gauss method from Chapter 1:
$\dfrac{({}^{-}40 + 40)81}{2} = 0$. The sum is 0.

24. Answers vary. If the east coast time is 0, then the Pacific time is $^{-}3$.

25. Write an equation:
$55\text{ mph} + 10\text{ mph over} + {}^{-}7\text{ mph under}$
$+ {}^{-}13\text{ mph under} = 45\text{ mph}.$

26. Leonine Trees shot the following:
Bogey, Par, Eagle, Double Bogey, Birdie, Hole-in-one $1 + 0 + {}^{-}2 + 2 + {}^{-}1 + {}^{-}3 = {}^{-}3$. He is 3 under par for these holes.

27. **(a)** **All negative integers**. If x is negative, then its additive inverse, ^{-}x, is positive, i.e. $^{-}({}^{-}x) = x$.
(b) **All positive integers**. If x is positive, then its additive inverse, ^{-}x, is negative.
(c) **All integers $< {}^{-}1$**. If $^{-}x - 1 > 0 \Rightarrow$ $^{-}x > 1 \Rightarrow x < {}^{-}1$.
(d) $\boldsymbol{x = 2}$ **or** $\boldsymbol{x = {}^{-}2}$. If $|x| = 2$, then x is a value on the number line that is 2 units away, in either direction, from 0.

28. **(a)** If $|x - 6| = 6$, then $x - 6 = 6$ or $x - 6 = {}^{-}6$:
(i) If $x - 6 = 6 \Rightarrow x - 6 + 6 = 6 + 6 \Rightarrow \boldsymbol{x = 12}$, or
(ii) If $x - 6 = {}^{-}6 \Rightarrow x - 6 + 6 = {}^{-}6 + 6 \Rightarrow \boldsymbol{x = 0}$.
(b) $|x| + 2 = 10 \Rightarrow |x| + 2 - 2 = 10 - 2$ $\Rightarrow |x| = 8 \Rightarrow \boldsymbol{x = 8}$ **or** $\boldsymbol{x = {}^{-}8}$.
(c) $|{}^{-}x| = |x|$ is **true for all integers**, since the distance from 0 to x on a number line is the same as the distance from 0 to ^{-}x.

29. To find the common difference (d) in an arithmetic sequence, subtract the first term from the second:
(a) $d = {}^{-}3 - 0 = \boldsymbol{{}^{-}3}$. The next two terms are $\boldsymbol{{}^{-}12, {}^{-}15}$.
(b) $d = x - (x + y) = \boldsymbol{{}^{-}y}$. The next two terms are $\boldsymbol{x - 2y}$, $\boldsymbol{x - 3y}$.

30. **(a)** **True**. Absolute value is always non-negative.
(b) **True**. Absolute value of the difference between the two elements is always non-negative.
(c) **True**. $^{-}x + {}^{-}y = {}^{-}(x + y)$, and $|{}^{-}(x + y)| = |x + y|$.

31. **(a)** $x + 7 = 3 \Rightarrow x + 7 - 7 = 3 - 7 \Rightarrow$ $x = 3 + {}^{-}7 \Rightarrow \boldsymbol{x = {}^{-}4}$.
(b) $^{-}10 + x = {}^{-}7 \Rightarrow {}^{-}10 + 10 + x =$ $^{-}7 + 10 \Rightarrow \boldsymbol{x = 3}$.
(c) $^{-}x = 5 \Rightarrow {}^{-}x + x = 5 + x \Rightarrow 0 =$ $5 + x \Rightarrow 0 + {}^{-}5 = 5 + {}^{-}5 + x \Rightarrow$ $^{-}5 = x \Rightarrow \boldsymbol{x = {}^{-}5}$.

32. Estimates may vary.
(a) $327 + {}^{-}52 - 398 \approx 325 + {}^{-}50 - 400 \approx {}^{-}125$.
The exact answer is $^{-}123$.
(b) $^{-}1772 + 2005 - 503 \approx {}^{-}1770 + 2000 - 500 \approx 270.$
The exact answer is $^{-}270$.
(c) $996 - {}^{-}10{,}007 - 102 \approx 1000 - {}^{-}10{,}000 - 100 \approx 10{,}900$
The exact answer is 10,901.
(d) $^{-}303 - {}^{-}1203 + 4997 \approx {}^{-}300 - {}^{-}1200 + 5000 \approx 5900$
The exact answer is 5897.

Assessment 5-1B

1. For every integer a, there exists a unique integer ^-a, such that $a + {}^-a = 0 = {}^-a + a$. ^-a is called the additive inverse of a.

(a) The unique integer $\mathbf{^-3}$ is the additive inverse of 3 because $3 + {}^-3 = 0$.

(b) The additive inverse of ^-a can be written as $^-({}^-a) = a$, or the number of the opposite sign. Thus the additive inverse of $^-4$ is $^-({}^-4) = 4$.

(c) The additive inverse of q is $\boldsymbol{^-q}$.

(d) The additive inverse of 6 is $\mathbf{^-6}$ because $6 + {}^-6 = 0$.

(e) The additive inverse of ^-n is $^-({}^-n) = \boldsymbol{n}$.

(f) The additive inverse of $(3 + x)$ is $^-(3 + x) = \mathbf{^-3} + \boldsymbol{^-x}$.

2. **(a)** $^-({}^-5)$ means the additive inverse of $^-5$, or **5**.

(b) $^-({}^-x) = \boldsymbol{x}$.

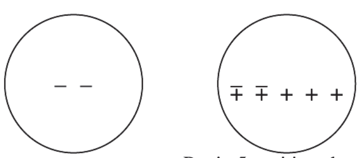
3. **(a)** Absolute value is the distance on a number line between the origin (0) and a specified number. Distance on the number line between 0 and $^-3$ is 3 units, so $|{}^-3| = \mathbf{3}$.

(b) Distance on the number line between 0 and 15 is 15 units, so $|15| = \mathbf{15}$.

(c) $^-|{}^-3|$ means the additive inverse of the absolute value of $^-3$, so $^-|{}^-3| = {}^-(3) = \mathbf{^-3}$.

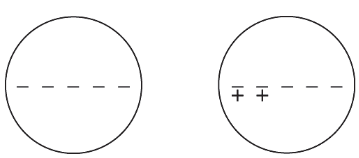
(d) $^-|6| = \mathbf{^-6}$

4. **(a)** Charged-field model:



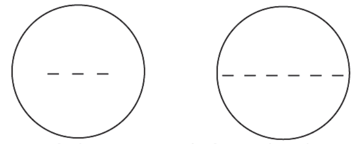
$^-2$ charge — Put in 5 positive charges. Net result: 3 positive charges; answer: 3

(b) Charged-field model:



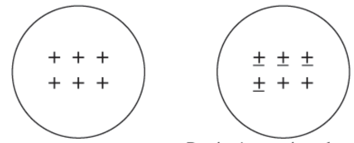
$^-5$ charge — Put in 2 positive charges. Net result: 3 negative charges; answer: $^-3$

(c) Charged-field model:



$^-3$ charge — Put in 3 negative charges: Net result: 6 negative charges; answer: $^-6$

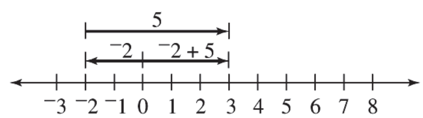
(d) Charged-field model:



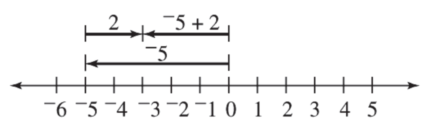
6 charge — Put in 4 negative charges. Net result: 2 positive charges; answer: 2

5. Movement on the number line in addition is always to the right for a positive number and to the left for a negative number.

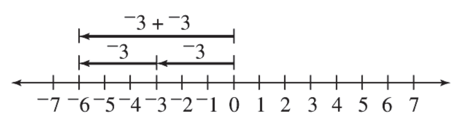
(a)



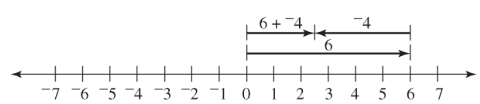
(b)



(c)



(d)



6. **(a)**
$$5 + {}^-31 = {}^-(|{}^-31| - |5|)$$
$$= {}^-(31 - 5)$$
$$= {}^-(26)$$
$$= {}^-26$$

(b)
$$^-5 + {}^-31 = {}^-(|{}^-5| + |{}^-31|)$$
$$= {}^-(5 + 31)$$
$$= {}^-(36)$$
$$= {}^-36$$

7. Using the theorem that $a - b = a + {}^-b$:

(a) ${}^-3 - 5 = {}^-3 + {}^-5 = \mathbf{{}^-8}$.

(b) $5 - {}^-3 = 5 + {}^-({}^-3) = 5 + 3 = \mathbf{8}$.

(c) ${}^-2 - ({}^-3) = {}^-2 + {}^-({}^-3) = -2 + 3 = \mathbf{1}$.

8. **(a)** ${}^-3 - 5 = n$ if and only if ${}^-3 = 5 + n$.
Thus $\boldsymbol{n} = \mathbf{{}^-8}$.

(b) $5 - ({}^-3) = n$ if and only if $5 = {}^-3 + n$.
Thus $\boldsymbol{n} = \mathbf{8}$.

(c) ${}^-\mathbf{2} - ({}^-\mathbf{3}) = n$ if and only if ${}^-\mathbf{2} = {}^-\mathbf{3} + n$.
Thus $\boldsymbol{n} = \mathbf{1}$.

9. **(a)** ${}^-\$200 + \$100 + {}^-\$50 = {}^-\$250 + \$100 =$
${}^-\$150$, or a **drop in net worth of $150**.

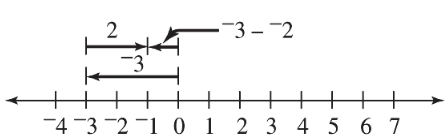
(b) ${}^-2 + 7 + 0 + {}^-8 = {}^-10 + 7 = {}^-3$, or a **loss of 3 yards**.

(c) $17 + {}^-8 + {}^-9 + 14 + 45 = 9 + {}^-9 +$
$14 + 45 = 0 + 14 + 45 =$ **59 points**.

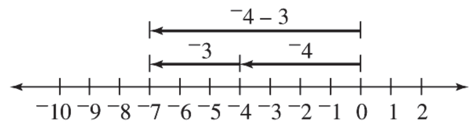
10. **(a)** ${}^-247 + {}^-11 + {}^-11 = {}^-258 + {}^-11$
$= \mathbf{{}^-269°\ C}$.

(b) $98 - {}^-94 = 98 + {}^-({}^-94) = 98 + 94 =$
$\mathbf{192°F}$.

11. **(a)** In the number-line model of ${}^-3 - {}^-2$, ${}^-3$ is depicted as a line that starts at 0 and moves 3 units to the left to ${}^-3$; the operation of *subtraction* now changes the direction of ${}^-2$ from moving left (as ${}^-2$ world during addition) to moving *right* 2 units, ending at ${}^-1$. Thus ${}^-3 - {}^-2 = \mathbf{{}^-1}$:



(b) ${}^-4$ is depicted as a line that starts at 0 and moves 4 units to the left; subtraction of 3 is a line that moves 3 units to the left, ending at ${}^-7$. Thus ${}^-4 - 3 = \mathbf{{}^-7}$:



12. **(a)** Starting with subtraction that is already known:

$${}^-2 - 1 = {}^-3$$

Continuing the pattern:

$${}^-2 - 0 = {}^-2$$
$${}^-2 - {}^-1 = {}^-1$$
$${}^-2 - {}^-2 = 0$$
$${}^-2 - {}^-3 = \mathbf{1}.$$

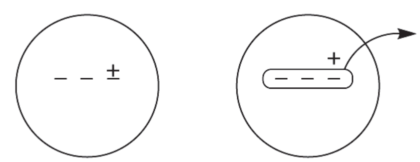
(b) Starting with subtraction that is already known:

$$3 - 2 = 1$$

Continuing the pattern:

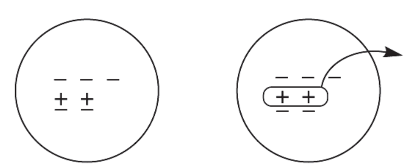
$$2 - 2 = 0$$
$$1 - 2 = {}^-1$$
$$0 - 2 = {}^-2$$
$${}^-1 - 2 = {}^-3$$
$${}^-2 - 2 = {}^-4$$
$${}^-3 - 2 = \mathbf{{}^-5}.$$

13. **(a)**



(a) ${}^-2$ charge — Take away 3 negative charges leaving 1 positive charge on field; answer: 1

(b)



(b) ${}^-3$ charge — Take away 2 positive charges leaving 5 negative charges; answer: ${}^-5$

14. **(a)** $^{-}2 - (7 + 10) = {}^{-}2 - (17) = {}^{-}2 + {}^{-}17 = \mathbf{^{-}19}$.

(b) $8 - 11 - 10 = 8 + {}^{-}11 + {}^{-}10 = {}^{-}3 + {}^{-}10 = \mathbf{^{-}13}$.

(c) $^{-}2 - 7 + 3 = {}^{-}2 + {}^{-}7 + 3 = {}^{-}9 + 3 = \mathbf{^{-}6}$.

15. Using values picked from the graph:

(a) 10W-30 or 10W-40.

(b) 5W-30.

(c) 5W-30, 10W-30, or 10W-40.

(d) None of the oils shown.

(e) 10W-30 or 10W-40.

16. **(a)** $4x - 2 - 3x = 4x + {}^{-}(3x) + {}^{-}(2) = x - 2$.

(b) $4x - (2 - 3x) = 4x + {}^{-}(2 + {}^{-}3x) = 4x + {}^{-}(2) + {}^{-}({}^{-}3x) = 4x + 3x + {}^{-}2 = \mathbf{7x - 2}$.

17. **(a)** $^{-}x - y = {}^{-}x + {}^{-}y = {}^{-}y + {}^{-}x = {}^{-}y - x$.

(b) **No**. $^{-}x - y = {}^{-}x + {}^{-}y$ and $^{-}y - x = {}^{-}y + {}^{-}x$, so the theorem used in part (a) is simply the commutative property of integer addition. In general, $x - y \neq y - x$.

18. **(a)** $W - I^{+} = \mathbf{\{0\}}$ $(0 \notin I^{+})$.

(b) $W - I^{-} = \boldsymbol{W}$. W and I^{-} are disjoint.

(c) $I \cap I = \boldsymbol{I}$ $(I \subseteq I)$.

(d) $I - W = I^{-}$

19. Adding all numbers yields $^{-}9$; dividing by 3 indicates that each row, column, and diagonal must total $^{-}3$. Possible solutions are:

8	$^{-}7$	$^{-}4$
$^{-}13$	$^{-}1$	11
2	5	$^{-}10$

or

2	$^{-}13$	8
5	$^{-}1$	$^{-}7$
$^{-}10$	11	$^{-}4$

20. If $y = {}^{-}3x - 2$:

(a) $y = {}^{-}3({}^{-}1) - 2 = 3 + {}^{-}2 = \mathbf{1}$, when $x = {}^{-}1$.

(b) $y = {}^{-}3(100) + {}^{-}2 = {}^{-}300 + {}^{-}2 = \mathbf{^{-}302}$, when $x = 100$.

(c) $y = {}^{-}3({}^{-}2) - 2 = 6 + {}^{-}2 = \mathbf{4}$, when $x = {}^{-}2$.

(d) $y = {}^{-}3({}^{-}a) - 2 = \mathbf{3a - 2}$, when $x = {}^{-}a$.

(e) $^{-}11 = {}^{-}3x - 2 \Rightarrow {}^{-}9 = {}^{-}3x \Rightarrow \boldsymbol{x = 3}$.

21. **(a)** The common difference is 1. The n^{th} term is $n + ({}^{-}28)$. Set the n^{th} term equal to the last term and solve for n.

$n + ({}^{-}28) = 52 \Rightarrow n = 80$. There are 80 terms in the arithmetic sequence.

(b) The common ratio is 3. The n^{th} term is $3^{(n+2)}$. Set the n^{th} term equal to the last term and solve for n. $3^{n+2} = 3^{10} \Rightarrow n + 2 = 10 \Rightarrow n = 8$. There are 8 terms in the geometric sequence.

22. The sum of $^{-}27 + {}^{-}26 + {}^{-}25 + ... + 50 + 51 + 52$
We know from 21(a) that there are 80 terms.
Using the Gauss method from Chapter 1:

$\frac{({}^{-}27 + 52)80}{2} = 1000$. The sum is 1000.

23. Write an equation:

$372 + {}^{-}9 + 4 + {}^{-}13 + {}^{-}24 = 330$ lbs.

24. Write an equation:

$65 + {}^{-}4 = 61$ seconds..

25. Answers vary. If x represents the amount of memory used $x + {}^{-}3.6 - {}^{-}7.2 = x + 3.6$ MB. She gained 3.6 MB of memory.

26. **(a)** There are **no values** of x for which this is true. If $^{-}|x| = 2$, then $|x| = {}^{-}2$, which is not possible.

(b) **All integers except 0.** Since $|x|$ is always non-negative, $^{-}|x|$ is always negative for all integers except 0; 0 is neither negative nor positive.

(c) There are **no values** of x for which this is true. $^{-}|x|$ is always negative for all integers except 0, which is neither negative nor positive.

(d) $^{-}x + 1$ is positive means $-x + 1 > 0$.
$-x + 1 > 0 \Rightarrow -x > -1 \Rightarrow \boldsymbol{x < 1}$.

(e) $^{-}x - 1$ is negative means $-x - 1 < 0$.
$-x - 1 < 0 \Rightarrow -x < 1 \Rightarrow \boldsymbol{x > -1}$.

27. If $y = |x - 5|$:

(a) $y = |(10) - 5| = |10 + {}^{-}5| = |5| = \mathbf{5}$, when $x = 10$.

(b) $f({}^{-}1) = |({}^{-}1) - 5| = |{}^{-}1 + {}^{-}5| = |{}^{-}6| = \mathbf{6}$, when $x = {}^{-}1$.

(c) If $7 = |x - 5| \Rightarrow x - 5 = 7$ or $x - 5 = {}^{-}7$:

(i) $x - 5 = 7 \Rightarrow x = 7 + 5 \Rightarrow \boldsymbol{x = 12}$.

(ii) $x - 5 = {}^{-}7 \Rightarrow x = {}^{-}7 + 5 \Rightarrow x = \mathbf{{}^{-}2}$.

28. $$y = \begin{cases} x - 6 & \text{if } x \geq 6 \\ {}^{-}x + 6 & \text{if } x < 6 \end{cases}$$

29. (a) $d = 3 - 7 = \mathbf{{}^{-}4}$. The next two terms are $\mathbf{{}^{-}9}, \mathbf{{}^{-}13}$.

(b) $d = (1 - x) - (1 - 3x) = 1 - x - 1 + 3x = \boldsymbol{2x}$. The next two terms are $\boldsymbol{1 + 3x}$, $\boldsymbol{1 + 5x}$.

30. (a) **True**. The square of integer is always non-negative.

(b) **False**. Let $x = {}^{-}2$. Then $|({}^{-}2)^3| = |{}^{-}8| = 8 \neq ({}^{-}2)^3 = {}^{-}8$.

(c) **True**. $|x^3| = |x^2 \cdot x| = |x^2| \cdot |x| = x^2 \cdot |x|$.

31. (a) $^{-}x + 5 = 7 \Rightarrow {}^{-}x + 5 + {}^{-}5 = 7 + {}^{-}5 \Rightarrow {}^{-}x = 2 \Rightarrow \boldsymbol{x = {}^{-}2}$.

(b) $1 - x = {}^{-}13 \Rightarrow 1 + {}^{-}1 + {}^{-}x = {}^{-}13 + {}^{-}1 \Rightarrow {}^{-}x = {}^{-}14 \Rightarrow \boldsymbol{x = 14}$.

(c) $^{-}x - 8 = {}^{-}9 \Rightarrow {}^{-}x + {}^{-}8 + 8 = {}^{-}9 + 8 \Rightarrow {}^{-}x = {}^{-}1 \Rightarrow \boldsymbol{x = 1}$.

32. Estimates may vary depending upon the method used:

(a) Estimate:
$343 + {}^{-}42 - 402 \approx 300 - 400 = \mathbf{{}^{-}100}$

Actual: $^{-}101$.

(b) Estimate:
$^{-}1992 + 3005 - 497 \approx$
$^{-}2000 + 3000 - 500 = \mathbf{500}$.
Actual: 516.

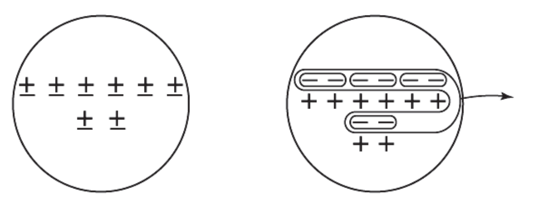
(c) Estimate: $992 - {}^{-}10{,}003 - 101 \approx 1000 + 10{,}000 - 100 = \mathbf{10{,}900}$.
Actual: 10,894.

(d) Estimate: $^{-}301 - {}^{-}1303 + 4993 \approx {}^{-}300 + 1300 + 5000 = \mathbf{6000}$.
Actual: 5995.

Assessment 5-2A: Multiplication and Division of Integers

1. $3 \cdot {}^{-}1 = {}^{-}1 + {}^{-}1 + {}^{-}1 = {}^{-}3$
$2 \cdot {}^{-}1 = {}^{-}1 + {}^{-}1 = {}^{-}2$
$1 \cdot {}^{-}1 = {}^{-}1$
$0 \cdot {}^{-}1 = 0$
$^{-}1 \cdot {}^{-}1 = 1$.

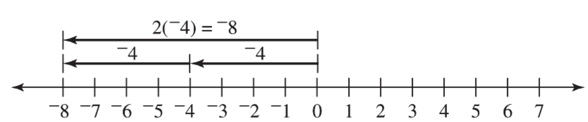
2.



0 charge

Take away 4 groups of 2 negative charges. Net result is 8 positive charges; answer 8

3. Move four units to the left twice to arrive at the product $2({}^{-}4) = {}^{-}8$:



4. **(a)** Take away three groups of three negative charges, leaving nine positive charges; i.e., $\mathbf{({}^{-}3)\cdot({}^{-}3) = 9}$.

(b) Take away five groups of two positive charges, leaving ten negative charges; i.e., $\mathbf{(5)({}^{-}2) = {}^{-}10}$.

5. **(a)** The number of students will decrease by 30 per year over the next four years, or $4\cdot{}^{-}30 = {}^{-}120$.

(b) There were 30 more students per year four years ago, or ${}^{-}4\cdot{}^{-}30 = 120$.

(c) The number of students will decrease by 30 per year over the next n years, or $\boldsymbol{n}\cdot\mathbf{{}^{-}30} = \mathbf{{}^{-}30\boldsymbol{n}}$ students.

(d) There were 30 more students per year during the past n years, or $\mathbf{{}^{-}\boldsymbol{n}\cdot{}^{-}30 = 30\boldsymbol{n}}$ students.

6. **(a)** ${}^{-}40 \div {}^{-}5 \Rightarrow {}^{-}40 = {}^{-}5\cdot 8$

$\Rightarrow {}^{-}40 \div {}^{-}5 = 8$.

(b) ${}^{-}143 = 13\cdot{}^{-}11 \Rightarrow {}^{-}143 \div 13 = \mathbf{{}^{-}11}$.

(c) ${}^{-}5 \neq 0\cdot$ any integer $\Rightarrow {}^{-}5 \div 0$ **is not defined** because no integer exists to satisfy the multiplication statement.

7. **(a)** $({}^{-}10 \div {}^{-}2)({}^{-}2) = 5\cdot{}^{-}2 = \mathbf{{}^{-}10}$.

(b) $({}^{-}10\cdot 5)\div 5 = {}^{-}50 \div 5 = \mathbf{{}^{-}10}$.

(c) ${}^{-}8 \div ({}^{-}8 + 8) = {}^{-}8 \div 0 \Rightarrow$ **not defined** because there is no integer that would make the multiplication statement true.

(d) $({}^{-}6 + 6) \div ({}^{-}2 + 2) = 0 \div 0 \Rightarrow$ **not defined** because too many integers can make the multiplication equation true.

(e) $|{}^{-}24| \div [4\cdot(9-15)] = 24 \div 4\cdot{}^{-}6 = {}^{-}1$. Order of operations specifies that multiplication and division be performed from left to right after the operations within the absolute value and parentheses are performed.

8. For each of the following, if $a\cdot b = c \Rightarrow c \div b = a$ and $c \div a = b$ ($a \neq 0$ and $b \neq 0$):

(a) **(*i*)** ${}^{-}6\cdot 5 = \mathbf{{}^{-}30}$.

(*ii*) ${}^{-}30 \div 5 = \mathbf{{}^{-}6}$.

(*iii*) ${}^{-}30 \div {}^{-}6 = \mathbf{5}$.

(b) **(*i*)** $({}^{-}5)\cdot({}^{-}4) = \mathbf{20}$.

(*ii*) $20 \div {}^{-}5 = \mathbf{{}^{-}4}$.

(*iii*) $20 \div {}^{-}4 = \mathbf{{}^{-}5}$.

(c) **(*i*)** ${}^{-}3\cdot 0 = \mathbf{0}$.

(*ii*) $0 \div {}^{-}3 = \mathbf{0}$.

(*iii*) $0 \div 0$ is undefined.

9. **(a)** $4x \div 4 = n$ if and only if $4x = 4n$, where n is an integer. Then $4x = 4n$ if and only if $\boldsymbol{n = x}$, so $4x \div 4 = x$.

(b) $({}^{-}xy) \div y = n$ if and only if $({}^{-}xy) = yn$. Then $({}^{-}xy) = yn$ if and only if $\boldsymbol{n = {}^{-}x}$ $(y \neq 0)$.

10. **(a)** $32°\text{C} + (30 \text{ minutes} \times {}^{-}1° \text{ per minute})$

$= 32° + {}^{-}30° = \mathbf{2°C}$.

(b) $0°\text{C} + ({}^{-}25 \text{ minutes} \times {}^{-}2° \text{ per minute})$

$= 0° + 50° = \mathbf{50°\,C}$.

(c) ${}^{-}20°\text{C} + ({}^{-}30 \text{ minutes} \times {}^{-}2° \text{ per minute})$

$= {}^{-}20° + 60° = \mathbf{40°\,C}$.

(d) $25°\text{C} + ({}^{-}20 \text{ minutes} \times 2° \text{ per minute})$

$= 25° + {}^{-}40° = {}^{-}15°\,\mathbf{C}$.

11. $({}^{-}12{,}000 \text{ acres per year} \times 8 \text{ years}) =$ ${}^{-}96{,}000$ acres, or 96,000 acres lost.

12. ${}^{-}1({}^{-}4 - {}^{-}2) = {}^{-}1\cdot{}^{-}4 - {}^{-}1\cdot{}^{-}2 = 4 - 2 = 2$

13. **(a)** $({}^{-}2)^3 = {}^{-}2\cdot{}^{-}2\cdot{}^{-}2 = ({}^{-}2\cdot{}^{-}2)\,{}^{-}2 =$ $4\cdot{}^{-}2 = \mathbf{{}^{-}8}$.

(b) $({}^{-}2)^4 = ({}^{-}2\cdot{}^{-}2)\cdot({}^{-}2\cdot{}^{-}2) =$ $4\cdot 4 = \mathbf{16}$.

(c) $({}^{-}10)^5 \div ({}^{-}10)^2 = {}^{-}100{,}000 \div 100 =$ $\mathbf{{}^{-}1000}$.

(d) $({}^{-}3)^5 \div ({}^{-}3) = {}^{-}243 \div {}^{-}3 = \mathbf{81}$.

(e) $({}^{-}1)^{50} = \mathbf{1}$ (i.e., ${}^{-}1$ taken to an even power).

(f) $({}^{-}1)^{151} = \mathbf{{}^{-}1}$ (i.e., ${}^{-}1$ taken to an odd power).

(g) $^-2 + 3 \cdot 5 - 1 = {^-2} + 15 - 1 = 13 - 1 =$ **12**. Note the order of operation; multiplication before addition.

(h) $10 - 3 \cdot 7 - 4(^-2) + 3 = 10 - 21 - {^-8} + 3 = (10 + 8 + 3) - 21 = \mathbf{0}$.

(i) $(^-2)^{64} - 2^{64} = 2^{64} - 2^{64} = \mathbf{0}$.

(j) $^-2^8 + 2^8 = \mathbf{0}$.

14. (a) Always **negative**. x^2 is always positive, so its additive inverse is always negative.

(b) Always **positive**. Any non-zero number taken to an even power is always positive.

(c) Always **positive**. Any non-zero number taken to an even power is always positive.

(d) **Positive when *x* is negative**; **negative when *x* is positive**. i.e., the additive inverse of x^3, which may be positive or negative.

(e) **Positive when *x* is negative** (the additive inverse of a negative integer is positive). **Negative when *x* is positive** (the additive inverse of a positive integer is negative).

15. (a) $^-x^2 = x^2$ for $x = 0$.

(b) $^-x^3 = (^-x)^3$ for all integers.

16. If $48 \div x$ is an integer then x equals all positive and negative divisors of 48. The factors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48. The negative divisors of 48 are

$^-1,\ ^-2,\ ^-3,\ ^-4,\ ^-6,\ ^-8,\ ^-12,\ ^-16,\ ^-24,$

and $^-48$.

17. (a) **Commutative property of multiplication**, or $a \cdot b = b \cdot a$.

(b) **Closure property of addition**. The sum of an integer plus another integer is an integer.

(c) **Associative property of multiplication**, or $a(b \cdot c) = (a \cdot b)c$.

(d) **Distributive property of multiplication over addition**, or $a(b + c) = ab + ac$.

18. (a) $(^-x)(^-y) = (^-1 \cdot x)(^-1 \cdot y)$

$= (^-1 \cdot {^-1}) \cdot (x \cdot y)$

$= 1(xy) = \boldsymbol{xy}$.

(b) $^-2x(^-y) = \mathbf{2}\boldsymbol{xy}$.

(c) $^-2(^-x + y) + x + y = 2x - 2y + x + y = \mathbf{3}\boldsymbol{x} - \boldsymbol{y}$.

(d) $^-1 \cdot x = {^-}\boldsymbol{x}$.

19. (a) $^-2(x - y) = {^-2}x - {^-2}y = {^-}\mathbf{2}\boldsymbol{x} + \mathbf{2}\boldsymbol{y}$.

(b) $x(x - y) = x \cdot x - x \cdot y = \boldsymbol{x}^{\mathbf{2}} - \boldsymbol{xy}$.

(c) $^-x(x - y) = {^-x^2} - {^-xy} = {^-}\boldsymbol{x}^{\mathbf{2}} + \boldsymbol{xy}$.

(d) $^-2(x + y - z) = {^-2}x + {^-2}y - {^-2}z = {^-}\mathbf{2}\boldsymbol{x} - \mathbf{2}\boldsymbol{y} + \mathbf{2}\boldsymbol{z}$.

20. (a) $^-3x = 6 \Rightarrow \frac{^-3x}{3} = \frac{6}{^-3} \Rightarrow \boldsymbol{x} = {^-}\mathbf{2}$.

(b) $^-2x = 0 \Rightarrow \frac{^-2x}{^-2} = \frac{0}{^-2} \Rightarrow \boldsymbol{x} = \mathbf{0}$.

(c) $x \div 3 = {^-12} \Rightarrow 3(x \div 3) = 3(^-12)$

$\Rightarrow \boldsymbol{x} = {^-}\mathbf{36}$.

(d) $x \div {^-3} = {^-2} \Rightarrow {^-3}(x \div {^-3}) = {^-3}(^-2)$

$\Rightarrow \boldsymbol{x} = \mathbf{6}$.

(e) $x \div (^-x) = {^-1} \Rightarrow {^-x}(x \div {^-x}) = {^-x}(^-1)$

$\Rightarrow x = x$; i.e., the equation is satisfied by **all integers except 0** (since $0 \div 0$ is not defined).

(f) $^-3x - 8 = 7 \Rightarrow {^-3}x - 8 + 8 = 7 + 8$

$\Rightarrow {^-3}x = 15 \Rightarrow \frac{^-3}{^-3}x = \frac{15}{^-3} \Rightarrow \boldsymbol{x} = {^-}\mathbf{5}$.

(g) $^-2(5x - 3) = 26 \Rightarrow {^-10}x + 6 = 26$

$\Rightarrow {^-10}x + 6 - 6 = 26 - 6 \Rightarrow {^-10}x = 20$

$\Rightarrow \frac{^-10}{^-10}x = \frac{20}{^-10} \Rightarrow \boldsymbol{x} = {^-}\mathbf{2}$.

(h) $3x - x - 2x = 3 \Rightarrow 0 = 3 \Rightarrow$ **no solution**.

(i) $^-2(5x - 6) - 30 = {^-x} \Rightarrow {^-10}x + 12$

$-30 = {^-x} \Rightarrow {^-10}x - 18 = {^-x} \Rightarrow$

$^-18 = 9x \Rightarrow \boldsymbol{x} = {^-}\mathbf{2}$.

(j) $x^2 = 4 \Rightarrow \boldsymbol{x} = \mathbf{2}$ or $\boldsymbol{x} = {^-}\mathbf{2}$, since $2^2 = 4$ or $(^-2)^2 = 4$.

(k) $(x - 1)^2 = 9 \Rightarrow$

(*i*) $(x - 1) = 3 \Rightarrow \boldsymbol{x} = \mathbf{4}$, or (*ii*) $(x-1)={}^-3 \Rightarrow \boldsymbol{x = {}^-2}$.

(l) $(x-1)^2=(x+3)^2 \Rightarrow$

(*i*) $(x-1)=(x+3) \Rightarrow 0=4 \Rightarrow$ not a viable solution.

(*ii*) $(x-1)={}^-(x+3) \Rightarrow x-1={}^-x-3$ $\Rightarrow 2x={}^-2 \Rightarrow \boldsymbol{x={}^-1}$.

(m) $(x-1)(x+3)=0 \Rightarrow$ either $(x-1)=0$ or $(x+3)=0$; i.e., the only way that the product of two numbers can be zero is that one or both must equal zero.

(*i*) **If** $(x-1)=0 \Rightarrow \boldsymbol{x=1}$.

(*ii*) **If** $(x+3)=0 \Rightarrow \boldsymbol{x={}^-3}$.

21. The difference-of-squares formula is: $(a+b)(a-b)=a^2-b^2$.

(a) $52 \cdot 48=(50+2)(50-2)=$ $50^2-2^2=2500-4=\mathbf{2496}$.

(b) $(5-100)(5+100)=5^2-100^2=$ $25-10{,}000=\mathbf{{}^-9975}$.

(c) $({}^-x-y)({}^-x+y)=({}^-x)^2-y^2=$ $\boldsymbol{x^2-y^2}$.

22. To *factor* an expression means to find an equivalent expression that is a product; i.e., if $N=ab$, then a and b are factors of N. Factoring may be said to undo the distributive property of multiplication over addition or subtraction.

(a) $3x+5x=x(3+5)=\boldsymbol{8x}$. The factor common to both terms, x, divides each and then multiplies their sum.

(b) $xy+x=x\cdot y+x\cdot 1=\boldsymbol{x(y+1)}$.

(c) $x^2+xy=x\cdot x+x\cdot y=\boldsymbol{x(x+y)}$.

(d) $3xy+2x-xz=\boldsymbol{x(3y+2-z)}$.

(e) $abc+ab-a=a(bc+b-1)=$ $\boldsymbol{a[b(c+1)-1]}$.

(f) $16-a^2=4^2-a^2=\boldsymbol{(4+a)(4-a)}$; i.e., the factorization of the difference-of-squares formula.

(g) $4x^2-25y^2=(2x)^2-(5y)^2$ $=\boldsymbol{(2x+5y)(2x-5y)}$.

23. $(a-b)^2=(a-b)(a-b)=a(a-b)$ $+{}^-b(a-b)=a^2-ab+{}^-ba-{}^-b^2=$ $a^2-2ab+b^2$.

24. **(a)** (*i*) Arithmetic sequence; **difference** $(d)=$ ${}^-7-{}^-10=\mathbf{3}$.

(*ii*) The next two terms are: $5+3=\mathbf{8}$ and $8+3=\mathbf{11}$.

(*iii*) $a_n=a_1+(n-1)d={}^-10+$ $(n-1)\cdot 3={}^-10+3n-3=$ $\boldsymbol{3n-13}$.

(b) (*i*) Geometric sequence; **ratio** $(r)=$ ${}^-4 \div {}^-2=\mathbf{2}$

(*ii*) The next two terms are: ${}^-64\cdot 2=\mathbf{{}^-128}$ and ${}^-128\cdot 2=\mathbf{{}^-256}$.

(*iii*) $a_n=a_1(r)^{n-1}={}^-2(2)^{n-1}=\boldsymbol{{}^-2^n}$.

(c) (*i*) Geometric sequence; $r={}^-2^2 \div 2=\mathbf{{}^-2}$.

(*ii*) The next two terms are: ${}^-2^6\cdot{}^-2=\mathbf{2^7}$ and $2^7\cdot{}^-2=\mathbf{{}^-2^8}$.

(*iii*) $a_n=2({}^-2)^{n-1}={}^-({}^-2)({}^-2)^{n-1}=$ ${}^-({}^-2)^{n-1+1}=\boldsymbol{{}^-({}^-2)^n}$.

25. **(a)** ${}^-3, {}^-9, \_, \_, {}^-243$ is a geometric sequence with a common ratio of 3. The missing terms are ${}^-27$ and ${}^-81$.

(b) $\_, 32, {}^-16, 8, \_$ is a geometric sequence. The common ratio is ${}^-\frac{1}{2}$. The missing terms are ${}^-64$ and ${}^-4$.

26. Write an equation: ${}^-5(6)+3(8)={}^-30+24={}^-6$. The net result is a descent of 6 feet.

27. Let n be the number of cups. The height of each cup is 7 and ${}^-6$ represents the unseen part of each cup when they are nested together. There are (n-1) cups with unseen parts. The total height of n cups can then be written as:

$7n+({}^-6)(n-1)=7n-6n+6=n+6$.

Assessment 5-2B

1. $3 \cdot {}^-2 = {}^-2 + {}^-2 + {}^-2 = {}^-6$

$2 \cdot {}^-2 = {}^-2 + {}^-2 = {}^-4$

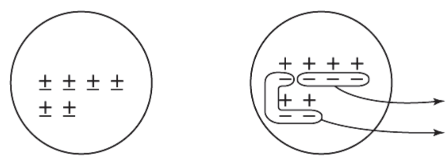
$1 \cdot {}^-2 = {}^-2$

$0 \cdot {}^-2 = 0$

${}^-1 \cdot {}^-2 = 2$

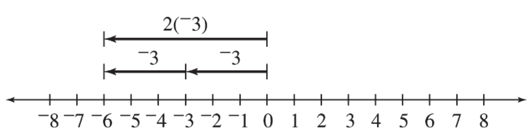
${}^-2 \cdot {}^-2 = 4.$

2.



0 charge   Take away 2 groups of 3 negative charges. Net result is 6 positive charges

3. Move three units to the left twice to arrive at the product $2({}^-3) = {}^-6$:



4. **(a)** Take away two groups of three negative charges, leaving six positive charges; i.e., $\mathbf{({}^-2) \cdot ({}^-3) = 6}$.

(b) Take away four groups of two positive charges, leaving eight negative charges; i.e., $\mathbf{({}^-4) \cdot 2 = {}^-8}$.

5. **(a)** $4({}^-20) = {}^-80$ **students**

(b) $({}^-5)({}^-20) = \mathbf{100}$ **students**

(c) $n({}^-20) = \mathbf{-20\boldsymbol{n}}$ **students**

(d) $({}^-n)({}^-20) = \mathbf{20\boldsymbol{n}}$ **students**

6. **(a)** $143 \div {}^-10 \Rightarrow 143 = {}^-10 \cdot x$. The quotient is undefined in the set of integers because there is no integer to multiply times ${}^-10$ to obtain 143.

(b) $0 \div ({}^-6) \Rightarrow 0 = {}^-6 \cdot 0 \Rightarrow 0 \div {}^-6 = 0.$

7. **(a)** Let $(a \div b)b = x$. Then $a \div b = x \div b \Rightarrow a = x$ and $(a \div b)b = \boldsymbol{a}$ for any integers a and b, $b \neq 0$.

(b) Let $(a \cdot b) \div b = x$. Then $a \cdot b = x \cdot b \Rightarrow a = x$ and $(a \cdot b) \div b = \boldsymbol{a}$ for any integers a and b, $b \neq 0$.

(c) $({}^-8 + 8) \div 8 = 0 \div 8 = \mathbf{0}$.

(d) $({}^-23 - {}^-7) \div 4 = {}^-16 \div 4 = {}^-4$ Order of operations specifies that the subtraction within the parentheses be performed first.

(e) $|{}^-28| \div [2|{}^-7|] = 28 \div [2(7)] = 28 \div 14 = \mathbf{2}$

8. For each of the following, if $a \cdot b = c \Rightarrow c \div b = a$ and $c \div a = b$ $(a \neq 0$ and $b \neq 0)$:

(a) ***(i)*** ${}^-5 \cdot 4 = {}^-20$

(ii) ${}^-20 \div 4 = {}^-5$

(iii) ${}^-20 \div {}^-5 = 4.$

(b) ***(i)*** $({}^-4) \cdot ({}^-3) = 12$

(ii) $12 \div ({}^-4) = {}^-3$

(iii) $12 \div ({}^-3) = {}^-4.$

(c) $0 \div 0$ is not defined. To understand why suppose there is one answer and name it x. If $0 \div 0 = x$, then $0 \cdot x = 0$. Notice that any real numbers could replace x and $0 \cdot x = 0$ would be true. For example, $0 \cdot 0 = 0$ and $0 \cdot 13 = 0$. Since the solution to $0 \cdot x = 0$ is not unique, $0 \div 0$ is not defined.

9. **(a)** $({}^-4x) \div x = n$ if and only if $({}^-4x) = xn$. Then $({}^-4x) = xn$ if and only if $\boldsymbol{n} = \mathbf{{}^-4}$ $(x \neq 0)$.

(b) $({}^-10x + 5) \div 5 = n$ if and only if ${}^-10x + 5 = 5n$. ${}^-10x + 5 = 5n$ if and only if $5 \cdot ({}^-2x + 1) = 5n$. $5 \cdot ({}^-2x + 1) = 5n$ if and only if $\boldsymbol{n} = \mathbf{{}^-2\boldsymbol{x} + 1}$.

10. **(a)** $^{-}5°\text{C} + (m \text{ minutes} \times d° \text{ per minute}) =$
$^{-}5° + md° =$ $\mathbf{(^{-}5 + \mathit{md})° \ C}$.

(b) $0°\text{C} + (^{-}m \text{ minutes} \times {}^{-}d° \text{ per minute}) =$
$0° + md° =$ $\boldsymbol{md}$**°C**.

(c) $20°\text{C} + (^{-}m \text{ minutes} \times d° \text{ per minute}) =$
$20° + {}^{-}md° =$ $\mathbf{(20 - \mathit{md})° \ C}$.

11. **(a)** $4 \cdot {}^{-}11 = {}^{-}44$, or **44 yards lost**.

(b) $^{-}66 \div 11 = {}^{-}6$, or **6 yards lost on average**.

12. $^{-}1(4 - {}^{-}2) = {}^{-}1 \cdot 4 - {}^{-}1 \cdot {}^{-}2 = {}^{-}4 - 2 = {}^{-}6$

13. **(a)** $10 - 3 - 12 = 7 - 12 =$ $\mathbf{^{-}5}$.

(b) $10 - (3 - 12) = 10 - (^{-}9) =$
$10 + 9 =$ **19**.

(c) $(^{-}3)^2 = {}^{-}3 \cdot {}^{-}3 =$ **9**. A negative integer taken to an even power is positive.

(d) $^{-}3^2 = {}^{-}(3 \cdot 3) =$ $\mathbf{^{-}9}$.

(e) $^{-}5^2 + 3(^{-}2)^2 = {}^{-}25 + 3 \cdot 4 = {}^{-}25$
$+ 12 =$ $\mathbf{^{-}13}$.

(f) $^{-}2^3 = {}^{-}(2 \cdot 2 \cdot 2) =$ $\mathbf{^{-}8}$.

(g) $(^{-}2)^5 = {}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 \cdot {}^{-}2 \cdot =$ $\mathbf{^{-}32}$.
A negative integer taken to an odd power is negative.

(h) $^{-}2^4 = {}^{-}(2 \cdot 2 \cdot 2 \cdot 2) =$ $\mathbf{^{-}16}$.

(i) $^{-}2^{63} + 2^{64} = 2^{63}(^{-}1 + 2) =$ $\mathbf{2^{63}}$.

(j) $7^5 + {}^{-}7^5 = 0$.

14. **(a)** Always **negative**. The additive inverse of a positive integer.

(b) Always **positive**. The product of either four positive or four negative integers is positive; i.e., any non-zero integer taken to an even power is positive.

(c) Always **positive**. Any non-zero integer taken to an even power is positive.

(d) **Positive when *x* is positive; negative when *x* is negative**.

(e) **Positive when *x* is negative; negative when *x* is positive** (i.e., additive inverses).

15. **(a)** **Closure property of multiplication**. The product of two integers is an integer.

(b) **Zero multiplication property**. For all integers a, $a \cdot 0 = 0 = 0 \cdot a$.

(c) **Distributive property of multiplication over addition**, or $a(b + c) = ab + ac$.

(d) **Commutative property of multiplication**, or $a \cdot b = b \cdot a$.

16. **(a)** $x - 2(^{-}y) = x - {}^{-}2y =$ $\mathbf{x + 2y}$.

(b) $a - (a - b)(^{-}1) = a - (^{-}a + b)$
$=$ $\mathbf{2a - b}$.

(c) $y - (y - x)(^{-}2) = y - (^{-}2y + 2x)$
$=$ $\mathbf{3y - 2x}$.

(d) $^{-}(x - y) + x = {}^{-}x - {}^{-}y + x =$ $\mathbf{y}$.

17. **(a)** $^{-}x(x - y - 3) = {}^{-}x^2 - {}^{-}xy - {}^{-}3x$
$=$ $\mathbf{^{-}x^2 + xy + 3x}$.

(b) $(^{-}5 - x)(5 + x) = {}^{-}5(5 + x) + {}^{-}x(5 + x)$
$= {}^{-}25 + {}^{-}5x + {}^{-}5x + {}^{-}x^2$
$=$ $\mathbf{^{-}25 - 10x - x^2}$.

(c) $(x - y - 1)(x + y + 1) = x(x + y + 1) -$
$y(x + y + 1) - 1(x + y + 1) = x^2 + xy$
$+ x + {}^{-}xy + {}^{-}y^2 + {}^{-}y + {}^{-}x + {}^{-}y + {}^{-}1 =$
$\mathbf{x^2 - y^2 - 2y - 1}$.

18. **(a)** $x^2 = 9 \Rightarrow (^{-}3)^2 = 9$ or $(3)^2 = 9$
$\Rightarrow$ $\mathbf{x = {}^{-}3}$ **or** $\mathbf{x = 3}$.

(b) $x^2 = {}^{-}9 \Rightarrow$ **no solution**. x^2 is always positive.

(c) $^{-}x \div {}^{-}x = 1 \Rightarrow x = x$; i.e., the equation is satisfied by **all integers except 0**.

(d) $^{-}x^2$ is negative $\Rightarrow x$ may be **any integer except 0**. 0 is neither positive nor negative.

(e) $^{-}(1 - x) = x - 1 \Rightarrow {}^{-}1 - {}^{-}x = x - 1 \Rightarrow x$ may be **any integer**.

(f) $x - 3x = {}^{-}2x \Rightarrow {}^{-}2x = {}^{-}2x \Rightarrow x$ may be **any integer**.

(g) ${}^{-}3(x + 2) = {}^{-}3x + 6 \Rightarrow {}^{-}3x + {}^{-}6 = {}^{-}3x + 6 \Rightarrow {}^{-}6 = 6 \Rightarrow$ **no solution**.

(h) $(2x - 1)^2 = (1 - 2x)^2$. True for **all integers**, since regardless of the value of x, the two sides of this equation will be additive inverses, and any number squared is non-negative.

(i) $x^3 = {}^{-}2^9 \Rightarrow x^3 = ({}^{-}2^3)^3$(i.e., ${}^{-}2^3 \cdot {}^{-}2^3 \cdot {}^{-}2^3. = {}^{-}2^9$) $\Rightarrow \boldsymbol{x = {}^{-}2^3 = {}^{-}8}$.

(j) ${}^{-}6x > {}^{-}x + 20 \Rightarrow {}^{-}5x > 20 \Rightarrow \boldsymbol{x < {}^{-}4}$. Note that the direction of the inequality changed under division by a negative number.

(k) ${}^{-}5(x - 3) > {}^{-}5 \Rightarrow x - 3 < 1 \Rightarrow \boldsymbol{x < 4}$.

(l) $x > {}^{-}2 \Rightarrow 5x > {}^{-}10 \Rightarrow 3 - 5x < 3 - {}^{-}10 \Rightarrow \boldsymbol{3 - 5x < 13}$.

(m) $x < 0 \Rightarrow 7x < 0 \Rightarrow 2 - 7x > 2 - 0 \Rightarrow \boldsymbol{2 - 7x > 2}$.

19. **(a)** $(2 + 3x)(2 - 3x) = 2^2 - (3x)^2 = \boldsymbol{4 - 9x^2}$.

(b) $(x - 1)(1 + x) = (x - 1)(x + 1) = x^2 - 1^2 = \boldsymbol{x^2 - 1}$.

(c) $213^2 - 13^2 = (213 + 13)(213 - 13) = 226 \cdot 200 = \boldsymbol{45{,}200}$.

20. To *factor* an expression means to find an equivalent expression that is a product; i.e., if $N = ab$, then *a* and *b* are factors of *N*. factoring may be said to undo the distributive property of multiplication over addition or subtraction.

(a) $ax + 2x = \boldsymbol{x(a + 2)}$.

(b) $ax - 2x = \boldsymbol{x(a - 2)}$.

(c) $3x - 4x + 7x = x(3 - 4 + 7) = \boldsymbol{6x}$.

(d) $3x^2 + xy - x = \boldsymbol{x(3x + y - 1)}$.

(e) $(a + b)(c + 1) - (a + b) = (a + b)[(c + 1) - 1] = \boldsymbol{(a + b)c}$.

(f) $x^2 - 9y^2 = x^2 - (3y)^2 = \boldsymbol{(x + 3y)(x - 3y)}$.

(g) $(x^2 - y^2) + x + y = (x + y)(x - y) + (x + y) = (x + y)[(x - y) + 1] = \boldsymbol{(x + y)(x - y + 1)}$.

21. $(a - 1)^2 = (a + {}^{-}1)(a + {}^{-}1)$

$= (a + {}^{-}1)a + (a + {}^{-}1)({}^{-}1)$

$= a^2 + {}^{-}1 \cdot a + a \cdot {}^{-}1 + {}^{-}1 \cdot {}^{-}1$

$= a^2 + {}^{-}a + {}^{-}a + 1$

$= a^2 + {}^{-}2a + 1$

$= a^2 - 2a + 1$

22. **(a)** **(*i*)** Arithmetic sequence; $\boldsymbol{d} = 7 - 10 = \boldsymbol{{}^{-}3}$.

(*ii*) The next two terms are: ${}^{-}5 + {}^{-}3 = \boldsymbol{{}^{-}8}$ and ${}^{-}8 + {}^{-}3 = \boldsymbol{{}^{-}11}$.

(*iii*) $a_n = 10 + (n - 1) \cdot {}^{-}3 = 10 - 3n + 3 = \boldsymbol{{}^{-}3n + 13}$.

(b) **(*i*)** Geometric sequence; $\boldsymbol{r} = 4 \div {}^{-}2 = \boldsymbol{{}^{-}2}$.

(*ii*) The next two terms are: $64 \cdot {}^{-}2 = \boldsymbol{{}^{-}128}$ and ${}^{-}128 \cdot {}^{-}2 = \boldsymbol{256}$.

(*iii*) $a_n = {}^{-}2({}^{-}2)^{n-1} = \boldsymbol{({}^{-}2)^n}$.

23. **(a)** ${}^{-}x = x^2$ for $x = {}^{-}1, 0$.

(b) ${}^{-}x^2 = ({}^{-}x)^3$ for $x = 0, 1$.

24. If $54 \div x$ is an integer then x equals all positive and negative divisors of 54. The factors of 54 are 1, 2, 3, 6, 9, 18, 27, and 54. The negative divisors of 54 are ${}^{-}1,\ {}^{-}2,\ {}^{-}3,\ {}^{-}6,\ {}^{-}9,\ {}^{-}18,\ {}^{-}27,$ and ${}^{-}54$.

25. **(a)** ${}^{-}4, {}^{-}16, \_, \_, {}^{-}1024$ is a geometric sequence with a common ratio of 4. The missing terms are ${}^{-}64$ and ${}^{-}256$.

(b) $\_, 27, {}^{-}9, 3, \_$ is a geometric sequence. The common ratio is ${}^{-}\dfrac{1}{3}$. The missing terms are ${}^{-}81$ and ${}^{-}1$.

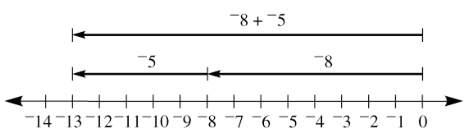
26. Write an equation: $^-4(6)+2(8)={}^-24+16={}^-8$. The net result is a descent of 8 feet.

27. Let n be the number of cups. The height of each cup is 8 and $^-6$ represents the unseen part of each cup when they are nested together. There are (n-1) cups with unseen parts. The total height of n cups can then be written as:

$8n+(^-6)(n-1)=8n-6n+6=2n+6$.

Mathematical Connections 5-2: Review Problems

18. Plot $^-$**8** on the number line then move 5 more units to the left to obtain $^-$**13**.



19. **(a)** Since $^-5+5=0$, **5** is the additive inverse.

(b) Since $7+{}^-7=0$, **7** is the additive inverse.

(c) Since $0+0=0$, **0** is the additive inverse.

20. **(a)** 14.
(b) 21.
(c) -4.
(d) 22.

21. **(a)** $|x-7|=0$ for $x=7$.

(b) $|x-7|<0$ has no solution because the absolute value can only be positive.

(c) $|x-7|>0$ is true for all integers other than 7.

22. Yes. When two integers are subtracted, their difference will always e another inter.

Chapter 5 Review

1. **(a)** The additive inverse of 3 is $^-$**3**.

(b) The additive inverse of ^-a is $^-(^-a)=\boldsymbol{a}$.

(c) The additive inverse of $^-2+3$ is $^-(^-2+3)={}^-1$

(d) The additive inverse of $x+y$ is $^-(x+y)={}^-x+{}^-y={}^-\boldsymbol{x}-\boldsymbol{y}$.

(e) The additive inverse of ^-x+y is $^-(^-x+y)=x+{}^-y=\boldsymbol{x}-\boldsymbol{y}$.

(f) The additive inverse of ^-x-y is $^-(^-x-y)=x-{}^-y=\boldsymbol{x}+\boldsymbol{y}$.

(g) $(^-2)^5={}^-32$, thus the additive inverse is **32**.

(h) $^-2^5={}^-32$, thus the additive inverse is **32**.

2. **(a)** $(^-2+{}^-8)+3=(^-10)+3={}^-\mathbf{7}$.

(b) $^-2-(^-5)+5={}^-2+5+5=\mathbf{8}$.

(c) $^-3(^-2)+2=6+2=\mathbf{8}$.

(d) $^-3(^-5+5)={}^-3(0)=\mathbf{0}$.

(e) $^-40\div{}^-5=\mathbf{8}$.

(f) $(^-25\div5)(^-3)=(^-5)(^-3)=\mathbf{15}$.

3. **(a)** $^-x+3=0\Rightarrow{}^-x+3-3=0-3\Rightarrow$ $^-x={}^-3\Rightarrow{}^-x(^-1)={}^-3(^-1)\Rightarrow x=3$.

(b) $^-2x=10\Rightarrow\frac{^-2x}{^-2}=\frac{10}{^-2}\Rightarrow x={}^-5$.

(c) $0\div(^-x)=0\Rightarrow\frac{0}{^-x}(^-x)=0(^-x)\Rightarrow$ $0=0$; i.e., x may be **any integer except 0**.

(d) $^-x\div0={}^-1\Rightarrow$ **no integer solution**. Division by 0 is undefined.

(e) $3x-1={}^-124\Rightarrow3x={}^-123\Rightarrow x=\frac{^-123}{3}$ $={}^-\mathbf{41}$

(f) $^-2x+3x=x\Rightarrow x(^-2+3)=x$ $\Rightarrow x=x$; i.e., x may be **any integer**.

4. $2\cdot{}^-3=(^-3)+(^-3)={}^-6$;

$1\cdot{}^-3={}^-3$;

$0\cdot{}^-3=0$; and if the pattern continues:

$^-1\cdot{}^-3=3$;

$^-2\cdot{}^-3=6$.

5. **(a)** Removing 5 black chips from $10 \Rightarrow 10 - 5 = \mathbf{5}$.

(b) When the 2 red chips are removed 3 black ones remain, thus $1 - {}^{-}2 = \mathbf{3}$.

6. **(a)** ${}^{-}1x = {}^{-}1 \cdot x = {}^{-}\boldsymbol{x}$.

(b) $({}^{-}1)(x - y) = {}^{-}1 \cdot x - {}^{-}1 \cdot y = {}^{-}x - {}^{-}y$
$= {}^{-}x + y = \boldsymbol{y - x}$.

(c) $2x - (1 - x) = 2x - 1 - {}^{-}x = 2x - 1 + x$
$= \mathbf{3}\boldsymbol{x} - \mathbf{1}$.

(d) $({}^{-}x)^2 + x^2 = ({}^{-}x \cdot {}^{-}x) + x^2 = x^2 + x^2$
$= \mathbf{2}\boldsymbol{x}^{\mathbf{2}}$.

(e) $({}^{-}x)^3 + x^3 = ({}^{-}x \cdot {}^{-}x \cdot {}^{-}x) + x^3$
$= {}^{-}x^3 + x^3 = \mathbf{0}$.

(f) $({}^{-}3 - x)(3 + x) = ({}^{-}3 - x)3 + ({}^{-}3 - x)x$
$= {}^{-}9 - 3x + {}^{-}3x - x^2 = {}^{-}9 - 6x - x^2$
$= {}^{-}\boldsymbol{x}^{\mathbf{2}} - \mathbf{6}\boldsymbol{x} - \mathbf{9}$.

(g) $({}^{-}2 - x)({}^{-}2 + x) = ({}^{-}2)^2 - x^2 = \mathbf{4} - \boldsymbol{x}^{\mathbf{2}}$.

7. **(a)** $x - 3x = x(1 - 3) = {}^{-}\mathbf{2}\boldsymbol{x}$.

(b) $x^2 + x = \boldsymbol{x(x + 1)}$.

(c) $x^2 - 36 = x^2 - 6^2 = \boldsymbol{(x + 6)(x - 6)}$.

(d) $81y^4 - 16x^4 = (9y^2 + 4x^2)(9y^2 - 4x^2)$
$= \boldsymbol{(9y^2 + 4x^2)(3y + 2x)(3y - 2x)}$.

(e) $5 + 5x = \boldsymbol{5(1 + x)}$.

(f) $(x - y)(x + 1) - (x - y)$
$= (x - y)[(x + 1) - 1] = \boldsymbol{(x - y)x}$.

8. **(a)** **False**. $|x|$ is not positive when $x = 0$.

(b) **False**. $|x + y| \neq |x| + |y|$ when x and y are of opposite sign.

(c) **False**. Let $a = 3$ and $b = {}^{-}4$. Then
$3 < {}^{-}({}^{-}4)$ and $3 \not< 0$.

(d) **True**. $x - y$ and $y - x$ both represent the difference between x and y. Squaring that difference produces a positive number in either case.

9. Answer may vary:

(a) $2 \div 1 \neq 1 \div 2$.

(b) $3 - (4 - 5) \neq (3 - 4) - 5$.

(c) $1 \div 2 \notin I$.

(d) $8 \div (4 - 2) \neq (8 \div 4) - (8 \div 2)$.

10. **(a)** $x + 3 = {}^{-}x - 17 \Rightarrow 2x = {}^{-}20$
$\Rightarrow x = {}^{-}\mathbf{10}$.

(b) $2x = {}^{-}2^{100} \Rightarrow x = \frac{{}^{-}2^{100}}{2} \Rightarrow x = {}^{-}\mathbf{2}^{\mathbf{99}}$.

(c) $2^{10}x = 2^{99} \Rightarrow x = \frac{2^{99}}{2^{10}} \Rightarrow x = \mathbf{2}^{\mathbf{89}}$.

(d) ${}^{-}x = x \Rightarrow {}^{-}2x = 0 \Rightarrow x = \mathbf{0}$.

(e) $|{}^{-}x| = 3 \Rightarrow x = 3$ or $x = {}^{-}\mathbf{3}$.

(f) $|x| = {}^{-}x \Rightarrow x = 0, {}^{-}1, {}^{-}2, \ldots$; i.e., $\boldsymbol{x} \leq \mathbf{0}$.

(g) $|x| > 3 \Rightarrow x = {}^{-}4, {}^{-}5, {}^{-}6, \ldots$, or
$x = 4, 5, 6, \ldots$; i.e., $\boldsymbol{x} > \mathbf{3}$ **or** $\boldsymbol{x} < {}^{-}\mathbf{3}$.

(h) $(x - 1)^2 = 100 \Rightarrow$

(i) $x - 1 = 10 \Rightarrow x = \mathbf{11}$, or

(ii) $x - 1 = {}^{-}10 \Rightarrow x = {}^{-}\mathbf{9}$.

11. **(a)** If $a_n = ({}^{-}1)^n$:

$a_1 = ({}^{-}1)^1 = {}^{-}\mathbf{1}$;
$a_2 = ({}^{-}1)^2 = \mathbf{1}$;
$a_3 = ({}^{-}1)^3 = {}^{-}\mathbf{1}$;
$a_4 = ({}^{-}1)^4 = \mathbf{1}$;
$a_5 = ({}^{-}1)^5 = {}^{-}\mathbf{1}$;
$a_6 = ({}^{-}1)^6 = \mathbf{1}$.

(b) If $a_n = ({}^{-}2)^n$:

$a_1 = ({}^{-}2)^1 = {}^{-}\mathbf{2}$;
$a_2 = ({}^{-}2)^2 = \mathbf{4}$;
$a_3 = ({}^{-}2)^3 = {}^{-}\mathbf{8}$;
$a_4 = ({}^{-}2)^4 = \mathbf{16}$;
$a_5 = ({}^{-}2)^5 = {}^{-}\mathbf{32}$;
$a_6 = ({}^{-}2)^6 = \mathbf{64}$.

(c) If $a_n = {}^-2 - 3n$:

$$a_1 = {}^-2 - 3 \cdot 1 = {}^-\mathbf{5};$$
$$a_2 = {}^-2 - 3 \cdot 2 = {}^-\mathbf{8};$$
$$a_3 = {}^-2 - 3 \cdot 3 = {}^-\mathbf{11};$$
$$a_4 = {}^-2 - 3 \cdot 4 = {}^-\mathbf{14};$$
$$a_5 = {}^-2 - 3 \cdot 5 = {}^-\mathbf{17};$$
$$a_6 = {}^-2 - 3 \cdot 6 = {}^-\mathbf{20}.$$

12. **(a)** Geometric; $r = {}^-1$.

(b) Geometric; $r = {}^-2$.

(c) Arithmetic; $d = 3$.

13. **(a)** Fibonacci-type sequence. The missing terms are 1, 1, 2.

(b) Geometric sequence with common ratio ⁻2. The missing terms are 392, ⁻784, 1568.

(c) Arithmetic sequence with common difference 147. The missing terms are 392, 539, 686.

14. **(a)** ${}^-1+1+0+1+1+2+3=7$.

(b) ${}^-49+98+{}^-196+392+{}^-784+1568=1029$.

(c) ${}^-49+98+245+392+539+686=1911$.

15. **(a)** This is an arithmetic sequence with common difference ⁻2. The n^{th} term is ⁻$2n$+103. We set the last term equal to the n^{th} term and solve for n: ${}^-2n+103={}^-103 \Rightarrow {}^-2n={}^-20$
$6 \Rightarrow n=103$. There are 103 terms total. Subtracting the 5 given terms leaves 98 terms missing by the indicated ellipse.

(b) Each term is the previous term multiplied by ⁻
5. This gives: $5, {}^-25, 125, {}^-625, 3125, {}^-15625$. There are 6 terms total. Subtracting the 4 given terms leaves 2 terms missing by the indicated ellipse.

16. **(a)** The common difference is ⁻2.

(b) The common ratio is ⁻5.

17. The questions answered correctly contribute ${}^+4$ points, and the questions answered incorrectly contribute ${}^-7$ points. Terry's score is $87(4) + 46({}^-7) = 348 - 322 = \mathbf{26\ points}$.

18. Oregon, Nevada, Iowa, California, Arizona, Ohio, West Virginia, Alabama.

19. **(a)** ${}^-40 - ({}^-62) = -40 + 62 = \mathbf{22}$ degrees Celsius.

(b) $8180 - 1100 = \mathbf{7080\ ft}$.

(c) The pattern above suggest that as one travels South from the North Pole elevations must be higher to achieve negative temperatures. Since ${}^-11 - ({}^-40) = 29$ degrees Celsius, we might guess that since Hawaii is closer to the equator than Alaska, the elevation where ${}^-11°\text{C}$ was recorded was 7000 or 8000 feet higher than the elevation where ${}^-40°\text{C}$ was recorded in Arizona. A reasonable guess is that the elevation in Hawaii is 14,000 to 15,000 ft.

20. **(a)** ${}^-8 - ({}^-282) = \mathbf{274\ ft}$.

(b) Answers may vary, since several states are at sea level. The most accurate answers are Death Valley, **California** (-282 ft) and **Louisiana**, which as a lowest point at -8 ft.

21. If sea level is considered 0 m elevation, then the average depth at the Pacific Ocean is $-\mathbf{3963\ m}$.

22. **(a)** **degree Celsius ≈ degree Kelvin $-273°$.**

(b) degree Kelvin $\approx 100° + 273° = \mathbf{373°}$.

(c) degree Kelvin $\approx {}^-40° + 273° = \mathbf{233°}$.

23. **(a)** If we consider the number of forces on the battlefield before the Union and Confederate force arrived as 0, we can describe the change in the number of forces engaged in the battle as ${}^+75{,}000$, ${}^+82{,}289$, ${}^-23{,}049$, and ${}^-28{,}063$.

(b) ${}^-23{,}049 + {}^-28{,}063 = {}^-51{,}112$. Since the ⁻ denotes casualties, there were **51,112** casualties.

24. If we interpret 0 to be the ground floor, -2 would mean that the elevator is below ground level, which might be described as a 2nd level basement.

25. **(a)** $^{-}2x < 14 \Rightarrow {}^{-}x < 7 \Rightarrow x > {}^{-}7$

$3x > {}^{-}15 \Rightarrow x > {}^{-}5$

The common solution is $x > {}^{-}5$.

(b) $x^2 < 9 \Rightarrow {}^{-}3 < x < 3$

$x^2 \le 1 \Rightarrow {}^{-}1 \le x \le 1$

The common solution is ${}^{-}1 \le x \le 1$

(c) $|x| > {}^{-}4$ and $|x| < {}^{-}1$ have no common solutions.

26. Write an equation:

$6 = -(-x + 30 - 14)$

$6 = x - 30 + 14$

$6 = x - 16$

$22 = x$

27. The difference of squares formula is: $a^2 - b^2 = (a - b)(a + b)$. Applying this to $96 \cdot 104$:

$$(100-4)(100+4) = 100^2 - 4^2 = 10{,}000 - 16 = 9984.$$

28. **(a)** $1^2 - 4 = {}^{-}3$

$2^2 - 4 = 0$

$3^2 - 4 = 5$

$4^2 - 4 = 12$

$5^2 - 4 = 21$

(b) ${}^{-}5(1) - ({}^{-}3) = {}^{-}2$

${}^{-}5(2) - ({}^{-}3) = {}^{-}7$

${}^{-}5(3) - ({}^{-}3) = {}^{-}12$

${}^{-}5(4) - ({}^{-}3) = {}^{-}17$

${}^{-}5(5) - ({}^{-}3) = {}^{-}22$

(c) ${}^{-}(1)^2 - 4 = {}^{-}5$

${}^{-}(2)^2 - 4 = {}^{-}8$

${}^{-}(3)^2 - 4 = {}^{-}13$

${}^{-}(4)^2 - 4 = {}^{-}20$

${}^{-}(5)^2 - 4 = {}^{-}29$

29. **(a)** $({}^{-}1)^{2n}$ will always be equal to 1 for all integer values of n because $2n$ will always be even.

(b) $({}^{-}1)^{2n+1}$ will always be equal to ${}^{-}1$ for all integer values of n because $2n+1$ will always be odd.

30. **(a)** The n^{th} term is $({}^{-}1)^{n+1}(n)$.The sequence is neither arithmetic or geometric but the signs alternate letting us know to use a ${}^{-}1$ in the n^{th} term. Each term number is also equal to the number in the term so we multiply the $({}^{-}1)$ by n. Finally we want the odd term numbers to be positive and the even terms numbers to be negative so we raise the ${}^{-}1$ to the power $n+1$ to allow for the correct signs.

(b) The sum of the first 100 terms of the sequence is:

$1 + {}^{-}2 + 3 + \ldots + {}^{-}98 + 99 + {}^{-}100$

$= (1 + 3 + 5 + \ldots + 99) + ({}^{-}2 + {}^{-}4 + {}^{-}6 + \ldots + {}^{-}100)$

$= \dfrac{(1+99)50}{2} + \dfrac{({}^{-}2 + {}^{-}100)50}{2}$

$= 2500 + {}^{-}2550$

$= {}^{-}50$

CHAPTER 6

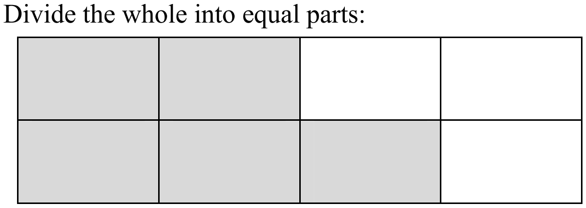
RATIONAL NUMBERS AND PROPORTIONAL REASONING

Assessment 6-1A: The Set of Rational Numbers

1. **(a)** The solution to $8x = 7$ is $\mathbf{\frac{7}{8}}$.
   (b) Joe ate seven of the eight apple slices.
   (c) The ratio of boys to girls in this math class is seven to eight.

2. Assume the shaded regions are symmetrical:
   (a) $\frac{1}{6}$. The ratio of shaded to unshaded areas is one to six.
   (b) $\frac{1}{4}$. The ratio of shaded to unshaded areas is one to four.
   (c) $\frac{2}{6} = \frac{1}{3}$. The circle is in six parts; two are shaded.
   (d) $\frac{7}{12}$. Seven of the twelve dots are shaded.

3. Divide the whole into equal parts:

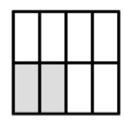


$\frac{5}{8}$ is shaded out of the whole.

4. The diagrams illustrate the Fundamental Law of Fractions; i.e., the value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero number.
   (a) $\frac{2}{3}$. Two of the three parts are shaded.
   (b) $\frac{4}{6} = \frac{2}{3}$. Four of the six parts are shaded.
   (c) $\frac{6}{9} = \frac{2}{3}$. Six of the nine parts are shaded.
   (d) $\frac{8}{12} = \frac{2}{3}$. Eight of the twelve parts are shaded.

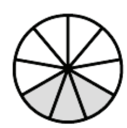
5. Assume the shaded regions are symmetrical:
   (a) **No**. The parts are not of equal area.
   (b) **Yes**. The sections are equal in area; three of the four are shaded.
   (c) **Yes**. The circle parts are equal in area; one of the two is shaded.

6. **(a)** One block of a square sectioned into four equal pieces is the same area as two blocks of a square sectioned into eight equal pieces; i.e., $\frac{1}{4} = \frac{2}{8}$:



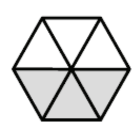
$\frac{2}{8}$

   (b) One arc of a circle sectioned into three equal pieces is the same area as three arcs of a circle sectioned into nine equal pieces; i.e., $\frac{1}{3} = \frac{3}{9}$:



$\frac{3}{9}$

   (c) One part of a hexagon sectioned into two equal pieces is the same area as three parts of a hexagon sectioned into six equal pieces; i.e., $\frac{1}{2} = \frac{3}{6}$:



$\frac{3}{6}$

7. **(a)** $\frac{\text{Dots in circle}}{\text{Total dots}} = \frac{9}{24} = \mathbf{\frac{3}{8}}$.

(b) $\frac{\text{Dots in rectangle}}{\text{Total dots}} = \frac{12}{24} = \mathbf{\frac{1}{2}}$.

(c) $\frac{\text{Dots in intersection}}{\text{Total dots}} = \frac{4}{24} = \mathbf{\frac{1}{6}}$.

(d) $\frac{\text{Dots in rectangle but outside circle}}{\text{Total dots}} = \frac{8}{24} = \mathbf{\frac{1}{3}}$.

8. **(a)** $A \cap B = \frac{4}{43}$. There are 4 people in both an arts class and a botany class out of the 43 people in the universal set $(17 + 4 + 13 + 9 = \quad 43)$.

(b) $A - B = \frac{17}{43}$. There are 21 people in an arts class but 4 of those are also in a botany class out of 43 people in the universal set $(21 - 4 = 17)$.

(c) $U = \frac{43}{43} = 1$. There are 43 people in the universal set. As a fraction of the universal set it would be 43 out of 43.

(d) $\overline{A \cap B} = \frac{43 - 4}{43} = \frac{39}{43}$. There are 43 people in the universal set and 4 people in set A and set B, so there are $43 - 4 = 39$ people who are not in A and B out of the the 43 people in the universal set.

9. Answers may vary (as long as the numerator and denominator of the given fractions are multiplied by the same number); some possibilities are:

(a) $\frac{4}{18}, \frac{6}{27}, \frac{10}{45}$.

(b) $\frac{^-4}{10}, \frac{2}{^-5}, \frac{^-10}{25}$.

(c) $\frac{0}{6}, \frac{0}{9}, \frac{0}{12}$.

(d) $\frac{2a}{4}, \frac{3a}{6}, \frac{4a}{8}$.

10. **(a)** $\frac{156}{93} = \frac{3 \cdot 52}{3 \cdot 31} = \mathbf{\frac{52}{31}}$.

(b) $\frac{27}{45} = \frac{9 \cdot 3}{9 \cdot 5} = \mathbf{\frac{3}{5}}$.

(c) $\frac{^-65}{91} = \frac{^-5 \cdot 13}{7 \cdot 13} = \mathbf{\frac{^-5}{7}}$.

11. **(a)** **Undefined**. Division by 0 is undefined.

(b) **Undefined**. Division by 0.

(c) **0**. $\frac{0}{5} = 0$ because $0 \cdot 5 = 0$.

(d) **Cannot be simplified**. $(2 + a)$ and a have no common factors other than 1.

(e) **Cannot be simplified**. $(15 + x)$ and $3x$ have no common factors other than 1.

12. In both parts (a) and (b), a restriction must be maintained so that the denominator cannot be zero.

(a) $\frac{a^2 - b^2}{3a + 3b} = \frac{(a+b)(a-b)}{3(a+b)} = \mathbf{\frac{a-b}{3}}\,(a \neq {}^-b)$.

(b) $\frac{14x^2y}{63xy^2} = \frac{7 \cdot 2 \cdot x \cdot x \cdot y}{7 \cdot 9 \cdot x \cdot y \cdot y} = \mathbf{\frac{2x}{9y}}\,(x, y \neq 0)$.

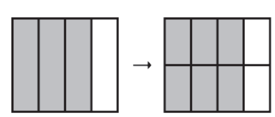
13. **(a)** **Equal**. $\frac{375}{1000} = \frac{125 \cdot 3}{125 \cdot 8} = \frac{3}{8}$ or $375 \cdot 8 = 1000 \cdot 3$.

(b) **Equal**. $\frac{18}{54} = \frac{18}{3 \cdot 18} = \frac{1}{3}$ and $\frac{23}{69} = \frac{1 \cdot 23}{3 \cdot 23} = \frac{1}{3}$ or $18 \cdot 69 = 54 \cdot 23$.

14. **(a)** **Not equal**. $16 = 2^4; 18 = 2 \cdot 3^2 \Rightarrow$ $LCM(16,18) = 2^4 \cdot 3^2 = 144$. Then $\frac{10}{16} = \frac{10 \cdot 9}{16 \cdot 9} = \frac{90}{144}$ and $\frac{12}{18} = \frac{12 \cdot 8}{18 \cdot 8} = \frac{96}{144}$; $\frac{90}{144} \neq \frac{96}{144}$.

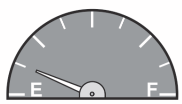
(b) **Not equal**. $86 = 2 \cdot 43; 215 = 5 \cdot 43 \Rightarrow$ $LCM(86,215) = 2 \cdot 5 \cdot 43 = 430$. Then $\frac{^-21}{86} = \frac{^-21 \cdot 5}{86 \cdot 5} = \frac{^-105}{430}$ and $\frac{^-51}{215} = \frac{^-51 \cdot 2}{215 \cdot 2} = \frac{^-102}{430}$; $\frac{^-105}{430} \neq \frac{^-102}{430}$.

15. The shaded area takes in three of the four columns and six of the eight small rectangles. Since the area in each case is the same, $\frac{3}{4} = \frac{6}{8}$.



16. To obtain equal fractions, multiply numerator and denominator by the same number. All fractions equal to $\frac{3}{4}$ will then be of the form $\frac{3n}{4n}$. $3n + 4n = 84 \Rightarrow n = 12$; thus $\frac{3}{4} = \frac{3 \cdot 12}{4 \cdot 12}. = \mathbf{\frac{36}{48}}$.

17. Mr. Gomez had $16 - 14 = 2$ gallons left. $\frac{2}{16} = \frac{1}{8}$ of a tank remained; the needle points to the 1st division of 8 as shown:



(b) He used 14 gallons: $\frac{14}{16} = \frac{7}{8}$.

18. **(a)** For equal fractions $\frac{a}{b} = \frac{c}{d}$, $ad = bc$. If $\frac{2}{3} = \frac{x}{16}$, then $2 \cdot 16 = 3 \cdot x \Rightarrow 32 = 3x \Rightarrow \boldsymbol{x = \frac{32}{3}}$.

(b) $3 \cdot x = 4 \cdot {}^-27 \Rightarrow 3x = {}^-108 \Rightarrow$ $\boldsymbol{x = {}^-36}$.

19. **(a)** $\frac{7}{8} > \frac{5}{6}$. $LCD(8,6) = 24 \Rightarrow \frac{7}{8} = \frac{21}{24}$ and $\frac{5}{6} = \frac{20}{24}$. Or (noting that if $\frac{a}{b} > \frac{c}{d}$ then $ad > bc$) $7 \cdot 6 > 8 \cdot 5$.

(b) $\frac{{}^-7}{8} < \frac{{}^-4}{5}$. $LCD\,(8,5) = 40 \Rightarrow \frac{{}^-7}{8} = \frac{{}^-35}{40}$ and $\frac{{}^-4}{5} = \frac{{}^-32}{40}$ (note that ${}^-35 < {}^-32$), or ${}^-7 \cdot 5 < 8 \cdot {}^-4$.

20. **(a)** $\frac{11}{13}, \frac{11}{16}, \frac{11}{22}$. When rational numbers have the same numerators, those with larger denominators, are of lesser value.

(b) $\frac{{}^-1}{5}, \frac{{}^-19}{36}, \frac{{}^-17}{30}$. $LCD(5,36,30) = 180 \Rightarrow$ $\frac{{}^-1}{5} = \frac{{}^-36}{180}, \frac{{}^-19}{36} = \frac{{}^-95}{180}$, and $\frac{{}^-17}{30} = \frac{{}^-102}{180}$. ${}^-36 > {}^-95 > {}^-102$.

21. Answers may vary; one method is to convert the given fractions into equal fractions having larger common denominators, thus creating spaces between them:

(a) $\frac{3}{7} = \frac{9}{21}$ and $\frac{4}{7} = \frac{12}{21}$. Two rational numbers between them could then be $\frac{10}{21}$ and $\frac{11}{21}$.

(b) $\frac{{}^-7}{9} = \frac{{}^-28}{36}$ and $\frac{{}^-8}{9} = \frac{{}^-32}{36}$. Two rational numbers between them could then be $\frac{{}^-30}{36}$ and $\frac{{}^-31}{36}$.

22. **(a)** **(*i*)** $6\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right) = \frac{6}{16}\text{ lb} = \boldsymbol{\frac{3}{8}}$ **pound**.

(*ii*) $6\text{ oz}\left(\frac{1\text{ lb}}{16\text{ oz}}\right)\left(\frac{1\text{ ton}}{2000\text{ lb}}\right) = \frac{6}{32{,}000}\text{ ton} = \boldsymbol{\frac{3}{16{,}000}}$ **ton**.

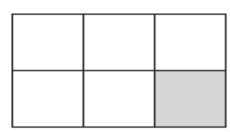
(b) $1\text{ dime}\left(\frac{1\text{ dollar}}{10\text{ dime}}\right) = \boldsymbol{\frac{1}{10}}$ **dollar**.

(c) $15\text{ min}\left(\frac{1\text{ hr}}{60\text{ min}}\right) = \frac{15}{60}\text{ hr} = \boldsymbol{\frac{1}{4}}$ **hour**.

(d) $8\text{ hr}\left(\frac{1\text{ day}}{24\text{ hr}}\right) = \frac{8}{24}\text{ day} = \boldsymbol{\frac{1}{3}}$ **day**.

23. If a, b, and c are integers and $b < 0$, then $\frac{a}{b} > \frac{c}{b}$ if, and only if, $a > c$. This is false. A counterexample would be $\frac{2}{{}^-5} > \frac{3}{{}^-5}$ but 2<3.

24. Even though the figure is divided in an inconsistent manner, we could redraw the figure as follows.



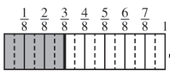
The shaded region is $\frac{1}{6}$ of the entire figure.

25. **(a)** Answers vary. For example $\frac{1}{3} < \frac{2}{4} < \frac{3}{5} < \frac{4}{6}$. The terms in this sequence go in ascending order: $\frac{20}{60} < \frac{30}{60} < \frac{36}{60} < \frac{40}{60}$.

(b) Answers vary. For example $\frac{{}^-3}{3} < \frac{{}^-4}{5} < \frac{{}^-5}{7} < \frac{{}^-6}{9}$. The terms in this sequence go in ascending order: $\frac{{}^-315}{315} < \frac{{}^-252}{315} < \frac{{}^-225}{315} < \frac{{}^-210}{315}$.

26. n cannot be zero otherwise there would be division by zero which is undefined. $\frac{0}{0}$ is undefined.

27. $n = \frac{n}{1}$ for all integers n, so every integer can be written as a fraction.

28. 

The drawing is not to scale.

29. $\frac{1}{5}$ of $10 = \frac{1}{5} \bullet 10 = 2.$ Two light bulbs were not shining.

30. $\frac{2}{5}$ were men so $\frac{3}{5}$ were women.

Assessment 6-1B

1. **(a)** The solution to $10x = 7$ is $\frac{7}{10}$.

(b) Joe ate seven of the ten apple slices.

(c) The ratio of boys to girls in this math class is seven to ten.

2. Assume the shaded regions are symmetrical:

(a) $\frac{1}{3}$. The figure highlights one of three parts.

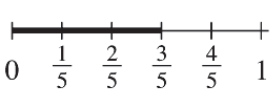
(b) $\frac{9}{12} = \frac{3}{4}$. Nine of the twelve dots are shaded.

(c) $\frac{5}{16}$. Five of the sixteen small triangles are shaded.

(d) $\frac{2}{16} = \frac{1}{8}$. If the large square were to be cut into pieces all the same size as those now shaded, there would be a total of sixteen, two of which are shaded.

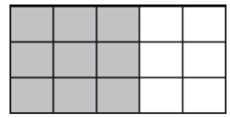
3. If the whole is divided into equal parts we have three triangles out of four shaded. $\frac{3}{4}$ of the whole is shaded.

4. **(a)** The figure highlights three of the five parts of the bar:

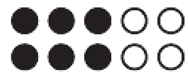


(b) Three of the five dots are filled:

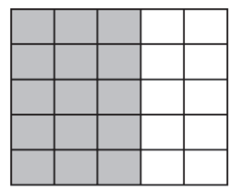
(c) Nine of the fifteen blocks are shaded $= \frac{3}{5}$ of the area:



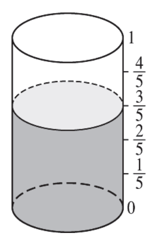
(d) Six of the ten dots are filled $= \frac{3}{5}$ of the dots:



(e) Fifteen of the 25 blocks are shaded $= \frac{3}{5}$ of the area:



(f) $\frac{3}{5}$ of the volume is shaded:

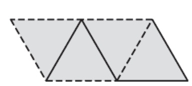


5. **(a)** **No**. The parts are not of equal area.

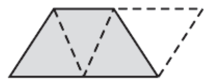
(b) **Yes**. The square can be divided into eight equal triangles, one of which is shaded.

(c) **Yes**. The circle can be divided into eight equal segments, two of which are shaded.

6. **(a)** Divide the figure into three equal parts, then add another equal part. The shaded area is then the whole:

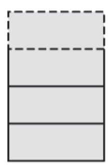


(b) Divide the figure into four equal parts, then remove one of the equal parts. The shaded area is then the whole:



(c) If two circles are $\frac{1}{5}$ or $\frac{2}{10}$ of the whole, then the whole is ten shaded circles:

(d) If the rectangle is $\frac{1}{4}$ of the whole, then add three other equal rectangles:



7. **(a)** $\frac{\text{Dots outside circle}}{\text{Total dots}} = \frac{15}{24} = \mathbf{\frac{5}{8}}.$

(b) $\frac{\text{Dots outside rectangle}}{\text{Total dots}} = \frac{12}{24} = \mathbf{\frac{1}{2}}.$

(c) $\frac{\text{Dots in union}}{\text{Total dots}} = \mathbf{\frac{17}{24}}.$

(d) $\frac{\text{Dots in circle but outside of rectangle}}{\text{Total dots}} = \mathbf{\frac{5}{24}}.$

8. **(a)** $A \cup B = \frac{17+4+13}{43} = \frac{34}{43}$. There are 34 people in either an algebra class or a biology class out of the 43 people in the universal set (17 + 4 + 13 + 9 = 43).

(b) $A - \overline{B} = \frac{4}{43}$. There are 21 people total in an algebra class and 17 of those are not in a biology class out of 43 people in the universal set (21 – 17 = 4).

(c) $\varnothing = \frac{0}{43} = 0$. There are 43 people in the universal set. There are no people in the empty set.

(d) $\overline{A \cup B} = \frac{43}{43} - \frac{17+4+13}{43} = \frac{43}{43} - \frac{34}{43} = \frac{9}{43}$. There are 43 people in the universal set and 9 people not in the union of set A and set B.

9. Answers may vary.

(a) $\mathbf{\frac{2}{6}, \frac{3}{9}, \frac{4}{12}}.$

(b) $\mathbf{\frac{8}{10}, \frac{12}{15}, \frac{16}{20}}.$

(c) $\mathbf{\frac{-6}{14}, \frac{-9}{21}, \frac{-12}{28}}.$

(d) $\mathbf{\frac{2a}{6}, \frac{3a}{9}, \frac{4a}{12}}.$

10. **(a)** $\frac{0}{68} = \frac{0.68}{1.68} = \mathbf{\frac{0}{1}}.$

(b) $\frac{84^2}{91^2} = \frac{(7\cdot 12)^2}{(7\cdot 13)^2} = \frac{7^2\cdot 12^2}{7^2\cdot 13^2} = \frac{12^2}{13^2} = \mathbf{\frac{144}{169}}.$

(c) $\frac{662}{703}$ is **already in reduced form**. There are no factors common to both numerator and denominator.

11. **(a)** **Cannot be simplified**. $\frac{6+x}{3x} \neq \frac{2+x}{x}$ because there are no factors common to each term.

(b) $\frac{2^6+2^5}{2^4+2^7} = \frac{2^5(2^1+1)}{2^4(1+2^3)} = \frac{2^5(3)}{2^4(9)} = \frac{32\cdot 3}{16\cdot 9} = \frac{2\cdot 16\cdot 3}{16\cdot 3\cdot 3} = \mathbf{\frac{2}{3}}.$

(c) $\frac{2^{100}+2^{98}}{2^{100}-2^{98}} = \frac{2^{98}(2^2+1)}{2^{98}(2^2-1)} = \frac{2^{98}\cdot 5}{2^{98}\cdot 3} = \mathbf{\frac{5}{3}}.$

12. In parts (a) and (b) both, a restriction must be maintained so that the denominator cannot be zero.

(a) $\frac{a^2+ab}{a+b} = \frac{a(a+b)}{a+b} = \mathbf{\frac{a}{1}} = \text{a}, (a+b \neq 0).$

(b) $\frac{a}{3a+ab} = \frac{a}{a(3+b)} = \mathbf{\frac{1}{3+b}} (a \neq 0, b \neq {}^{-}3).$

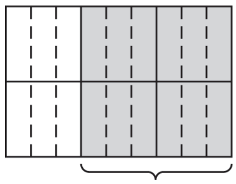
13. **(a)** $\frac{6}{16} = \frac{6}{4^2} = \frac{6}{4^2} \cdot \frac{25}{25} \cdot \frac{25}{25} = \frac{6\cdot 25\cdot 25}{4\cdot 25\cdot 4\cdot 25} = \frac{3{,}750}{10{,}000}$. The fractions are **equal**.

(b) **Not equal**. $\frac{17}{27}$ is in its reduced form and $\frac{25}{45} = \frac{5\cdot 5}{5\cdot 9} = \frac{5}{9}$ or $17 \cdot 45 \neq 27 \cdot 25$.

14. **(a)** **Equal**. $\frac{3}{{}^{-}12} = \frac{{}^{-}3}{12} = \frac{{}^{-}3\cdot 12}{12\cdot 12} = \frac{{}^{-}36}{144}.$

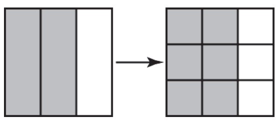
(b) **Not equal**.
$\frac{{}^{-}51}{215} = \frac{{}^{-}51\cdot 2}{215\cdot 2} = \frac{{}^{-}102}{430}; \frac{{}^{-}105}{430} \neq \frac{{}^{-}21}{430}.$

15.



$\frac{2}{3} = \frac{6}{9}$

Alternative:



16. **Yes**. $\frac{3}{8} = \frac{3 \cdot 4}{8 \cdot 4} = \frac{12}{32} > \frac{11}{32}$, so the board is thick enough. Shaving $\frac{1}{32}$ **inch** will bring it to the required thickness.

17. **Meter A by 3 minutes**. Meter A has $\frac{4}{10}$ of one hour $= 24$ minutes remaining; meter B has $\frac{7}{10}$ of $\frac{1}{2}$ hour $= \frac{7}{20}$ of one hour $= 21$ minutes remaining.

18. **(a)** For equal fractions $\frac{a}{b} = \frac{c}{d}$, $ad = bc$.
If $\frac{2}{3} = \frac{x}{18}$, then $2 \cdot 18 = 3 \cdot x \Rightarrow 36 = 3x \Rightarrow \boldsymbol{x = 12}$.

(b) **Any nonzero rational number except 0**.
$3 \cdot x^2 = x \cdot 3x \Rightarrow 3x^2 = 3x^2$; but 0 cannot be a solution because of division by x and x^2 in the original equation.

19. **(a)** $\frac{1}{^-7} < \frac{1}{^-8}$. $LCD(7, 8) = 56 \Rightarrow$
$\frac{1}{^-7} = \frac{^-1}{7} = \frac{^-8}{56}$ and $\frac{1}{^-8} = \frac{^-1}{8} = \frac{^-7}{56}$,
or $1 \cdot {}^-8 < {}^-7 \cdot 1$.

(b) $\frac{2}{5} = \frac{4}{10} \cdot 2 \cdot 10 = 5 \cdot 4$.

(c) $\frac{0}{7} = \frac{0}{17} \cdot \frac{0}{7} = 0 = \frac{0}{17}$.

20. Answers may vary; one method is to convert the given fractions into equal fractions having larger common denominators, thus creating spaces between them:

(a) $\frac{5}{6} = \frac{1000}{1200}$ and $\frac{83}{100} = \frac{996}{1200}$. Two rational numbers between them could then be $\frac{997}{1200}$ and $\frac{998}{1200}$.

(b) $\frac{^-1}{3} = \frac{^-4}{12}$ and $\frac{3}{4} = \frac{9}{12}$. Two rational numbers between them could then be **0** and $\boldsymbol{\frac{1}{2}}$.

21. **(a)** $12 \text{ oz}\left(\frac{1 \text{ lb}}{16 \text{ oz}}\right) = \frac{12}{16} \text{ lb} = \boldsymbol{\frac{3}{4}}$ **lb**.

(b) $1 \text{ nickel}\left(\frac{1 \text{ dollar}}{20 \text{ nickel}}\right) = \boldsymbol{\frac{1}{20}}$ **dollar**.

(c) $25 \text{ min}\left(\frac{1 \text{ hr}}{60 \text{ min}}\right) = \frac{25}{60} \text{ hr} = \boldsymbol{\frac{5}{12}}$ **hr**.

(d) $16 \text{ hr}\left(\frac{1 \text{ day}}{24 \text{ hr}}\right) = \frac{16}{24} \text{ day} = \boldsymbol{\frac{2}{3}}$ **day**.

22. **(a)** $\boldsymbol{2\frac{7}{8}}$ **inch**, or 2 inches plus 14 of the 16 divisions between 2 and 3; $2\frac{14}{16}$ inch $=$ $2\frac{7}{8}$ inch.

(b) $\boldsymbol{2\frac{3}{8}}$ **inch**, or 2 inches plus 6 of the 16 divisions between 2 and 3; $2\frac{6}{16}$ inch $=$ $2\frac{3}{8}$ inch.

(c) $\boldsymbol{1\frac{3}{8}}$ **inch**, or 1 inch plus 6 of the 16 divisions between 1 and 2; $1\frac{6}{16}$ inch $= 1\frac{3}{8}$ inch.

(d) $\boldsymbol{\frac{15}{16}}$ **inch**, or 15 of the 16 divisions between 0 and 1 inch.

23. **(a)** Answers vary. For example
$\frac{1}{3} > \frac{0}{4} > \frac{^-1}{5} > \frac{^-2}{6}$. The terms in this sequence go in descending order:
$\frac{20}{60} > \frac{0}{60} > \frac{^-12}{60} > \frac{^-20}{60}$.

(b) Answers vary. For example
$\frac{^-3}{6} > \frac{^-4}{5} > \frac{^-5}{4} > \frac{^-6}{3}$. The terms in this sequence go in descending order:
$\frac{^-30}{60} > \frac{^-48}{60} > \frac{^-75}{60} > \frac{^-120}{60}$.

24. $n = \frac{n}{1}$ where n is an integer. Because the set of integers is infinite, so is this representation.

25.

$n = \frac{n}{1}$ for all negative integers n,
so every negative integer can be written as a fraction with a positive denominator of 1.

26. If $1\text{mm} = \frac{1}{1000}\text{m}$ then $1000\text{mm} = 1\text{m}$ and $5\text{m} = 5000\text{mm}$.

27. $\frac{3}{5}$ were shining so $\frac{2}{5}$ were not shining.

28. $\frac{1}{5}$ were men so $\frac{4}{5}$ were women.

29.

(a)	(b)
Area of $a = \frac{1}{4}$	Area of $a = 1$
Area of $b = \frac{1}{4}$	Area of $b = 1$
Area of $c = \frac{1}{16}$	Area of $c = \frac{1}{4}$
Area of $d = \frac{1}{8}$	Area of $d = \frac{1}{2}$
Area of $e = \frac{1}{16}$	Area of $e = \frac{1}{4}$
Area of $f = \frac{1}{8}$	Area of $f = \frac{1}{2}$
Area of $g = \frac{1}{8}$	Area of $g = \frac{1}{2}$

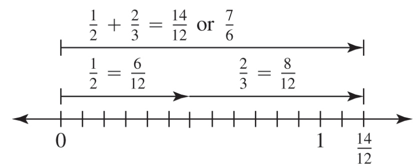
Assessment 6-2A: Addition, Subtraction, and Estimation with Rational Numbers

1. **(a)** Three possible methods are illustrated below:

(*i*) $\frac{1}{2} + \frac{2}{3} = \frac{1\cdot 3 + 2\cdot 2}{2\cdot 3} = \frac{3+4}{6} = \frac{7}{6}$; or

(*ii*) $LCD(2,3) = 6 \Rightarrow \frac{1}{2} + \frac{2}{3} = \frac{1}{2}\cdot\frac{3}{3} + \frac{2}{3}\cdot\frac{2}{2} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$; or

(*iii*)



(b) $LCD(12,3) = 12 \Rightarrow \frac{4}{12} - \frac{2}{3} = \frac{4}{12} - \frac{2}{3}\cdot\frac{4}{4} = \frac{4}{12} - \frac{8}{12} = \frac{^-4}{12} = \mathbf{\frac{^-1}{3}}$.

(c) $\frac{5}{x} + \frac{-3}{y} = \frac{5\cdot y - x\cdot 3}{x\cdot y} = \mathbf{\frac{5y-3x}{xy}}$.

(d) $LCD(2x^2y, 2xy^2, x^2) = 2x^2y^2 \Rightarrow$
$\frac{^-3}{2x^2y} + \frac{5}{2xy^2} + \frac{7}{x^2} = \frac{^-3}{2x^2y}\cdot\frac{y}{y} + \frac{5}{2xy^2}\cdot\frac{x}{x} + \frac{7}{x^2}\cdot\frac{2y^2}{2y^2} = \mathbf{\frac{^-3y+5x+14y^2}{2x^2y^2}}$.

(e) $\frac{5}{6} + 2\frac{1}{8} = \frac{5}{6} + \frac{17}{8} = \frac{5\cdot 8 + 6\cdot 17}{6\cdot 8} = \frac{40+102}{48} = \frac{142}{48} = \frac{71}{24} = \mathbf{2\frac{23}{24}}$.

(f) $^-4\frac{1}{2} - 3\frac{1}{6} = {^-4}\frac{3}{6} - 3\frac{1}{6} = {^-7}\frac{4}{6} = {^-7}\frac{2}{3}$ $\left(\text{or } \mathbf{\frac{^-23}{3}}\right)$.

(g) $7\frac{1}{4} + 3\frac{5}{12} - 2\frac{1}{3} = \frac{29}{4} + \frac{41}{12} - \frac{7}{3} = \frac{87}{12} + \frac{41}{12} - \frac{28}{12} = \frac{128}{12} - \frac{28}{12} = \frac{100}{12} = 8\frac{4}{12} = 8\frac{1}{3}$

2. **(a)** Two possible methods are illustrated below:

(*i*) $\frac{56}{3} = \frac{3\cdot 18 + 2}{3} = \frac{3\cdot 18}{3} + \frac{2}{3} = 18 + \frac{2}{3} = \mathbf{18\frac{2}{3}}$; or

(*ii*) $56 \div 3 = 18$, remainder $2 \Rightarrow \frac{56}{3} = \mathbf{18\frac{2}{3}}$.

(b) $-\frac{293}{100} = -\left(\frac{2\cdot 100 + 93}{100}\right) = {^-}\mathbf{2\frac{93}{100}}$.

3. **(a)** $6\frac{3}{4} = \frac{6}{1} + \frac{3}{4} = \frac{6\cdot 4 + 1\cdot 3}{1\cdot 4} = \frac{24+3}{4} = \mathbf{\frac{27}{4}}$.

(b) $^{-}3\frac{5}{8} = {}^{-}\left(\frac{3}{1} + \frac{5}{8}\right) = {}^{-}\left(\frac{3\cdot 8 + 1\cdot 5}{1\cdot 8}\right) =$
$-\left(\frac{24+5}{8}\right) = \mathbf{\frac{^{-}29}{8}}$.

4. **(a)** $\frac{15}{46} \approx \frac{15}{45} = \mathbf{\frac{1}{3}}$. 45 is smaller than the actual denominator so the estimate is **high**.

(b) $\frac{7}{41} \approx \frac{7}{42} = \mathbf{\frac{1}{6}}$. 42 is larger than the actual denominator so the estimate is **low**.

(c) $\frac{62}{80} \approx \frac{60}{80} = \mathbf{\frac{3}{4}}$. 60 is smaller than the actual numerator so the estimate is **low**.

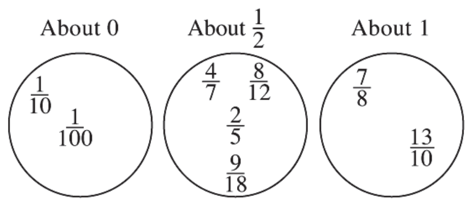
(d) $\frac{9}{19} \approx \frac{9}{18} = \mathbf{\frac{1}{2}}$. The denominator in the estimate is smaller than the actual denominator, so the estimate is **high**.

5. **(a)** **Beavers**. $\frac{10}{19} > \frac{1}{2}$.

(b) **Ducks**. $\frac{10}{22} < \frac{1}{2}$ by the least margin.

(c) **Bears**. $\frac{8}{23} > \frac{1}{3}$ by the least margin.

6.



Note that $\frac{8}{12} = \frac{3}{4}$ so it could be placed in either the "about $\frac{1}{2}$" or the "about 1" oval.

7. **(a)** $\mathbf{\frac{1}{2}}$**; high**. $\frac{19}{38} = \frac{1}{2} \Rightarrow \frac{19}{39} < \frac{1}{2}$.

(b) **0; low**. $\frac{3}{197} > 0$.

(c) $\mathbf{\frac{3}{4}}$**; high**. $\frac{150}{200} = \frac{3}{4} \Rightarrow \frac{150}{201} < \frac{3}{4}$.

(d) **1; high**. $\frac{8}{9} < 1$.

8. **(a)** **2**. Each addend is about $\frac{1}{2}$; thus the best approximation would be $4 \cdot \frac{1}{2} = 2$.

(b) $\mathbf{\frac{3}{4}}$. $\frac{30}{41}$ is about $\frac{3}{4}$; the other two addends are negligible compared to $\frac{3}{4}$.

9. Possible thought processes could be:

(a) $\frac{4}{4} - \frac{3}{4} = \mathbf{\frac{1}{4}}$.

(b) $\left(3 + 2 + \frac{3}{8} + \frac{2}{8}\right) - 5\frac{5}{8} = 5\frac{5}{8} - 5\frac{5}{8} = \mathbf{0}$.

10. **(a)** Region ***A***. $\frac{20}{8} = \frac{10}{4}$ is between 2 and 3.

(b) Region ***H***. $\frac{36}{8} = \frac{18}{4}$ is between 4 and 5.

(c) Region ***T***. $\frac{60}{16} = \frac{15}{4}$ is between 3 and 4.

(d) Region ***H***. $\frac{18}{4}$ is between 4 and 5.

11. About $4 \cdot 3 = \mathbf{12}$. The numbers cluster around 3.

12. The entire student population is represented by 1; subtract to obtain the senior's fraction; i.e., seniors make up $1 - \frac{2}{5} - \frac{1}{4} - \frac{1}{10}$ of the class. Thus $1 - \frac{2}{5} - \frac{1}{4} - \frac{1}{10} = \frac{20}{20} - \frac{8}{20} - \frac{5}{20} - \frac{2}{20} = \frac{5}{20} = \mathbf{\frac{1}{4}}$.

13. $\frac{1}{3} + 2\frac{3}{4} + 3\frac{1}{2} = \frac{4}{12} + 2\frac{9}{12} + 3\frac{6}{12} = 5\frac{19}{12} = 5 + 1\frac{7}{12} = \mathbf{6\frac{7}{12}}$ **yards**.

14. The amount of fabric to be used is $1\frac{7}{8} + 2\frac{3}{8} + 1\frac{2}{3} = 1\frac{21}{24} + 2\frac{9}{24} + 1\frac{16}{24} = 4\frac{46}{24} = 5\frac{22}{24}$ yards. The amount left over is $8\frac{3}{4} - 5\frac{22}{24} = 8\frac{18}{24} - 5\frac{22}{24} = 7\frac{42}{24} - 5\frac{22}{24} = 2\frac{20}{24} = \mathbf{2\frac{5}{6}}$ **yards**.

15. Examples may vary:

(a) Closure property: If two rational numbers are added, the sum should also be rational; e.g., $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$, which is rational.

(b) Commutative property: $\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$; e.g., $\frac{1}{2} + \frac{3}{4} = \frac{5}{4} = \frac{3}{4} + \frac{1}{2}$.

(c) Associative property: $\frac{a}{b} + \left(\frac{c}{d} + \frac{c}{f}\right) = \left(\frac{a}{b} + \frac{c}{d}\right) + \frac{c}{f}$; e.g., $\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{13}{12} = \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4}$.

16. Let the sequence be $1, a_2, a_3, a_4, a_5, a_6, 2$.

Thus $2 = 1 + (7 - 1)d \Rightarrow 6d = 1 \Rightarrow d = \frac{1}{6}$.

Then $a_2 = 1 + \frac{1}{6} = \frac{7}{6}$,

$a_3 = \frac{7}{6} + \frac{1}{6} = \frac{8}{6}$,

$a_4 = \frac{8}{6} + \frac{1}{6} = \frac{9}{6}$,

$a_5 = \frac{9}{6} + \frac{1}{6} = \frac{10}{6}$,

and $a_6 = \frac{10}{6} + \frac{1}{6} = \frac{11}{6}$.

The sequence is $\mathbf{1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}, \frac{10}{6}, \frac{11}{6}, 2}$.

17. **(a)** (*i*) $\frac{1}{4} + \frac{1}{3\cdot 4} = \frac{1}{4} + \frac{1}{12} = \frac{3}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$.

(*ii*) $\frac{1}{5} + \frac{1}{4\cdot 5} = \frac{1}{5} + \frac{1}{20} = \frac{4}{20} + \frac{1}{20} = \frac{5}{20} = \frac{1}{4}$.

(*iii*) $\frac{1}{6} + \frac{1}{5\cdot 6} = \frac{1}{6} + \frac{1}{30} = \frac{5}{30} + \frac{1}{30} = \frac{6}{30} = \frac{1}{5}$.

(b) $\mathbf{\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}}$.

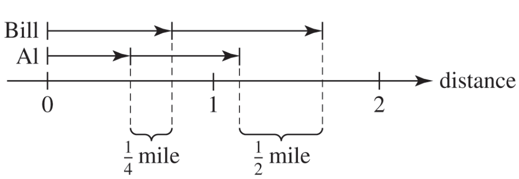
18. **(a)** $x + 2\frac{1}{2} = 3\frac{1}{3} \Rightarrow x = 3\frac{1}{3} - 2\frac{1}{2} = 2\frac{8}{6} - 2\frac{3}{6}$. $\mathbf{x = \frac{5}{6}}$.

(b) $x - 2\frac{2}{3} = \frac{5}{6} \Rightarrow x = \frac{5}{6} + \frac{16}{6}$.

$\mathbf{x = \frac{21}{6} = 3\frac{1}{2}}$.

19. In 10 minutes, the difference of the two runners' distances is $\frac{7}{8} - \frac{5}{8} = \frac{2}{8} = \frac{1}{4}$ mile. In 20 minutes the difference will be twice as large. Answer: $2\left(\frac{1}{4}\right) = \frac{2}{4} = \mathbf{\frac{1}{2}}$ **mile**.

Alternative:



20. **No**. To make both recipes, you need $1\frac{3}{4} + 1\frac{1}{2} = 1 + \frac{3}{4} + 1 + \frac{1}{2} = 2 + \frac{3}{4} + \frac{1}{2} = 2 + 1\frac{1}{4} = 3\frac{1}{4}$ cups. $3\frac{1}{4} - 3 = \frac{1}{4}$ cups less than what is needed.

21. Answers vary.

(a) $\frac{76,432 + 76,503 + 78,672}{150,155 + 149,904 + 154,144} = \frac{231,607}{454,203} \approx \frac{23}{45}$

(b) $\frac{73,723 + 73,401 + 75,472}{150,155 + 149,904 + 154,144} = \frac{222,596}{454,203} \approx \frac{22}{45}$

(c) $\frac{231,607}{12,742,886} \approx \frac{2}{127}$

22. $\frac{633,000}{1,193,000} \approx \frac{63}{119}$ and $\frac{1,186,000}{1,259,000} \approx \frac{118}{125}$. Since $\frac{118}{125} > \frac{63}{119}$ there are more 16- and 17-year olds in school.

23. $40^\circ + 5^\circ = 45^\circ$ and $\frac{45^\circ}{360^\circ} = \frac{1}{8}$ so rent and saving represent $\frac{1}{8}$ of the total.

24. $\frac{37}{100} + \frac{3}{10} + \frac{19}{100} + \frac{7}{100} = \frac{37}{100} + \frac{30}{100} + \frac{19}{100} + \frac{7}{100} = \frac{93}{100}$ so $1 - \frac{93}{100} = \frac{7}{100}$ is unaccounted for.

Assessment 6-2B

1. Various methods may be used:

(a) $\frac{^-1}{2} + \frac{2}{3} \Rightarrow \text{LCD}(2,3) = 6 \Rightarrow \frac{^-1}{2}\cdot\frac{3}{3} + \frac{2}{3}\cdot\frac{2}{2} = \frac{^-3}{6} + \frac{4}{6} = \mathbf{\frac{1}{6}}$.

(b) $\frac{5}{12} - \frac{2}{3} = \frac{5\cdot 3 - 2\cdot 12}{12\cdot 3} = \frac{15 - 24}{36} = \frac{^-9}{36} = \mathbf{\frac{^-1}{4}}$.

(c) $\frac{5}{4x} + \frac{^-3}{2y} = \frac{5\cdot 2y + {^-3}\cdot 4x}{4x\cdot 2y} = \frac{10y - 12x}{8xy} = \mathbf{\frac{5y - 6x}{4xy}}$.

(d) $\frac{^{-}3}{2x^2y^2} + \frac{5}{2xy^2} + \frac{7}{x^2y} \Rightarrow$
$\text{LCD}(2x^2y^2, 2xy^2, x^2y) =$
$2x^2y^2 \Rightarrow \frac{^{-}3}{2x^2y^2} + \frac{5}{2xy^2} \cdot \frac{x}{x} +$
$\frac{7}{x^2y} \cdot \frac{2y}{2y} = \mathbf{\frac{^{-}3+5x+14y}{2x^2y^2}}.$

(e) $\frac{5}{6} - 2\frac{1}{8} = \frac{5}{6} - \frac{17}{8} = \frac{5\cdot 8 - 6\cdot 17}{6\cdot 8} =$
$\frac{40-102}{48} = \frac{^{-}62}{48} = \mathbf{\frac{^{-}31}{24}} = \mathbf{^{-}1\frac{7}{24}}.$

(f) $^{-}4\frac{1}{2} + 3\frac{1}{6} \Rightarrow \text{LCD}(2,6) = 6 \Rightarrow$
$^{-}4\frac{1}{2} \cdot \frac{3}{3} + 3\frac{1}{6} = {^{-}4}\frac{3}{6} + 3\frac{1}{6} =$
$^{-}1\frac{2}{6} = \mathbf{^{-}1\frac{1}{3}} = \mathbf{\frac{^{-}4}{3}}.$

(g) $5\frac{1}{3}+5\frac{5}{6}-3\frac{1}{9}=\frac{16}{3}+\frac{35}{6}-\frac{28}{9}=$
$\frac{96}{18}+\frac{105}{18}-\frac{56}{18}=\frac{201}{18}-\frac{56}{18}=\frac{145}{18}=8\frac{1}{18}$

2. (a) $14 \div 5 = 2$, remainder $4 \Rightarrow \frac{14}{5} = \mathbf{2\frac{4}{5}}.$

(b) $^{-}47 \div 8 = {^{-}5}$, remainder $7 \Rightarrow -\frac{47}{8} =$
$\mathbf{^{-}5\frac{7}{8}}.$

3. (a) $7\frac{1}{2} = \frac{7}{1} + \frac{1}{2} = \frac{7\cdot 2 + 1\cdot 1}{1\cdot 2} = \frac{14+1}{2} = \mathbf{\frac{15}{2}}.$

(b) $^{-}4\frac{2}{3} = {^{-}}\left(\frac{4}{1} + \frac{2}{3}\right) = {^{-}}\left(\frac{4\cdot 3 + 1\cdot 2}{1\cdot 3}\right) =$
$^{-}\left(\frac{12+2}{3}\right) = \mathbf{\frac{^{-}14}{3}}.$

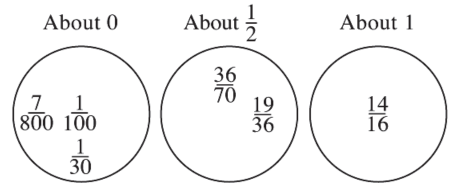
4. $\frac{\boxed{2}}{\boxed{6}} + \frac{\boxed{5}}{\boxed{8}} = \frac{23}{24}$. A denominator of 24 in the sum eliminates 5 as a denominator in either of the addends; 2 in either would give an answer > 1; trial and error yields 6 paired with 2 and 8 paired with 5.

5. (a) **Tigers**. $\frac{9}{28} < \frac{1}{3}$ by the least margin.

(b) **Lions**. $\frac{7}{27} > \frac{1}{4}$ by the least margin.

(c) **Wildcats** and **Badgers**. $\frac{6}{25}$ and $\frac{5}{21} < \frac{1}{4}$.

6.



7. (a) **1; low**. $\frac{113}{100} > 1.$

(b) **0; low**. $\frac{3}{1978} > 0.$

(c) $\mathbf{\frac{3}{4}}$**; low**. $\frac{150}{198} > \frac{150}{200} = \frac{3}{4}.$

(d) **1; high**. $\frac{8}{9} < 1.$

8. (a) The first addend is about 0, the next two addends are about $\frac{1}{2}$, and the third is about 1. **2** best approximates the sum.

(b) The third addend is about 0 and the second is close to one-fourth. The first is estimated at about three-forths. **1** is the best approximation.

9. Possible thought processes could be:

(a) $(5 + 1) - \frac{7}{8} = 5 + \left(\frac{8}{8} - \frac{7}{8}\right) = \mathbf{5\frac{1}{8}}.$

(b) $\left(2 + 4 + 3 + \frac{6}{10} + \frac{1}{10} + \frac{3}{10}\right) =$
$9 + \frac{10}{10} = \mathbf{10}.$

10. (a) Region ***M***. $\frac{9}{8} = 1\frac{1}{8}$ is between 1 and 2.

(b) Region ***A***. $\frac{18}{8} = \frac{9}{4} = 2\frac{1}{4}$ is between 2 and 3.

(c) Region ***T***. $\frac{50}{16} = \frac{25}{8} = 3\frac{1}{8}$ is between 3 and 4.

(d) Region ***H***. $\frac{17}{4} = 4\frac{1}{4}$ is between 4 and 5.

11. The entire student population is represented by 1; subtract to obtain the senior's fraction; i.e., seniors make up $1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{10}$ of the class. Thus
$1 - \frac{1}{4} - \frac{1}{5} - \frac{1}{10} = \frac{20}{20} - \frac{5}{20} - \frac{4}{20} - \frac{2}{20} = \mathbf{\frac{9}{20}}.$

12. **(a)** $\frac{1}{5}$ (Japan) $-\frac{1}{6}$ (Canada) $=\frac{6}{30}-\frac{5}{30}=\mathbf{\frac{1}{30}}$.

(b) $\frac{7}{20}$ (United States) $-\frac{1}{4}$ (England) $=$ $\frac{7}{20}-\frac{5}{20}=\frac{2}{20}=\mathbf{\frac{1}{10}}$.

(c) $\frac{7}{20}(2012)-\frac{1}{3}(2009)=\frac{21}{60}-\frac{20}{60}=\mathbf{\frac{1}{60}}$.

(d) **No**. The total number of dollars might have been greater in 2012 than in 2014, but the fraction of the total dollars $\left(\text{i.e., } \frac{1}{10} \text{ versus } \frac{1}{20}\right)$ was greater in 2014.

13. $3\frac{1}{2}-1\frac{3}{4}=3\frac{2}{4}-1\frac{3}{4}=\left(2\frac{2}{4}+\frac{4}{4}\right)-1\frac{3}{4}$
$=2\frac{6}{4}-1\frac{3}{4}=\mathbf{1\frac{3}{4}}$ **cups**.

14. $38\frac{1}{4}-\left(15\frac{3}{4}+\frac{3}{8}\right)=38\frac{2}{8}-\left(15\frac{6}{8}+\frac{3}{8}\right)=$
$38\frac{2}{8}-16\frac{1}{8}=\mathbf{22\frac{1}{8}}$ **inches**.

15. **(a)** Team **4** collected $35\frac{3}{16}+41\frac{1}{2}=$ $35\frac{3}{16}+41\frac{8}{16}=\mathbf{76\frac{11}{16}}$ **pounds**.

(b) Collections in April were $28\frac{3}{4}+32\frac{7}{8}+28\frac{1}{2}+35\frac{3}{16}=28\frac{12}{16}+32\frac{14}{16}+28\frac{8}{16}+35\frac{3}{16}=123\frac{37}{16}=125\frac{5}{16}$ pounds.

Collections in May were $33\frac{1}{3}+28\frac{5}{12}+25\frac{3}{4}+41\frac{1}{2}=33\frac{4}{12}+28\frac{5}{12}+25\frac{9}{12}+41\frac{6}{12}=127\frac{24}{12}=129$ pounds.

The difference was $129-125\frac{5}{16}=$ $128\frac{16}{16}-125\frac{5}{16}=\mathbf{3\frac{11}{16}}$ **pounds**.

16. Let the sequence be $1, a_2, a_3, a_4, a_5, 3$. Thus $3=1+(6-1)d \Rightarrow 5d=2 \Rightarrow d=\frac{2}{5}$. Then

$a_2=1+\frac{2}{5}=\frac{7}{5}$,

$a_3=\frac{7}{5}+\frac{2}{5}=\frac{9}{5}$,

$a_4=\frac{9}{5}+\frac{2}{5}=\frac{11}{5}$, and

$a_5=\frac{11}{5}+\frac{2}{5}=\frac{13}{5}$.

The sequence is $\mathbf{1,\frac{7}{5},\frac{9}{5},\frac{11}{5},\frac{13}{5},3}$.

17. **(a)** $x-\frac{5}{6}=\frac{2}{3}\Rightarrow x=\frac{5}{6}+\frac{2}{3}=\frac{5}{6}+\frac{4}{6}=\frac{9}{6}=\frac{3}{2}$. $\boldsymbol{x=1\frac{1}{2}}$.

(b) $x-\frac{7}{2^3\cdot 3^2}=\frac{5}{2^2\cdot 3^2}\Rightarrow x=\frac{5}{2^2\cdot 3^2}+\frac{7}{2^3\cdot 3^2}=\frac{5}{2^2\cdot 3^2}\cdot\frac{2}{2}+\frac{7}{2^3\cdot 3^2}=\frac{10+7}{2^3\cdot 3^2}\Rightarrow \boldsymbol{x=\frac{17}{2^3\cdot 3^2}=\frac{17}{72}}$.

18. Write the equivalent operations of using (take away) paint and adding paint:

$$\frac{3}{4}-\frac{1}{3}+\frac{1}{2}=\frac{9}{12}-\frac{4}{12}+\frac{6}{12}=\mathbf{\frac{11}{12}}\ \textbf{cup}.$$

19. **(a)** $\frac{1}{2}+\frac{1}{4}=\frac{2}{4}+\frac{1}{4}=\mathbf{\frac{3}{4}}$.

(b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{3}{4}+\frac{1}{8}=\frac{6}{8}+\frac{1}{8}=\mathbf{\frac{7}{8}}$.

(c) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{7}{8}+\frac{1}{16}=\frac{14}{16}+\frac{1}{16}=\mathbf{\frac{15}{16}}$.

(d) **No**. Notice that in each of the sums in a,b, and c, the numerator is one less than the denominator. In fact, $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$.

Alternative: Note that each new sum in the pattern is created by adding a value that is $\frac{1}{2}$ the distance between the previous sum and 1. For example, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$. $1-\frac{15}{16}=\frac{1}{16}$, so this sum is $\frac{1}{16}$ units from 1. The next sum in the pattern is $1+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$. Since $\frac{1}{32}$ is $\frac{1}{2}$ of $\frac{1}{16}$, the new sum is less than 1.

20. Answers vary.

(a) $\frac{73{,}723}{150{,}155}\approx\frac{74}{150}\approx\frac{148}{300}$ and $\frac{75{,}472}{154{,}144}\approx\frac{75}{154}\approx\frac{150}{300}$ so approximately $\frac{2}{300}$ or $\frac{1}{150}$ more.

(b) $\frac{73,723+73,401}{150,155+149,904}=\frac{147,124}{300,059}\approx\frac{147}{300}\approx\frac{1}{2}$

(c)
$\frac{73,723+73,401+75,472}{12,742,886}=\frac{222,596}{12,742,886}\approx\frac{2}{127}$.

21. $\frac{633,000}{1,193,000}\approx\frac{63}{119}$ and $\frac{1,186,000}{1,259,000}\approx\frac{118}{125}$. Since $\frac{63}{119}<\frac{118}{125}$ there are fewer 3- and 4-year olds in school.

22. $50^\circ+7^\circ=57^\circ$ and $\frac{360^\circ}{360^\circ}-\frac{57^\circ}{360^\circ}=\frac{303}{360}=\frac{101}{120}$ so the fractional part not represented by rent and saving is $\frac{101}{120}$ of the total.

23. $\frac{37}{100}=\frac{3}{10}+x\Rightarrow\frac{37}{100}=\frac{30}{100}+\frac{7}{100}$ so $\frac{7}{100}$ more coal than natural gas is used.

Mathematical Connections 6-2: Review Problems

15. **(a)** $\frac{14}{21}=\frac{7\cdot 2}{7\cdot 3}=\mathbf{\frac{2}{3}}$.

(b) $\frac{117}{153}=\frac{9\cdot 13}{9\cdot 17}=\mathbf{\frac{13}{17}}$.

(c) $\frac{5^2}{7^2}=\mathbf{\frac{25}{49}}$, which cannot be further simplified.

(d) $\frac{a^2+a}{1+a}=\frac{a(a+1)}{a+1}=\frac{a}{1}=\boldsymbol{a}$.

(e) $\frac{a^2+1}{a+1}$ **cannot be further simplified**. There are no factors common to both the numerator and denominator.

(f) $\frac{a^2-b^2}{a-b}=\frac{(a+b)(a-b)}{a-b}=\boldsymbol{a+b}$ (if $a\neq b$).

16. **(a)** **Equal**. $\frac{a^2b^2}{b^3}=\frac{a^2\cdot b^2}{b\cdot b^2}=\frac{a^2}{b}$ $(b\neq 0)$.

(b) **Not equal**. $377\cdot 401=151,177\neq 400\cdot 378=151,200$.

(c) **Equal**. $\frac{0}{10}=0=\frac{0}{^-10}$.

(d) **Not equal**. $a(b+1)=ab+a\neq b(a+1)=ab+b$.

17. Yes. $\frac{27}{103}=\frac{54}{206}$ so there could be 54 bones in both hands or 27 bones in each hand.

18. There are infinitely many fractions equivalent to $\frac{3}{5}$ because the equivalent fractions can be written in the form $\frac{3n}{5n}$ with n an integer.Since the set of integers is infinite, the set of fractions equivalent to $\frac{3}{5}$ is infinite.

19. There are $1-\frac{3}{5}=\frac{5}{5}-\frac{3}{5}=\frac{2}{5}$ of the cookies left for Suzanne.

20. $\frac{-1}{10}=\frac{^-10}{100}$ and $\frac{^-10}{100}$ is $\frac{10}{100}^{th}$ away from 0 and $\frac{-1}{100}$ is $\frac{1}{100}^{th}$ away from 0 so $\frac{^-1}{100}$ is greater than $\frac{^-1}{10}$.

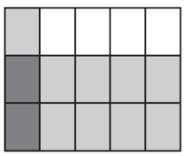
Assessment 6-3A: Multiplication, Division, and Estimation with Rational Numbers

1. **(a)** The blue-shaded vertical region represents $\frac{1}{3}$ of the total area; the yellow-shaded horizontal region represents $\frac{1}{4}$ of the total area. The blue-yellow region therefore represents $\frac{1}{4}$ of $\frac{1}{3}$, or the product of the two fractions. Since one of the twelve blocks is blue-yellow it represents $\mathbf{\frac{1}{4}\cdot\frac{1}{3}=\frac{1}{12}}$.

(b) The blue-shaded vertical region represents $\frac{3}{5}$ of the total area; the yellow-shaded horizontal region represents $\frac{2}{4}$ of the total area. The blue-yellow region therefore represents $\frac{2}{4}$ of $\frac{3}{5}$, or the product of the two fractions. Since six of the twenty blocks are blue-yellow it represents $\mathbf{\frac{2}{4}\cdot\frac{3}{5}=\frac{6}{20}}$.

2. **(a)** There are 12 boxes in the figure below. The dark-shaded region represents $\frac{9}{12}$ of $\frac{4}{12}$, or $\frac{3}{4} \cdot \frac{1}{3} = \frac{3}{12} = \frac{1}{4}$:

(b) There are 15 boxes in the figure below. The dark-shaded region represents $\frac{3}{15}$ of $\frac{10}{15}$, or $\frac{1}{5} \cdot \frac{2}{3} = \mathbf{\frac{2}{15}}$:



3. **(a)** $\frac{49}{65} \cdot \frac{26}{98} = \frac{1274}{6370} = \frac{1 \cdot 1274}{5 \cdot 1274} = \mathbf{\frac{1}{5}}$.

(b) $\frac{a}{b} \cdot \frac{b^2}{a^2} = \frac{ab^2}{a^2 b} = \frac{b \cdot ab}{a \cdot ab} = \boldsymbol{\frac{b}{a}}$.

(c) $\frac{xy}{z} \cdot \frac{z^2 a}{x^3 y^2} = \frac{axyz^2}{x^3 y^2 z} = \frac{az \cdot xyz}{x^2 y \cdot xyz} = \boldsymbol{\frac{az}{x^2 y}}$.

4. **(a)**
$$\begin{aligned} 4\tfrac{1}{2} \cdot 2\tfrac{1}{3} &= \left(4 + \tfrac{1}{2}\right) \cdot \left(2 + \tfrac{1}{3}\right) \\ &= 4\left(2 + \tfrac{1}{3}\right) + \tfrac{1}{2}\left(2 + \tfrac{1}{3}\right) \\ &= 8 + \tfrac{4}{3} + 1 + \tfrac{1}{6} \\ &= 9 + \tfrac{8}{6} + \tfrac{1}{6} = \mathbf{10\tfrac{1}{2}}. \end{aligned}$$

(b)
$$\begin{aligned} 3\tfrac{1}{3} \cdot 2\tfrac{1}{2} &= \left(3 + \tfrac{1}{3}\right) \cdot \left(2 + \tfrac{1}{2}\right) \\ &= 3\left(2 + \tfrac{1}{2}\right) + \tfrac{1}{3}\left(2 + \tfrac{1}{2}\right) \\ &= 6 + \tfrac{3}{2} + \tfrac{2}{3} + \tfrac{1}{6} \\ &= 6 + \tfrac{9}{6} + \tfrac{4}{6} + \tfrac{1}{6} = \mathbf{8\tfrac{1}{3}}. \end{aligned}$$

5. **(a)** Multiplicative inverse of $\frac{^{-}1}{3}$ is $\frac{3}{^{-}1} = \mathbf{^{-}3}$.

(b) Multiplicative inverse of $3\frac{1}{3} = \frac{10}{3}$ is $\mathbf{\frac{3}{10}}$.

(c) Multiplicative inverse of $\frac{x}{y}$ is $\boldsymbol{\frac{y}{x}}\,(x, y \neq 0)$.

(d) Multiplicative inverse of $^{-}7 = \frac{^{-}7}{1}$ is $\frac{1}{^{-}7} = \mathbf{\frac{^{-}1}{7}}$.

6. **(a)** $\frac{2}{3}x = \frac{7}{6} \Rightarrow x = \frac{7}{6} \div \frac{2}{3} = \frac{7}{6} \cdot \frac{3}{2} \Rightarrow \boldsymbol{x = \frac{7}{4}}$.

(b) $\frac{3}{4} \div x = \frac{1}{2} \Rightarrow \frac{3 \cdot 1}{4 \cdot x} = \frac{1}{2} \Rightarrow 3 \cdot 1 \cdot 2 = 4 \cdot x \cdot 1 \Rightarrow 4x = 6 \Rightarrow \boldsymbol{x = \frac{6}{4} = \frac{3}{2}}$.

(c) $\frac{5}{6} + \frac{2}{3}x = \frac{3}{4} \Rightarrow \frac{2}{3}x = \frac{3}{4} - \frac{5}{6} = \frac{^{-}1}{12} \Rightarrow x = \frac{^{-}1}{12} \div \frac{2}{3} = \frac{^{-}1}{12} \cdot \frac{3}{2} \Rightarrow \boldsymbol{x = \frac{^{-}1}{8}}$.

(d) $\frac{2x}{3} - \frac{1}{4} = \frac{x}{6} + \frac{1}{2} \Rightarrow \frac{2}{3}x - \frac{1}{6}x = \frac{1}{2} + \frac{1}{4} \Rightarrow \frac{1}{2}x = \frac{3}{4} \Rightarrow x = \frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \cdot \frac{2}{1} \Rightarrow \boldsymbol{x = \frac{3}{2}}$.

7. Answers may vary; e.g.,

(a) $\frac{1}{2} \div \frac{1}{4} \neq \frac{1}{4} \div \frac{1}{2}$.

(b) $\left(\frac{2}{3} \div \frac{1}{2}\right) \div \frac{3}{5} \neq \frac{2}{3} \div \left(\frac{1}{2} \div \frac{3}{5}\right)$

8. Possible thought processes are described; all provide exact answers:

(a) $3 \cdot 8 = 24, \frac{1}{4} \cdot 8 = 2$, and $24 + 2 = \mathbf{26}$.

(b) $7 \cdot 4 = 28, \frac{1}{4} \cdot 4 = 1$, and $28 + 1 = \mathbf{29}$.

(c) $9 \cdot 10 = 90, \frac{1}{5} \cdot 10 = 2$, and $90 + 2 = \mathbf{92}$.

(d) $8 \cdot 2 = 16, 8 \cdot \frac{1}{4} = 2$, and $16 + 2 = \mathbf{18}$.

9. **(a)** **20**. $3\frac{11}{12} \cdot 5\frac{3}{100} \approx 4 \cdot 5 = 20$.

(b) **16**. $2\frac{1}{10} \cdot 7\frac{7}{8} \approx 2 \cdot 8 = 16$.

(c) **1**. $\frac{1}{101}$ and $\frac{1}{103}$ are approximately equal.

10. **(a)** **Less than 1**. $\frac{13}{14} \cdot \frac{17}{19}$ is the product of two proper fractions; i.e., each is less than 1. The product of two positive proper fractions will be smaller than either, and therefore will always be less than 1.

(b) **Less than 1**. $3\frac{2}{7} \div 5\frac{1}{9}$ is a positive number divided by a larger positive number. Their quotient, a proper fraction, would always be less than 1.

(c) **Greater than 2**. If the quotient was 2, then checking would show that $2\left(2\frac{3}{100}\right) = 4\frac{6}{100}$, which is less than $4\frac{1}{3}$. Thus the true quotient must be greater than 2.

11. **(d)**, between \$33 and \$40. Estimation gives a cost of approximately
$6 \cdot \$4.00 + 3 \cdot \$3.00 = \$24.00 + \$9.00 = \$33.00$
All rounding was down, so the estimate is low.

12. Let p be the student population. The 6000 students living in dorms are $\frac{5}{8}$ of p; i.e., $6000 = \frac{5}{8}p \Rightarrow p = \frac{8}{5}(6000) = $ **9600 students**.

13. Alberto has $\frac{5}{9}$ of the stock; Renatta has $\frac{1}{2} \cdot \frac{5}{9} = \frac{5}{18}$ of the stock. Thus $1 - \frac{5}{9} - \frac{5}{18} = \mathbf{\frac{1}{6}}$ **of the stock** is not owned by them.

14. Let p be the original price. Then $p - \frac{1}{4}p = 180 \Rightarrow \frac{3}{4}p = 180 \Rightarrow p = \frac{4}{3}(180) = \mathbf{\$240}$.

15. Let a be the amount of money in the account. After spending \$50 there was $a - 50$ remaining. He spent $\frac{3}{5}$ of that, or $\frac{3}{5}(a - 50)$, leaving him $\frac{2}{5}(a - 50)$; half goes back into the bank, or $\frac{1}{2} \cdot \frac{2}{5}(a - 50) = \frac{1}{5}(a - 50)$. The other half was \$35; or $\frac{1}{5}(a - 50) = 35 \Rightarrow \frac{1}{5}a - 10 = 35 \Rightarrow \frac{1}{5}a = 45 \Rightarrow a = 5 \cdot 45 = \mathbf{\$225}$.

16. Al's marbles are halved three times in the process of Dani receiving 4 marbles, i.e., if m is the number of marbles Al had originally, then $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot m = 4 \Rightarrow \frac{1}{8}M = 4 \Rightarrow M = 8 \cdot 4 = $ **32 marbles**.

17. If a is any number and m and n are natural numbers, then $a^m \cdot a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, and $a^{-m} = \frac{1}{a^m}$:

(a) $3^{^-7} \cdot 3^{^-6} = 3^{^-7+^-6} = 3^{^-13} = \frac{1}{3^{13}} = \mathbf{\left(\frac{1}{3}\right)^{13}}$.

(b) $3^7 \cdot 3^6 = 3^{7+6} = \mathbf{3^{13}}$.

(c) $5^{15} \div 5^4 = 5^{15-4} = \mathbf{5^{11}}$.

(d) $5^{15} \div 5^{^-4} = 5^{15-^-4} = 5^{15+4} = \mathbf{5^{19}}$.

(e) $(^-5)^{^-2} = \frac{1}{(^-5)^2} = \mathbf{\frac{1}{5^2}} = \mathbf{\left(\frac{1}{5}\right)^2}$.

(f) $\frac{a^2}{a^{^-3}} = a^{2-^-3} = a^{2+3} = \boldsymbol{a^5}$.

18. **(a)** $\left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^{3+7} = \mathbf{\left(\frac{1}{2}\right)^{10}}$.

(b) $\left(\frac{1}{2}\right)^9 \div \left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^{9-6} = \mathbf{\left(\frac{1}{2}\right)^{3}}$.

(c) $\left(\frac{2}{3}\right)^5 \cdot \left(\frac{4}{9}\right)^2 = \left(\frac{2}{3}\right)^5 \cdot \left[\left(\frac{2}{3}\right)^2\right]^2 = \left(\frac{2}{3}\right)^{5+4} = \mathbf{\left(\frac{2}{3}\right)^{9}}$.

(d) $\left(\frac{3}{5}\right)^7 \div \left(\frac{3}{5}\right)^7 = \left(\frac{3}{5}\right)^{7-7} = \left(\frac{3}{5}\right)^0 = \mathbf{1}$.

19. Counterexamples may vary:

(a) **False**. $2^3 \cdot 3^2 = 8 \cdot 9 = 72 \neq (2 \cdot 3)^{3+2} = 6^5 = 7776$.

(b) **False**. $2^3 \cdot 3^2 = 72 \neq (2 \cdot 3)^{2 \cdot 3} = 6^6 = 46{,}656$.

(c) **False**. $2^3 \cdot 3^3 = (2 \cdot 3)^3 = 216 \neq (2 \cdot 3)^{2 \cdot 3} = 46{,}656$.

(d) **False**. Any number (except 0) to the 0 power $= 1$; 0^0 is undefined.

(e) **False**. $(2 + 3)^2 = (2 + 3)(2 + 3) = 25 \neq 2^2 + 3^2 = 13$.

(f) **False**. $(2 + 3)^{^-2} = \frac{1}{(2+3)^2} = \frac{1}{25} \neq \frac{1}{2^2} + \frac{1}{3^2} = \frac{13}{36}$.

20. **(a)** $2^n = 32 \Rightarrow 2^n = 2^5 \Rightarrow \boldsymbol{n = 5}$.

(b) $n^2 = 36 \Rightarrow n^2 = (\pm 6)^2 \Rightarrow \boldsymbol{n = 6}$ or $\mathbf{^-6}$.

(c) $2^n \cdot 2^7 = 2^5 \Rightarrow 2^{n+7} = 2^5 \Rightarrow n + 7 = 5 \Rightarrow \boldsymbol{n = {}^-2}$.

(d) $2^n \cdot 2^7 = 8 \Rightarrow 2^{n+7} = 2^3 \Rightarrow n + 7 = 3 \Rightarrow \boldsymbol{n = {}^-4}$.

21. **(a)** $3^x \le 9 \Rightarrow 3^x \le 3^2 \Rightarrow \boldsymbol{x \le 2}$, where x is an integer.

(b) $25^x < 125 \Rightarrow (5^2)^x < 5^3 \Rightarrow$
$5^{2x} < 5^3 \Rightarrow 2x < 3 \Rightarrow \boldsymbol{x < \frac{3}{2}}$,
$\Rightarrow x < 2$
where x is an integer.

(c) $3^{2x} > 27 \Rightarrow 3^{2x} > 3^3 \Rightarrow 2x > 3 \Rightarrow$
$\boldsymbol{x > \frac{3}{2}} \Rightarrow x \ge 2$ where x is an integer.

(d) $4^x > 1 \Rightarrow 4^x > 4^0$
$\Rightarrow x > 0 \Rightarrow x \ge 1$, where x is an integer.

22. **(a)** $\left(\boldsymbol{\frac{1}{2}}\right)^{\boldsymbol{3}} > \left(\frac{1}{2}\right)^4$. $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ while $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$; $\frac{1}{8}$ is greater.

(b) $\left(\frac{3}{4}\right)^8 > \left(\frac{3}{4}\right)^{10}$. $\left(\frac{3}{4}\right)^8 \div \left(\frac{3}{4}\right)^{10} = \left(\frac{3}{4}\right)^{^-2} = \left(\frac{4}{3}\right)^2$, which is greater than $1 \Rightarrow$ an improper fraction $\Rightarrow \left(\frac{3}{4}\right)^8$ is greater.

(c) $\left(\boldsymbol{\frac{4}{3}}\right)^{\boldsymbol{10}} > \left(\frac{4}{3}\right)^8$. $\left(\frac{4}{3}\right)^{10} \div \left(\frac{4}{3}\right)^8 = \left(\frac{4}{3}\right)^2$, which is greater than $1 \Rightarrow \left(\frac{4}{3}\right)^{10}$ is greater.

(d) $\left(\boldsymbol{\frac{4}{5}}\right)^{\boldsymbol{10}} > \left(\frac{3}{4}\right)^{10}$. $\frac{4}{5} \div \frac{3}{4} > 1 \Rightarrow \frac{4}{5} > \frac{3}{4} \Rightarrow$
$\left(\frac{4}{5}\right)^{10} > \left(\frac{3}{4}\right)^{10}$.

(e) $32^{50} > 4^{100} \Rightarrow 32^{50} = (2^5)^{50} = 2^{250}$
$\Rightarrow 4^{100} = (2^2)^{100} = 2^{200}$
$2^{250} > 2^{200}$
$(^-3)^{^-75} > (^-27)^{^-15}$

(f) $(^-27)^{^-15} = (^-3^3)^{^-15} = (^-3)^{^-45}$
$(^-3)^{^-75} > (^-3)^{^-45}$

23. If either $\frac{a}{b}$ and $\frac{c}{d}$ is negative, it can be written with the negative signs associated with the numerators. Thus, $\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc$. Adding ad to both sides of the inequalities results in $2ad < ad + bc$. Adding bc to both sides of the inequalities results in $ad + bc < 2bc$. Dividing both sides of the inequalities by $2bd$ gives us $\frac{a}{b} < \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) < \frac{c}{d}$.

24. **(a)** $1 - \frac{17}{25} = \frac{25}{25} - \frac{17}{25} = \frac{8}{25}$. There were $\frac{8}{25}$ of the students who were female.
(b) No, the number of male students depends on the total number of students. If there were 25 students total then there would be 17 male students.

25. **(a)** The restaurant tax minus the retail tax is approximately
$120,000,000 - 60,000,000 = 60,000,000$ which is approximately $\frac{60,000,000}{670,000,000} \approx \frac{1}{11}$ of the total tax.
(b) The sales taxes add up to 187,656,061 ≈ 190,000,000. So $\frac{190,000,000}{680,000,000} \approx \frac{2}{7}$ would be lost.

26. 2004 -> 2005: -2
2005 -> 2006: -27
2006 -> 2007: +12
2007 -> 2008: +5
2008 -> 2009: -42
2009 -> 2010: -12
2010 -> 2011: -24
2011 -> 2012: -20
2012 -> 2013: +16

(a) The greatest fractional increase from the previous year happened from 2012 to 2013.
(b) The greatest fractional decrease from the previous year happened from 2008 to 2009.

27. $\frac{1}{33} + \frac{2}{7} = \frac{7}{231} + \frac{66}{231} = \frac{73}{231}$ is the fractional part of the deer population that is in Mississippi and in Montana.

28. $\frac{1}{4} \div 8 = \frac{1}{32}$ of the entire estate is what each cousin receives.

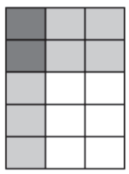
Assessment 6-3B

1. **(a)**

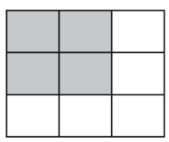
The figure above represents $\frac{2}{3}$. Divide this figure horizontally into two equal parts. This divides $\frac{2}{3}$ into 4 pieces of 6. Only 2 of the 6 have both shadings. This represents $\boldsymbol{\frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6}}$.

(b) The figure shows $\frac{1}{3}$ divided into two equal parts. $\mathbf{\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}}$.

2. **(a)** There are 15 boxes in the figure below. The dark-shaded region represents $\frac{6}{15}$ of $\frac{5}{15}$, or $\frac{2}{5} \cdot \frac{1}{3} = \mathbf{\frac{2}{15}}$:



(b) There are 9 boxes in the figure below. The dark-shaded region represents $\frac{6}{9}$ of $\frac{6}{9}$, or $\frac{2}{3} \cdot \frac{2}{3} = \mathbf{\frac{4}{9}}$.



3. **(a)** $2\frac{1}{3} \cdot 3\frac{3}{4} = \frac{7}{3} \cdot \frac{15}{4} = \frac{105}{12} = \frac{35 \cdot 3}{4 \cdot 3} = \frac{35}{4} = \mathbf{8\frac{3}{4}}$.

(b) $\frac{22}{7} \cdot 4\frac{2}{3} = \frac{22}{7} \cdot \frac{14}{3} = \frac{308}{21} = \frac{44 \cdot 7}{3 \cdot 7} = \frac{44}{3} = \mathbf{14\frac{2}{3}}$.

(c) $\frac{{}^{-}5}{2} \cdot 2\frac{1}{2} = \frac{{}^{-}5}{2} \cdot \frac{5}{2} = \frac{{}^{-}25}{4} = \mathbf{{}^{-}6\frac{1}{4}}$.

(d) $2\frac{3}{4} \cdot 2\frac{1}{3} = \frac{11}{4} \cdot \frac{7}{3} = \frac{77}{12} = \mathbf{6\frac{5}{12}}$.

(e) $\frac{a^2}{b^3} \cdot \frac{b^2}{a^3} = \frac{a^2b^2}{a^3b^3} = \frac{a^2b^2 \cdot 1}{a^2b^2 \cdot ab} = \mathbf{\frac{1}{ab}}$.

(f) $\frac{x^3y^2}{z} \cdot \frac{z}{x^2y} = \frac{x^3y^2z}{x^2yz} = \frac{x^2yzxy}{x^2yz} = \boldsymbol{xy}$.

4. **(a)** $2\frac{1}{3} \cdot 4\frac{3}{5} = \left(2 + \frac{1}{3}\right) \cdot \left(4 + \frac{3}{5}\right) = 2\left(4 + \frac{3}{5}\right) + \frac{1}{3}\left(4 + \frac{3}{5}\right) = 8 + \frac{6}{5} + \frac{4}{3} + \frac{3}{15} = \frac{120+18+20+3}{15} = \frac{161}{15} = \mathbf{10\frac{11}{15}}$.

(b) $\left(\frac{x}{y} + 1\right)\left(\frac{y}{x} - 1\right) = \frac{x}{y}\left(\frac{y}{x} - 1\right) + 1\left(\frac{y}{x} - 1\right) = 1 - \frac{x}{y} + \frac{y}{x} - 1 = \boldsymbol{\frac{y^2 - x^2}{xy}}$.

(c) $248\frac{2}{5} \cdot 100\frac{1}{8} = \left(248 + \frac{2}{5}\right) \cdot \left(100 + \frac{1}{8}\right) = 24{,}800 + 31 + 40 + \frac{1}{20} = \mathbf{24{,}871\frac{1}{20}}$.

5. **(a)** $\mathbf{\frac{7}{6}}$. $\frac{6}{7} \cdot \frac{7}{6} = 1$.

(b) $\mathbf{\frac{1}{8}}$. $8 \cdot \frac{1}{8} = 1$.

(c) $4\frac{1}{5} = \frac{21}{5}$. $\frac{21}{5} \cdot \mathbf{\frac{5}{21}} = 1$.

(d) ${}^{-}1\frac{1}{2} = \frac{{}^{-}3}{2}$. $\frac{{}^{-}3}{2} \cdot \left(\mathbf{\frac{{}^{-}2}{3}}\right) = 1$.

6. **(a)** $\frac{2}{3}x = \frac{11}{6} \Rightarrow x = \frac{11}{6} \div \frac{2}{3} = \frac{11}{6} \cdot \frac{3}{2} \Rightarrow x = \frac{33}{12} = \mathbf{\frac{11}{4}} = \mathbf{2\frac{3}{4}}$.

(b) $\frac{3}{4} \div x = \frac{1}{3} \Rightarrow \frac{3 \cdot 1}{4 \cdot x} = \frac{1}{3} \Rightarrow 3 \cdot 1 \cdot 3 = 4 \cdot x \cdot 1 \Rightarrow 4x = 9 \Rightarrow \boldsymbol{x = \frac{9}{4} = 2\frac{1}{4}}$.

(c) $\frac{5}{6} - \frac{2}{3}x = \frac{3}{4} \Rightarrow \frac{2}{3}x = \frac{5}{6} - \frac{3}{4} = \frac{1}{12} \Rightarrow x = \frac{1}{12} \div \frac{2}{3} = \frac{1}{12} \cdot \frac{3}{2} \Rightarrow \boldsymbol{x = \frac{1}{8}}$.

(d) $\frac{2x}{3} + \frac{1}{4} = \frac{x}{6} - \frac{1}{2} \Rightarrow \frac{2}{3}x - \frac{1}{6}x = {}^{-}\left(\frac{1}{2}\right) + {}^{-}\left(\frac{1}{4}\right) \Rightarrow \frac{1}{2}x = {}^{-}\left(\frac{3}{4}\right)$

$\Rightarrow x = {}^{-}\left(\frac{3}{4}\right) \div \frac{1}{2} = {}^{-}\left(\frac{3}{4}\right) \cdot \frac{2}{1}$

$\Rightarrow \boldsymbol{x = \frac{{}^{-}3}{2} = {}^{-}1\frac{1}{2}}$.

7. Let the fraction be $\frac{a}{b}$. Then $\frac{a+b}{b} = 3\left(\frac{a}{b}\right)$ or $\frac{a+b}{b} = \frac{3a}{b}$. The denominators are equal, so $a + b = 3a \Rightarrow b = 2a$. The fraction then becomes $\frac{a}{2a} = \mathbf{\frac{1}{2}}$.

8. Possible thought processes are described; all provide exact answers:

(a) $3 \cdot 8 = 24$ and $\frac{1}{2} \cdot 8 = 4$. So $3\frac{1}{2} \cdot 8 = 24 + 4 = \mathbf{28}$.

(b) $7 \cdot 4 = 28$ and $\frac{3}{4} \cdot 4 = 3$. So $7\frac{3}{4} \cdot 4 = 28 + 3 = \mathbf{31}$.

(c) $9 \cdot 6 = 54$ and $\frac{1}{5} \cdot 6 = \frac{6}{5} = 1\frac{1}{5}$. So $9\frac{1}{5} \cdot 6 = 54 + 1\frac{1}{5} = \mathbf{55\frac{1}{5}}$.

(d) $8 \cdot 2 = 16$ and $8 \cdot \frac{1}{3} = \frac{8}{3} = 2\frac{2}{3}$. So $8 \cdot 2\frac{1}{3} = 16 + 2\frac{2}{3} = \mathbf{18\frac{2}{3}}$.

(e) $3 \div \frac{1}{2} = 3 \cdot 2 = \mathbf{6}$.

(f) $3\frac{1}{2} \div \frac{1}{2} = 3\frac{1}{2} \cdot 2 = \mathbf{7}$.

(g) $3 \div \frac{1}{3} = 3 \cdot 3 = \mathbf{9}$.

(h) $4\frac{1}{2} \div 2 = 4\frac{1}{2} \cdot \frac{1}{2} \Rightarrow 4 \cdot \frac{1}{2} = 2$, $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and $2 + \frac{1}{4} = \mathbf{2\frac{1}{4}}$.

9. **(a)** **2**. $20\frac{2}{3} \div 9\frac{7}{8} \approx 20 \div 10 = 2$.

(b) **24**. $3\frac{1}{20} \cdot 7\frac{77}{100} \approx 3 \cdot 8 = 24$.

(c) **1**. $\frac{1}{10^3} \div \frac{1}{1001} \approx \frac{1}{1000} \div \frac{1}{1000} = 1$.

10. **(a)** **Greater than 2**. If the quotient was 2, then checking would show that $2\left(2\frac{13}{100}\right) = 4\frac{26}{100}$, which is less than $4\frac{1}{3}$. Thus the true quotient must be greater than 2.

(b) **Less than 4**. $16 \div 4\frac{3}{18}$ is 16 divided by a number more than 4.

(c) **Greater than 4**. $16 \div 3\frac{8}{9}$ is 16 divided by a number less than 4.

11. If n is the number, then $3n - \frac{7}{18} = 2n + \frac{5}{12} \Rightarrow n = \frac{5}{12} + \frac{7}{18} = \frac{5 \cdot 18 + 12 \cdot 7}{12 \cdot 18} = \mathbf{\frac{29}{36}}$.

12. Let f be the original number of faculty members. Then $f - \frac{1}{5}f = 320 \Rightarrow \frac{4}{5}f = 320 \Rightarrow \left(\frac{5}{4}\right)\frac{4}{5}f = \left(\frac{5}{4}\right)320 \Rightarrow f = $ **400 members**.

13. **(a)** Let u be the number of uniforms to be made. Assuming no waste, $u = 29\frac{1}{2} \div \frac{3}{4} = \frac{59}{2} \cdot \frac{4}{3} = \frac{236}{6} = \frac{118}{3} = 39\frac{1}{3}$, or **39 uniforms** can be made.

(b) Enough material for $\frac{1}{3}$ uniform will be left over. Each uniform needs $\frac{3}{4}$ yard material, so $\frac{1}{3} \cdot \frac{3}{4} = \frac{3}{12} = \mathbf{\frac{1}{4}}$ **yard** will remain.

14. **(a)** Increasing a salary by $\frac{1}{10}$ means the salary will be $1\frac{1}{10} = \frac{11}{10}$ of its previous value. With two such raises Martha will make $\left(100{,}000 \cdot \frac{11}{10}\right) \cdot \frac{11}{10} = 110{,}000 \cdot \frac{11}{10} =$ **$121,000**.

(b) \$99,000 is $\frac{11}{10}$ of **what Aaron made last year**, i.e., $99{,}000 = \frac{11}{10}s$ (where s is last year's salary) $\Rightarrow s = \frac{10}{11}(99{,}000) =$ **$90,000**.

(c) Let s be Juanita's salary two years ago. Then $363{,}000 = \frac{11}{10}\left(\frac{11}{10}s\right)$, since two raises brought her to that point. $\frac{121}{100}s = 363{,}000 \Rightarrow s = \frac{100}{121}(363{,}000) =$ **$300,000**.

15. Let b be the number of pages in the book. Jasmine has read $\frac{3}{4}b$, so she has $b - \frac{3}{4}b = \frac{1}{4}b$ yet to read. If 82 pages $= \frac{1}{4}b$ then multiply each side of the equation by 3 to yield $246 = \frac{3}{4}b$, or she has read **246 pages**.

16. **(a)** Peter, $\frac{1}{2} \cdot 60 =$ **30 minutes**.

Paul, $\frac{5}{12} \cdot 60 =$ **25 minutes**.

Mary, $\frac{1}{3} \cdot 60 =$ **20 minutes**.

(b) $LCD(30, 25, 20) = 300$ minutes.
$300 \div 30 =$ **10 laps for Peter**.
$300 \div 25 =$ **12 laps for Paul**.
$300 \div 20 =$ **15 laps for Mary**.

17. **(a)** $\left(\frac{1}{3}\right)^{-1} = \left(\frac{3}{1}\right)^{1} = \mathbf{3}$.

(b) $\frac{a^{-3}}{a} = \frac{\left(\frac{1}{a}\right)^3}{a} = \frac{1}{a^3} \cdot \frac{1}{a} = \mathbf{\frac{1}{a^4}}$.

(c) $\frac{(a^{-4})^3}{a^{-4}} = \frac{a^{-12}}{a^{-4}} = a^{-12-(-4)} = a^{-8} = \mathbf{\frac{1}{a^8}}$.

(d) $\frac{a}{a^{-1}} = \frac{a}{\left(\frac{1}{a}\right)} = \frac{a}{1} \cdot \frac{a}{1} = \mathbf{a^2}$.

(e) $\frac{a^{-3}}{a^{-2}} = \frac{a^2}{a^3} = \mathbf{\frac{1}{a}}$.

18. **(a)** $\left(\frac{1}{2}\right)^{10} \div \left(\frac{1}{2}\right)^{2} = \left(\frac{1}{2}\right)^{10-2} = \boldsymbol{\left(\frac{1}{2}\right)^{8}}$.

(b) $\left(\frac{2}{3}\right)^{5} \cdot \left(\frac{4}{9}\right)^{-2} = \left(\frac{2}{3}\right)^{5} \cdot \left[\left(\frac{2}{3}\right)^{2}\right]^{-2} =$

$\left(\frac{2}{3}\right)^{5-4} = \boldsymbol{\frac{2}{3}}$.

(c) $\left(\frac{3}{5}\right)^{7} \div \left(\frac{5}{3}\right)^{4} = \left(\frac{3}{5}\right)^{7} \div \left(\frac{3}{5}\right)^{-4} =$

$\left(\frac{3}{5}\right)^{7-{}^-4} = \boldsymbol{\left(\frac{3}{5}\right)^{11}}$.

(d) $\left[\left(\frac{5}{6}\right)^{7}\right]^{3} = \left(\frac{5}{6}\right)^{7\cdot 3} = \boldsymbol{\left(\frac{5}{6}\right)^{21}}$.

19. False counterexamples may vary:

(a) **False**. $\frac{2^3}{3^2} = \frac{8}{9} \neq \left(\frac{2}{3}\right)^{3-2} = \frac{2}{3}$.

(b) **True**.
$(ab)^{-m} = \frac{1}{(ab)^m} = \frac{1}{a^m b^m} = \frac{1}{a^m} \cdot \frac{1}{b^m}$.

(c) **False**. $\left(\frac{2}{2^{-1}+3^{-1}}\right)^{-1} = \frac{\left(\frac{1}{2}+\frac{1}{3}\right)}{2} = \frac{\left(\frac{5}{6}\right)}{2} =$

$\frac{5}{12} \neq \frac{1}{2} \cdot \frac{1}{2+3} = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$.

(d) **True**. $2(a^{-1} + b^{-1})^{-1} = 2 \cdot \frac{1}{a^{-1}+b^{-1}} =$

$2 \cdot \frac{1}{\frac{1}{a}+\frac{1}{b}} = 2 \cdot \frac{1}{\left(\frac{a+b}{ab}\right)} = \frac{2ab}{a+b}$.

(e) **False**. $3^{2\cdot 3} = 3^6 = 729 \neq 3^2 \cdot 3^3 = 243$.

(f) **True**. $\left(\frac{a}{b}\right)^{-1} = \frac{1}{\left(\frac{a}{b}\right)} = \frac{b}{a}$.

20. **(a)** $2^n = {}^-32 \Rightarrow$ **no integer solution** because 2^n will always be a positive number regardless of the value of n.

(b) $n^3 = {}^-\left(\frac{1}{27}\right) \Rightarrow n^3 = {}^-\left(\frac{1}{3}\right)^3$
$\Rightarrow \boldsymbol{n} = {}^-\left(\frac{1}{3}\right) \Rightarrow$ **no integer solution**.

(c) $2^n \cdot 2^7 = 1024 \Rightarrow 2^{n+7} = 2^{10}$
$\Rightarrow n + 7 = 10 \Rightarrow \boldsymbol{n = 3}$.

(d) $2^n \cdot 2^7 = 64 \Rightarrow 2^{n+7} = 2^6$
$\Rightarrow n + 7 = 6 \Rightarrow \boldsymbol{n = {}^-1}$.

(e) $(2 + n)^2 = 2^2 + n^2 \Rightarrow 2^2 + 4n + n^2 =$
$2^2 + n^2 \Rightarrow 4n = 0 \Rightarrow \boldsymbol{n = 0}$.

(f) $3^n = 27^5 \Rightarrow 3^n = (3^3)^5 \Rightarrow 3^n = 3^{15}$
$\Rightarrow \boldsymbol{n = 15}$.

21. **(a)** $3^x \geq 81 \Rightarrow 3^x \geq 3^4 \Rightarrow \boldsymbol{x \geq 4}$.

(b) $4^x \geq 8 \Rightarrow (2^2)^x \geq 2^3 \Rightarrow 2x \geq 3$
$\Rightarrow x \geq \frac{3}{2} \Rightarrow \boldsymbol{x \geq 2}$, where x is an integer.

(c) $3^{2x} \leq 27 \Rightarrow 3^{2x} \leq 3^3 \Rightarrow 2x \leq 3$
$\Rightarrow x \leq \frac{3}{2} \Rightarrow \boldsymbol{x \leq 1}$, where x is an integer.

(d) $2^x < 1 \Rightarrow 2^x < 2^0 \Rightarrow$
$\boldsymbol{x < 0} \Rightarrow \boldsymbol{x \leq {}^-1}$.

22. **(a)** $\boldsymbol{\left(\frac{4}{3}\right)^{10}}$. $\left(\frac{4}{3}\right)^{10} = \frac{4}{3} \cdot \frac{4}{3} \cdot \left(\frac{4}{3}\right)^{8}$; i.e., multiplying $\left(\frac{4}{3}\right)^{8}$ by a number greater than 1.

(b) $\boldsymbol{\left(\frac{4}{5}\right)^{10}}$. $\frac{4}{5} > \frac{3}{4} \Rightarrow \left(\frac{4}{5}\right)^{10} > \left(\frac{3}{4}\right)^{10}$.

(c) $\boldsymbol{\left(\frac{4}{3}\right)^{10}}$. $\frac{4}{3} > \frac{5}{4}$ (because $4 \cdot 4 > 3 \cdot 5$)
$\Rightarrow \left(\frac{4}{3}\right)^{10} > \left(\frac{5}{4}\right)^{10}$.

(d) $\boldsymbol{\left(\frac{3}{4}\right)^{100}}$. $\frac{3}{4} > \frac{3\cdot 9}{4\cdot 10} = \frac{9}{10} \cdot \frac{3}{4}$
$\Rightarrow \left(\frac{3}{4}\right)^{100} > \left(\frac{3\cdot 9}{4\cdot 10}\right)^{100}$.

23. **(a)** $\boldsymbol{32^{100}}$. $32^{100} = (2^5)^{100} = 2^{500}$, while $4^{200} = (2^2)^{200} = 2^{400}$.

(b) $\boldsymbol{({}^-3)^{-50}}$. $({}^-27)^{-15} = [({}^-3)^3]^{-15} =$
$({}^-3)^{-45} = \frac{1}{({}^-3)^{45}} = {}^-\left(\frac{1}{3^{45}}\right)$, while
$({}^-3)^{-50} = \frac{1}{({}^-3)^{50}} = \left(\frac{1}{3^{50}}\right)$, a positive number.

So $({}^-3)^{-50} > ({}^-27)^{-15}$
($\frac{1}{3^{45}} > \frac{1}{3^{50}}$, but the negative reverses the direction of the inequality).

24. Let x = the amount Brandy spent on the horse's keep. Counting the amount she paid for the horse, Brandy spent $270 + x$. Her profit or loss can be found by $270 + x - 540$. In the second sentence we are told Brandy lost $\frac{1}{2}(270) + \frac{1}{4}x$. Thus,

$$\frac{1}{2}(270) + \frac{1}{4}x = 270 + x - 540$$
$$135 + \frac{1}{4}x = x - 270$$
$$135 + 270 = x - \frac{1}{4}x$$
$$405 = \frac{3}{4}x$$
$$x = \frac{4}{3} \cdot 405 = 540 \text{ dollars.}$$

Since x represents the money Brandy spent on keep, her loss is $270 + 540 - 540 =$ **270 dollars**.

25. $1 - \frac{8}{25} = \frac{25}{25} - \frac{8}{25} = \frac{17}{25}$ of the students were male.

26. **(a)** The hotel tax minus the property tax is approximately $210{,}000{,}000 - 180{,}000{,}000 = 40{,}000{,}000$ which is approximately $\frac{40{,}000{,}000}{670{,}000{,}000} \approx \frac{4}{67}$ of the total tax.

(b) The sales tax increased by $\frac{1}{11}$ would be approximately $\frac{1}{11}(187{,}000{,}000) \approx 17{,}000{,}000$. That is approximately $\frac{17{,}000{,}000}{670{,}000{,}000} \approx \frac{20}{700} \approx \frac{2}{70}$.

27. **(a)** The fractional decrease in homicides from 2005 to 2012 was $\frac{196-88}{196} = \frac{108}{196} = \frac{27}{49}$.

(b) The fractional increase in homicides from 2012 to 2013 was $\frac{104-88}{88} = \frac{16}{88} = \frac{2}{11}$.

28. $1 - \frac{1}{33} - \frac{2}{7} = \frac{231}{231} - \frac{7}{231} - \frac{66}{231} = \frac{158}{231}$ of the deer population in the United States is not in Mississippi and Montana.

29. $\frac{2}{5}$ of $\frac{1}{4} = \frac{2}{20} = \frac{1}{10}$ is stocks and $\frac{3}{5}$ of $\frac{1}{4} = \frac{3}{20}$ is for bonds.

Mathematical Connections 6-3: Review Problems

18. **(a)** $\frac{^-3}{16} + \frac{7}{4} = \frac{^-3}{16} + \frac{28}{16} = \frac{25}{16} = \mathbf{1\frac{9}{16}}$.

(b) $\frac{1}{6} + \frac{^-4}{9} + \frac{5}{3} = \frac{3}{18} + \frac{^-8}{18} + \frac{30}{18} = \frac{25}{18} = \mathbf{1\frac{7}{18}}$.

(c) $\frac{^-5}{2^3 \cdot 3^2} - \frac{^-5}{2 \cdot 3^3} = \frac{^-5 \cdot 3}{2^3 \cdot 3^3} - \frac{^-5 \cdot 2^2}{2^3 \cdot 3^3} = \frac{^-15}{216} + \frac{20}{216} = \mathbf{\frac{5}{216}}$.

(d) $3\frac{4}{5} + 4\frac{5}{6} = 3\frac{24}{30} + 4\frac{25}{30} = 7\frac{49}{30} = \mathbf{8\frac{19}{30}}$.

(e) $5\frac{1}{6} - 3\frac{5}{8} = 5\frac{4}{24} - 3\frac{15}{24} = 4\frac{28}{24} - 3\frac{15}{24} = \mathbf{1\frac{13}{24}}$.

(f) $^-4\frac{1}{3} - 5\frac{5}{12} = \frac{^-13}{3} - \frac{65}{12} = \frac{^-52}{12} - \frac{65}{12} = \frac{^-117}{12} = \frac{^-39}{4} = \mathbf{^-9\frac{3}{4}}$.

19. The portion of students that take Spanish, French, or German is $\frac{2}{3} + \frac{1}{9} + \frac{1}{18} = \frac{5}{6}$. The portion of students not taking one of these languages is $1 - \frac{5}{6} = \frac{1}{6}$. That number is $\frac{1}{6} \cdot 720 =$ **120 students**.

20. **(a)** $\frac{3x}{xy^2} + \frac{y}{x^2} = \frac{3x^2}{x^2y^2} + \frac{y^3}{x^2y^2} = \frac{3x^2 + y^3}{x^2y^2}$.

(b) $\frac{a}{xy^2} - \frac{b}{xyz} = \frac{az}{xy^2z} - \frac{by}{xy^2z} = \frac{az - by}{xy^2z}$.

(c)
$$\frac{a^2}{a^2 - b^2} - \frac{a-b}{a+b} = \frac{a^2}{(a-b)(a+b)} - \frac{(a-b)(a-b)}{(a-b)(a+b)}$$
$$= \frac{a^2 - (a^2 - ab - ab + b^2)}{(a-b)(a+b)} = \frac{2ab - b^2}{a^2 - b^2}.$$

21. **(a)** This is not correct. For example, $\frac{2 \bullet 3 + 5}{3} \neq 2 + 5$ because $\frac{11}{3} \neq 7$.

(b) This is not correct. For example, because $\frac{5}{7} \neq \frac{3}{5}$.

(c) This is correct as long as a≠0. For example, $\frac{2\bullet 3+2\bullet 5}{2\bullet 5}=\frac{3+5}{5}$ because $\frac{6+10}{10}=\frac{3+5}{5}\Rightarrow\frac{16}{10}=\frac{8}{5}$.

Assessment 6-4A: Proportional Reasoning

1. (a) There are five vowels and 21 consonants. Their ratio is $\frac{5 \text{ vowels}}{21 \text{ consonants}} = \frac{5}{21}$, or **5:21**.

(b) $\frac{21 \text{ consonants}}{5 \text{ vowels}} = \frac{21}{5}$, or **21:5**.

(c) $\frac{21 \text{ consonants}}{26 \text{ alphabet letters}} = \frac{21}{26}$, or **21:26**.

(d) Answers may vary. "Break" (2 vowels, 3 consonants) or "minor" (2 vowels, 3 consonants) are two.

2. $\frac{a}{b} = \frac{c}{d}$ only if $ad = bc$:

(a) $\frac{12}{x} = \frac{18}{45} \Rightarrow 18x = 12 \cdot 45 \Rightarrow x = \frac{12 \cdot 45}{18} = \mathbf{30}$.

(b) $\frac{x}{7} = \frac{^-10}{21} \Rightarrow 21x = 7 \cdot {^-10} \Rightarrow x = \frac{7 \cdot {^-10}}{21} = \frac{^-10}{3} = \mathbf{^-3\frac{1}{3}}$.

(c) $\frac{5}{7} = \frac{3x}{98} \Rightarrow 7 \cdot 3x = 5 \cdot 98 \Rightarrow x = \frac{5 \cdot 98}{7 \cdot 3} = \frac{70}{3} = \mathbf{23\frac{1}{3}}$.

(d) $3\frac{1}{2}$ is to 5 as x is to 15 $\Rightarrow \frac{3\frac{1}{2}}{5} = \frac{x}{15} \Rightarrow$
$5 \cdot x = 3\frac{1}{2} \cdot 15 \Rightarrow x = \frac{3\frac{1}{2} \cdot 15}{5} = \frac{21}{2} = \mathbf{10\frac{1}{2}}$.

3. (a) Because the ratio is 2:3, there are $2x$ boys and $3x$ girls. The ratio of boys to all students is then $\frac{2x}{2x+3x} = \frac{2x}{5x} = \frac{2}{5} = \mathbf{2:5}$.

(b) $\boldsymbol{m:(m+n)}$. See part (a) above.

(c) Because the ratio of girls to all students is $\frac{3}{5}$, there are 3 girls to every 2 boys, or a ratio of girls to boys of **3:2**.

4. $\frac{2 \text{ pounds muscle}}{5 \text{ pounds body weight}} = \frac{x \text{ pounds muscle}}{90 \text{ pounds body weight}} \Rightarrow$
$5 \cdot x = 2 \cdot 90 \Rightarrow x = \frac{2 \cdot 90}{5} = \mathbf{36\ pounds}$.

5. $\frac{4 \text{ grapefruit}}{80¢}$ and $\frac{12 \text{ grapefruit}}{\$2.00}$ $\Rightarrow \frac{1 \text{ grapefruit}}{20¢}$ and $\frac{1 \text{ grapefruit}}{16.\overline{6}¢}$ so **\$2.00 for 12** is a better buy.

6. $\frac{\frac{1}{3} \text{ inch}}{5 \text{ miles}} = \frac{18 \text{ inches}}{x \text{ miles}} \Rightarrow \frac{1}{3} \cdot x = 5 \cdot 18 \Rightarrow x = \frac{5 \cdot 18}{\frac{1}{3}} = \mathbf{270\ miles}$.

7. $\frac{40 \text{ pages}}{50 \text{ minutes}} = \frac{x \text{ pages}}{80 \text{ minutes}} \Rightarrow 50 \cdot x = 40 \cdot 80 \Rightarrow x = \frac{40 \cdot 80}{50} = \mathbf{64\ pages}$.

8. (a) Let u be a unit value. Then $3u + 4u = 98 \Rightarrow 7u = 98 \Rightarrow u = 14$. $3u = \mathbf{42}$ and $4u = \mathbf{56}$.

(b) Let u be a unit value. $3u \cdot 4u = 768 \Rightarrow 12u^2 = 768 \Rightarrow u^2 = 64 \Rightarrow u = 8$ or $u = {^-8}$
[since $8^2 = 64$ and $(^-8)^2 = 64$]. Then $3u = \mathbf{24}$ and $4u = \mathbf{32}$ **or** $3u = \mathbf{^-24}$ and $4u = \mathbf{^-32}$.

9. (*i*) There are a total of $2 + 3 + 5 = 10$ shares. Using a proportion, and letting G represent the amount of money Gary would receive, we write: $\frac{2}{10} = \frac{G}{82{,}000} \Rightarrow 10G = 2 \cdot 82{,}000 \Rightarrow G = \mathbf{\$16{,}400}$.

Alternative thinking: Common Core State Standards suggests we consider using rates to solve this type of problem. Gary's rate is 2 to 10. Thus, $\frac{2}{10} \cdot 82{,}000 = \frac{2 \cdot 82{,}000}{10} = \mathbf{\$16{,}400}$.

(*ii*) $\frac{3}{10} = \frac{\text{Bill's amount}}{82{,}000} \Rightarrow 10(\text{Bill's}) = 3 \cdot 82{,}000 \Rightarrow \text{Bill's amount} = \mathbf{\$24{,}600}$.

(*iii*) $\frac{5}{10} = \frac{\text{Carmella's amount}}{82{,}000} \Rightarrow$
$10 \cdot \text{Carmella's} = 5 \cdot 82{,}000 \Rightarrow$
$\text{Carmella's amount} = \mathbf{\$41{,}000}$.

10. The ratio of Sheila's hours to Dora's is $3\frac{1}{2}$ to $4\frac{1}{2}$. Then $3\frac{1}{2}x + 4\frac{1}{2}x = \$176 \Rightarrow 8x = 176 \Rightarrow x = 22$. Thus **Sheila's** earnings are $3\frac{1}{2} \cdot 22 =$ **\$77** and **Dora's** are $4\frac{1}{2} \cdot 22 =$ **\$99**.

11. Success:failure $= 5{:}4 \Rightarrow \frac{5 \text{ successes}}{4 \text{ failures}} = \frac{75 \text{ successes}}{x \text{ failures}} \Rightarrow 5 \cdot x = 4 \cdot 75 \Rightarrow x = \frac{4 \cdot 75}{5} =$ 60 failures. 75 successes + 60 failures = **135 attempts**.

12. **(a)** $\frac{1}{6}{:}1 \Rightarrow (6)\frac{1}{6}{:}(6)1 \Rightarrow 1{:}6 = \mathbf{\frac{1}{6}}$.

(b) $\frac{1}{3}{:}\frac{1}{3} \Rightarrow (3)\frac{1}{3}{:}(3)\frac{1}{3} \Rightarrow 1{:}1 = \mathbf{\frac{1}{1}}$.

(c) $\frac{1}{6}{:}\frac{2}{7} \Rightarrow (42)\frac{1}{6}{:}(42)\frac{2}{7} \Rightarrow 7{:}12 = \mathbf{\frac{7}{12}}$.

13. The proportion implies $12¢ \cdot 48\,\text{oz} = 16¢ \cdot 36\,\text{oz}$. Other equivalent proportions are thus:

(i) $\frac{12¢}{16¢} = \frac{36 \text{ ounces}}{48 \text{ ounces}}$.

(ii) $\frac{48 \text{ ounces}}{16¢} = \frac{36 \text{ ounces}}{12¢}$.

(iii) $\frac{16¢}{12¢} = \frac{48 \text{ oz}}{36 \text{ oz}}$.

14. **(a)** $\frac{\text{Rise}}{\text{half-span}} = \frac{10}{14} = \frac{5}{7}$ or **5:7**.

(b) $\frac{\text{Rise}}{\text{half-span}} =$ pitch. $\frac{\text{Rise}}{8} = \frac{3}{4} \Rightarrow \text{rise} \cdot 4 = 8 \cdot 3 \Rightarrow \text{rise} = \frac{8 \cdot 3}{4} =$ **6 feet**.

15. **(a)** $\frac{4 \text{ rpm on large gear}}{6 \text{ rpm on small gear}} = \frac{18 \text{ teeth on small gear}}{x \text{ teeth on large gear}} \Rightarrow 4 \cdot x = 6 \cdot 18 \Rightarrow x = \frac{6 \cdot 18}{4} =$ **27 teeth**.

(b) $\frac{200 \text{ rpm on large gear}}{600 \text{ rpm on small gear}} = \frac{x \text{ teeth on small gear}}{60 \text{ teeth on large gear}} \Rightarrow 600 \cdot x = 200 \cdot 60 \Rightarrow x = \frac{200 \cdot 60}{600} =$ **20 teeth**.

16. $\frac{230 \text{ feet length}}{195 \text{ feet wingspan}} = \frac{40 \text{ cm length}}{x \text{ cm wingspan}} \Rightarrow 230 \cdot x = 195 \cdot 40 \Rightarrow x = \frac{195 \cdot 40}{230} = 33\frac{21}{23}$, or **about 34 cm**.

17. **(a)** $\frac{3 \text{ cups tomato sauce}}{2 \text{ cups tomato sauce}} = \frac{3}{2}$, so:

(i) $\frac{1 \text{ tsp mustard seed}}{x \text{ tsp mustard seed}} = \frac{3}{2} \Rightarrow 3 \cdot x = 1 \cdot 2 \Rightarrow x = \mathbf{\frac{2}{3}}$ **tsp mustard seed**.

(ii) $\frac{1\frac{1}{2} \text{ cups scallions}}{x \text{ cups scallions}} = \frac{3}{2} \Rightarrow 3 \cdot x = 1\frac{1}{2} \cdot 2 \Rightarrow x = \frac{1\frac{1}{2} \cdot 2}{3} =$ **1 cup scallions**.

(iii) $\frac{3\frac{1}{4} \text{ cups beans}}{x \text{ cups beans}} = \frac{3}{2} \Rightarrow 3 \cdot x = 3\frac{1}{4} \cdot 2 \Rightarrow x = \frac{3\frac{1}{4} \cdot 2}{3} =$ $\mathbf{2\frac{1}{6}}$ **cups beans**.

(b) $\frac{1\frac{1}{2} \text{ cups sacallions}}{1 \text{ cup sacallions}} = \frac{3}{2}$, which is the same ratio as in (a) above, or $\mathbf{\frac{2}{3}}$ **tsp mustard seed**, **2 cups tomato sauce**, $\mathbf{2\frac{1}{6}}$ **cups beans**.

(c) $\frac{3\frac{1}{4} \text{ cups beans}}{1\frac{3}{4} \text{ cups beans}} = \frac{13}{7}$, so:

(i) $\frac{1 \text{ tsp mustard seed}}{x \text{ tsp mustard seed}} = \frac{13}{7} \Rightarrow 13 \cdot x = 1 \cdot 7 \Rightarrow x = \frac{1 \cdot 7}{13} =$ $\mathbf{\frac{7}{13}}$ **tsp mustard seed**.

(ii) $\frac{3 \text{ cups tomato sauce}}{x \text{ cups tomato sauce}} = \frac{13}{7} \Rightarrow 13 \cdot x = 3 \cdot 7 \Rightarrow x = \frac{3 \cdot 7}{13} =$ $\mathbf{1\frac{8}{13}}$ **cups tomato sauce**.

(iii) $\frac{1\frac{1}{2} \text{ cups scallions}}{x \text{ cups scallions}} = \frac{13}{7} \Rightarrow 13 \cdot x = 1\frac{1}{2} \cdot 7 \Rightarrow x = \frac{1\frac{1}{2} \cdot 7}{13} =$ $\mathbf{\frac{21}{26}}$ **cups scallions**.

18. $\frac{4 \text{ ohms}}{5 \text{ feet}} = \frac{x \text{ ohms}}{20 \text{ feet}} \Rightarrow 5 \cdot x = 4 \cdot 20 \Rightarrow x = \frac{4 \cdot 20}{5} = 16$ **ohms**.

19. $\frac{2 \text{ cm (daughter)}}{6 \text{ cm (father)}} = \frac{x \text{ cm (daughter)}}{183 \text{ cm (father)}} \Rightarrow 6x = 2 \cdot 183 \Rightarrow x = \frac{2 \cdot 183}{6} = 61\,\text{cm}$.

20. The ratio between the mass of the gold in the ring to the mass of the ring is 18:24. If x is the number of ounces of pure gold in a ring that weighs 0.4 ounces, then $\frac{18 \text{ ounces of gold}}{24 \text{ ounces of ring}} = \frac{x \text{ ounces of gold}}{4 \text{ ounces of ring}} \Rightarrow 24x = 18 \cdot 4 \Rightarrow x = \frac{18 \cdot 4}{24} = 3$ ounces of gold in the ring.

3 ounces at $1800 per ounce of $= \$5400$.

21. **(a)** $\frac{5 \text{ hours}}{\$40} = \frac{40 \text{ hours}}{\$x} \Rightarrow 5x = 40 \cdot 40 \Rightarrow x = \frac{40 \cdot 40}{5} \Rightarrow x = \mathbf{\$320}$. The same result could have been obtained by using any of the other ratios; e.g., $16 for 2 hours.

(b) The constant of proportionality is **8**, representing $8 dollars per hour.

22. **(a)** The total number of men in all three rooms is $1 + 2 + 5 = 8$; the total number of women in all three rooms is $2 + 4 + 10 = 16$. The ratio of men to women is $\frac{8}{16} = \mathbf{\frac{1}{2}}$, or **1:2**.

(b) Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = r$.

Then $a = br$;

$c = dr$;

$e = fr$.

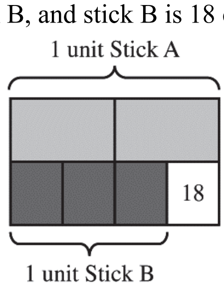
So $a + c + e = br + dr + fr \Rightarrow$

$a + c + e = r(b + d + f) \Rightarrow$

$r = \frac{a+c+e}{b+d+f}$

Thus $r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}$.

23. As seen in the drawing, $\frac{1}{2}$ of stick A is $\frac{2}{3}$ of stick B, and stick B is 18 cm shorter than stick A.



We have the following: Stick B is 3 • 18 unit sections whikle stick A is 4 • 18 unit sections. Thus the lenths of the sticks are A, 72 cm, and B, 54 cm.

24. $\frac{26 miles}{1 gallon}$ for the car and $\frac{250 miles}{14 gallons} \approx \frac{17.85 miles}{1 gallon}$ for the truck. The car gets the better gas mileage because it travels 26 miles on 1 gallon and the truck only travels 17.85 miles on 1 gallon.

25. $\frac{20 miles}{2 hours} = \frac{10 miles}{1 hour}$ for Susan and $\frac{32 miles}{3 hours} \approx \frac{10.\overline{6} miles}{1 hour}$ for Nick. So Nick travels faster.

26. The desired ratio of students to computers is 3:1. If there are 40 students then the ratio should be 40:13.3. So there would need to be 14 computers.

27. The ratio of oblong tables to round tables is 5:1. The ratio of oblong tables to total tables is then 5:6. If there are 102 total tables then the ratio of oblong tables to total tables is $\frac{5}{6} = \frac{85}{102}$. There are 85 oblong tables. 102- 85 = 17, so there are 17 round tables.

28. $1\frac{1}{2}$ cups for 2 dozen cookies

3 cups for 4 dozen cookies

$4\frac{1}{2}$ for 6 dozen cookies

4 cups would not be enough for 6 dozen cookies because it would take $4\frac{1}{2}$ cups for 6 dozen cookies.

Assessment 6-4B

1. **(a)** There are four vowels and seven consonants in "Mississippi." Their ratio is $\frac{4 \text{ vowels}}{7 \text{ consonants}} = \frac{4}{7}$, or **4:7**.

(b) $\frac{7 \text{ consonants}}{4 \text{ vowels}} = \frac{7}{4}$, or **7:4**.

(c) There are seven consonants in the eleven letters, so $\frac{7 \text{ consonants}}{11 \text{ letters}} = \frac{7}{11}$, or **7:11**.

2. $\frac{a}{b} = \frac{c}{d}$ only if $ad = bc$:

(a) $\frac{5}{x} = \frac{30}{42} \Rightarrow 30x = 5 \cdot 42 \Rightarrow x = \frac{5 \cdot 42}{30} = \frac{210}{30} = \mathbf{7}$.

(b) $\frac{x}{8} = \frac{^-12}{32} \Rightarrow 32x = {^-12} \cdot 8 \Rightarrow x = \frac{^-12 \cdot 8}{32} = \mathbf{^-3}$.

(c) $\frac{7}{8} = \frac{3x}{48} \Rightarrow 3x \cdot 8 = 7 \cdot 48 \Rightarrow x = \frac{7 \cdot 48}{3 \cdot 8} = \mathbf{14}$.

(d) $3\frac{1}{2}$ is to 8 as x is to 24 $\Rightarrow \frac{3\frac{1}{2}}{8} = \frac{x}{24} \Rightarrow 8x = 3\frac{1}{2} \cdot 24 \Rightarrow x = \frac{3\frac{1}{2} \cdot 24}{8} = \mathbf{10\frac{1}{2}}$.

3. $\frac{5 \text{ adults}}{1 \text{ teen}} = \frac{12{,}345 \text{ adults}}{x \text{ teens}} \Rightarrow 5 \cdot x = 1 \cdot 12{,}345 \Rightarrow x = \frac{1 \cdot 12{,}345}{5} =$ **2469 teen drivers**.

4. The candle has burned 5 inches in 12 minutes. $\frac{5 \text{ inches}}{12 \text{ minutes}} = \frac{30 \text{ inches}}{x \text{ minutes}} \Rightarrow 5 \cdot x = 12 \cdot 30 \Rightarrow x = \frac{12 \cdot 30}{5} =$ **72 minutes** for a 30-inch candle.

5. Let $5x$ represent width and $9x$ represent length. The perimeter of the rectangle is $2 \cdot 5x + 2 \cdot 9x = 2800 \Rightarrow 28x = 2800 \Rightarrow x = 100$. Thus width $= 5 \cdot 100 =$ **500 feet** and length $= 9 \cdot 100 =$ **900 feet**.

6. If the proportion of jump:length is 20:1 $\Rightarrow \frac{\text{jump}}{6 \text{ feet}} = \frac{20}{1} \Rightarrow 1 \cdot \text{jump} = 6 \cdot 20 \Rightarrow \text{jump} =$ **120 feet**, assuming the length of a grasshopper corresponds to the height of a human.

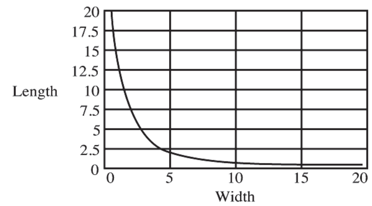
7. $\frac{9 \text{ months}}{6 \text{ vacation days}} = \frac{12 \text{ months}}{x \text{ vacation days}} \Rightarrow 9 \cdot x = 6 \cdot 12 \Rightarrow x = \frac{6 \cdot 12}{9} =$ **8 days**.

8. $\frac{1 \text{ teacher}}{30 \text{ students}} = \frac{x \text{ teachers}}{1200 \text{ students}} \Rightarrow 30x = 1200 \Rightarrow x = 40$ teachers presently.

Then $\frac{1 \text{ teacher}}{20 \text{ students}} = \frac{40 + t \text{ additional teachers}}{1200 \text{ students}} \Rightarrow 20(40 + t) = 1200 \Rightarrow 800 + 20t = 1200 \Rightarrow 20t = 400 \Rightarrow t = \frac{400}{20} =$ **20 additional teachers**.

9. $\frac{30 \text{ feet tall}}{12 \text{ foot shadow}} = \frac{x \text{ feet tall}}{14 \text{ foot shadow}} \Rightarrow 12 \cdot x = 30 \cdot 14 \Rightarrow x = \frac{30 \cdot 14}{12} =$ **35 feet**.

10. (a) Plotting lengths against widths from the table:



(b) $\boldsymbol{WL = 10}$; or width $\times$ length $=$ area $= 10 \text{ ft}^2$.

11. Solutions may vary. The ratio of 4 to 20 is the same in each case; x (the number of tickets) and y (the cost) could therefore be $\frac{8 \text{ tickets}}{\$40}, \frac{12 \text{ tickets}}{\$60}, \frac{16 \text{ tickets}}{\$80}, \ldots$.

12. $\frac{\$850}{2 \text{ weeks}} = \frac{\$x}{7 \text{ weeks}} \Rightarrow 2x = 850 \cdot 7 \Rightarrow x = \frac{850 \cdot 7}{2} = 2975$, or the rent for seven weeks would be **$2975**.

13. (a) $\frac{\text{Length of hand}}{\text{Length of big toe}} = \frac{14}{3} \Rightarrow \frac{14 \text{ cm}}{3 \text{ cm}} = \frac{x \text{ cm}}{6 \text{ cm}} \Rightarrow 3x = 14 \cdot 6 \Rightarrow x = \frac{14 \cdot 6}{3} =$ **28 cm**.

(b) $\frac{\text{Length of hand}}{\text{Length of foot}} = \frac{7}{9} \Rightarrow \frac{7 \text{ cm}}{9 \text{ cm}} = \frac{21 \text{ cm}}{x \text{ cm}} \Rightarrow 7x = 21 \cdot 9 \Rightarrow x = \frac{21 \cdot 9}{7} =$ **27 cm**.

14. The ratio of length to width of the park is $\frac{300 \text{ ft}}{200 \text{ ft}} = \frac{3}{2}$. Then $\frac{3 \text{ in}}{2 \text{ in}} = \frac{4 \text{ in}}{x \text{ in}} \Rightarrow 3x = 4 \bullet 2 \Rightarrow x = \frac{4(2)}{3} = \frac{8}{3} = 2\frac{2}{3}$ in

15. $\frac{240 \text{ mi}}{15 \text{ gallons}} = \frac{x \text{ mi}}{3 \text{ gallons}} \Rightarrow 15x = 240 \cdot 3 \Rightarrow x = \frac{240 \cdot 3}{15} =$ **48 mi**.

16. $\frac{1 \text{ in}}{48 \text{ in}} = \frac{18 \text{ in}}{x \text{ in}} \Rightarrow 1x = 18 \cdot 48 \Rightarrow 864 \text{ in} =$ **72 ft**.

17. (a) There are 50 stars and 13 stripes in the flag, or a ratio of $\frac{50 \text{ stars}}{13 \text{ stripes}} =$ **50:13**.

(b) $\frac{13 \text{ stripes}}{50 \text{ stars}} =$ **13:50**.

18. **(a)** Given $\frac{\text{length}}{\text{width}} = \frac{19}{10} \Rightarrow \frac{19 \text{ ft}}{10 \text{ ft}} = \frac{9\frac{1}{2} \text{ ft}}{x \text{ ft}} \Rightarrow$

$19x = 9\frac{1}{2} \cdot 10 \Rightarrow x = \frac{9\frac{1}{2} \cdot 10}{19} = \mathbf{5\ ft}.$

(b) **No**. $\frac{5 \text{ ft}}{3 \text{ ft}} = \frac{5}{3} \neq \frac{19}{10}$ because $5 \cdot 10 \neq 19 \cdot 3$.

19. $\frac{x}{y} = \frac{a}{b} \Rightarrow xb = ya \Rightarrow \frac{b}{a} = \frac{y}{x}$, $\frac{a}{x} = \frac{b}{y}$, and $\frac{x}{a} = \frac{y}{b}$ $(x, y, a, b \neq 0)$.

20. Let x represent the amount of milk needed in the recipe if only 1 cup of flour is used.

$\frac{1\frac{1}{2} \text{ cups flour}}{4 \text{ cups milk}} = \frac{1 \text{ cup flour}}{x \text{ cups milk}} \Rightarrow 1\frac{1}{2}x = 1 \cdot 4 \Rightarrow$

$\frac{3}{2}x = 4 \Rightarrow x = 4 \cdot \frac{2}{3} = \frac{8}{3} = \mathbf{2\frac{2}{3}}$ **cups**.

21. Set up a proportion:

$\frac{21 \text{ tagged fish}}{68 \text{ fish}} = \frac{173 \text{ tagged fish}}{x \text{ fish}} \Rightarrow 21x =$

$68 \cdot 173 \Rightarrow x = \frac{68 \cdot 173}{21} \approx$ **560 fish**.

22. $\frac{36 miles}{1 gallon}$ for the car and $\frac{200 miles}{14 gallons} \approx \frac{14.3 miles}{1 gallon}$ for the truck. The car gets the better gas mileage because it travels 36 miles on 1 gallon and the truck only travels 14.3 miles on 1 gallon.

23. $\frac{18 miles}{2 hours} = \frac{9 miles}{1 hour}$ for Susan and $\frac{30 miles}{3 hours} \approx \frac{10 miles}{1 hour}$ for Nick. So Nick travels faster.

24. The desired ratio of students to computers is 3:1. If there are 32 students then the ratio should be 32:10.6. So there would need to be 11 computers.

25. The ratio of oblong tables to round tables is 6:1. The ratio of oblong tables to total tables is then 6:7. If there are 112 total tables then the ratio of oblong tables to total tables is $\frac{6}{7} = \frac{96}{112}$. There are 96 oblong tables. 112- 96 = 16, so there are 16 round tables.

26. $1\frac{1}{2}$ cups for 3 dozen cookies

3 cups for 6 dozen cookies

4 cups would be enough for 6 dozen cookies because it would take 3 cups for 6 dozen cookies.

27. $\frac{160 lb}{416 lb} = \frac{40 lb}{104 lb} = \frac{120 lb}{312 lb}$ so Amy would weigh 312 lb on Jupiter if she weighs 120lb on Earth.

Mathematical Connections 6-4: Review Problems

18. If the numerator is 6 times the denominator then $n = 6d$. If the numerator is also 5 more than the denominator then $n = d + 5$. Solving for d, $6d = d + 5 \Rightarrow 5d = 5 \Rightarrow d = 1$. Solving for n, $n = 6d \Rightarrow n = 6 \bullet 1 \Rightarrow n = 6$. The numerator is 6 and the denominator is 1.

19. $\frac{3}{4}$ is a proper fraction because the numerator and denominator are both positive with the numerator less than the denominator.

20. Any integer can be written as a rational number with the integer as the numerator and the denominator 1.

21. The statement is not true in general; for example, $\frac{25}{35} \neq \frac{2}{3}$ because $75 \neq 70$.

22. At 6:00, the hands of the clock form a straight line approximately every 1hr, 5 min, and 27 sec. So after 6:00, the minute and hour hands form a straight line at 7:05:27, 8:10:55, 9:16:22, 10:21:49, 11:27:16, 12:32:44, 1:38:11, 2:43:38, 3:49:05, and 4:54:33.

23. No, $\frac{2}{3}$ of a class is not equivalent to $\frac{4}{5}$ of the school being absent because $2 \bullet 5 \neq 3 \bullet 4$, $10 \neq 12$.

24. $99 \bullet 96 < 98 \bullet 97$, $9504 < 9506$, so $\frac{99}{98} < \frac{97}{96}$.

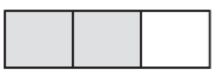
25. **(a)** $\frac{1}{\frac{1}{2}}+\frac{1}{\frac{1}{4}}=\frac{1}{t}\Rightarrow 2+4=\frac{1}{t}\Rightarrow 6=\frac{1}{t}\Rightarrow t=\frac{1}{6}$ of a day.

(b) $\frac{ab}{a+b}$ hours. In 1 hr Amal can finish $\frac{1}{a}$ of the job, while Sharif can finish $\frac{1}{b}$ of the job in 1 hr. If they work together, they can finish $\frac{1}{a}+\frac{1}{b}$ of the job in 1 hr. If it takes them x hr to finish the job working together, then $x(\frac{1}{a}+\frac{1}{b})=1$ or $(\frac{a+b}{ab})x=1$. Hence, $x=\frac{ab}{a+b}$.

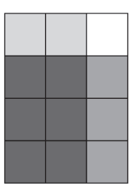
Chapter 6 Review

1. **(a)** Answers may vary. Shade three of four parts:

(b) Two of three blocks shaded:



(c) Three of four horizontal bars are shaded, meshed with two of three vertical bars. Six of the twelve bars are dark-shaded; i.e., $\frac{3}{4}\cdot\frac{2}{3}=\frac{6}{12}$:



2. Answers may vary; three are $\frac{10}{12}$, $\frac{15}{18}$, and $\frac{50}{60}$.

3. **(a)** $\frac{24}{28}=\frac{6\cdot 4}{7\cdot 4}=\mathbf{\frac{6}{7}}$.

(b) $\frac{ax^2}{bx}=\frac{ax\cdot x}{b\cdot x}=\boldsymbol{\frac{ax}{b}}$.

(c) $\frac{0}{17}=\frac{0\cdot 17}{1\cdot 17}=\mathbf{\frac{0}{1}}=0$.

(d) $\frac{45}{81}=\frac{5\cdot 9}{9\cdot 9}=\mathbf{\frac{5}{9}}$.

(e) $\frac{bx^2+bx}{b+x}=\frac{b\cdot(b+x)}{1\cdot(b+x)}=\boldsymbol{\frac{b}{1}}=b$.

(f) $\frac{16}{216}=\frac{2\cdot 8}{27\cdot 8}=\mathbf{\frac{2}{27}}$.

(g) $\frac{x+a}{x-a}$ **cannot be further reduced**. There are no factors common to both numerator and denominator (x and a are terms).

(h) $\frac{xa}{x+a}$ **cannot be further reduced**.

4. **(a)** $=$. $\frac{6\cdot 20}{10\cdot 20}=\frac{120}{200}$.

(b) $>$. $\frac{{}^{-}3}{4}=\frac{{}^{-}18}{24}>\frac{{}^{-}5}{6}=\frac{{}^{-}20}{24}$.

(c) $>$. $\left(\frac{4}{5}\right)^{10}>\left(\frac{4}{5}\right)^{20}$.

(d) $<$. $\left(1+\frac{1}{3}\right)^2=\left(\frac{4}{3}\right)^2<\left(1+\frac{1}{3}\right)^3=\left(\frac{4}{3}\right)^3$.

5. Additive inverse: $n+$ inverse $=0$.
Multiplicative inverse: $n\cdot$ inverse $=1$.

	n	Additive Inverse	Multiplicative Inverse
(a)	3	${}^{-}3$	$\frac{1}{3}$
(b)	$3\frac{1}{7}=\frac{22}{7}$	${}^{-}3\frac{1}{7}$	$\frac{7}{22}$
(c)	$\frac{5}{6}$	$\frac{{}^{-}5}{6}$	$\frac{6}{5}$
(d)	$\frac{{}^{-}3}{4}$	$\frac{3}{4}$	$\frac{{}^{-}4}{3}$

6. ${}^{-}2\frac{1}{3}<{}^{-}1\frac{7}{8}<0<\left(\frac{71}{140}\right)^{300}<\frac{69}{140}<\frac{1}{2}<\frac{71}{140}<\left(\frac{74}{73}\right)^{300}$.

7. **Yes**. Apply the laws of multiplication and the commutative and associative laws of multiplication to find: $\frac{4}{5}\cdot\frac{7}{8}\cdot\frac{5}{14}=\frac{4\cdot 7\cdot 5}{5\cdot 8\cdot 14}=\frac{4\cdot 7\cdot 5}{8\cdot 14\cdot 5}=\frac{4}{8}\cdot\frac{7}{14}\cdot\frac{5}{5}$.

8. Methods may vary:

(a) $\frac{1}{3}\cdot(8\cdot 9)=\left(\frac{1}{3}\cdot 9\right)\cdot 8=3\cdot 8=\mathbf{24}$.

(b) $36\cdot 1\frac{5}{6}=36\cdot\frac{11}{6}=6\cdot 11=\mathbf{66}$.

9. (*a*) Assuming no waste, $54\frac{1}{4} \div 3\frac{1}{12} = 17\frac{22}{37}$, so **17 pieces** can be cut.

(b) $\frac{22}{37} \cdot 3\frac{1}{12} = \mathbf{\frac{11}{6}} = \mathbf{1\frac{5}{6}}$ **yards left over**.

10. (a) **15**.
$\frac{30\frac{3}{8}}{4\frac{1}{9}} \cdot \frac{8\frac{1}{3}}{3\frac{8}{9}} \approx \frac{30\frac{2}{8}}{4\frac{1}{8}} \cdot \frac{8}{4} = \frac{30}{4} \cdot 2 = 7\frac{1}{2} \cdot 2.$

(b) **15**. $\left(\frac{3}{800} + \frac{4}{5000} + \frac{15}{6}\right) \cdot 6 \approx \frac{15}{6} \cdot 6.$

(c) **4**. $\frac{1}{407} \div \frac{1}{1609} \approx \frac{1}{400} \div \frac{1}{1600} = \frac{1}{400} \cdot \frac{1600}{1}.$

11. Answers may vary; for instance: A carpenter has a board measuring $4\frac{5}{8}$ feet. How many 6-inch pieces can be cut from it (disregarding the width of the cuts)? The solution is then $4\frac{5}{8} \div \frac{1}{2}$; a diagram would show four foot-long pieces plus $\frac{5}{8}$ of another, with $\frac{1}{2}$-foot size pieces shaded.

12. Answers may vary. $\frac{3}{4} = \frac{60}{80}$ and $\frac{4}{5} = \frac{64}{80}$ so $\frac{61}{80}$ and $\frac{62}{80}$ are between $\frac{3}{4}$ and $\frac{4}{5}$.

13. Think of $504792 \div 23$ as $504792 \cdot \frac{1}{23}$ and enter:

$\boxed{5}\,\boxed{0}\,\boxed{4}\,\boxed{7}\,\boxed{9}\,\boxed{2}\,\boxed{\times}\,\boxed{2}\,\boxed{3}\,\boxed{\frac{1}{x}}\,\boxed{=}$

14. Jim ate $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$ pizza. $\frac{1}{6} \cdot 2000 = \mathbf{333\frac{1}{3}}$ **calories**.

15. Let t be the number of times the coin was flipped. Then $\frac{1}{2}t = 376 \Rightarrow t = 376 \div \frac{1}{2} =$ **752 times**.

16. $\frac{240 \text{ heads}}{1000 \text{ flips}} = \frac{6 \cdot 40}{25 \cdot 40} = \mathbf{\frac{6}{25}}$ **of the time**.

17. **Not reasonable**. The statement implies that the University won $\frac{3}{4} + \frac{5}{8} = \frac{11}{8}$ of its games, or that it won more games than it played.

18. We could convert to the same units (for example inches to feet); however, this is not necessary when using proportional reasoning provide corresponding measures are in the same units.

Let $x =$ the distance between pupils on the carring of George Washington's head.

$\frac{2\frac{1}{2} \text{ in}}{9 \text{ in}} = \frac{x \text{ ft}}{60 \text{ ft}} \Rightarrow 9x = 2\frac{1}{2} \cdot 60 \Rightarrow x = \frac{2\frac{1}{2} \cdot 60}{9} = \frac{\frac{5}{2} \cdot 60}{9} = \frac{150}{9} = \frac{50}{3} = \mathbf{16\frac{2}{3}}$ **ft**.

19. $\frac{\left(\frac{2}{3}\right)}{\left(\frac{3}{4}\right)} = \frac{\frac{2}{3} \cdot \frac{4}{3}}{\frac{3}{4} \cdot \frac{4}{3}} = \frac{\left(\frac{8}{9}\right)}{1} = \frac{8}{9}$, which is the quotient of two integers.

20. 12 acres $\cdot\, 9\frac{1}{3}$ bags per acre $= 108 + 4 =$ **112 bags**.

21. $\frac{2}{3}$ female $\cdot$ $\frac{2}{5}$ blond $= \mathbf{\frac{4}{15}}$ **blond females**; i.e., a fraction of a fraction of the student population.

22. $\frac{^-12}{10}$ is greater than $\frac{^-11}{9}$ because $^-12 \cdot 9 > {}^-11 \cdot 10$. Alternatively, $\frac{^-12}{10} - \frac{^-11}{9} = \frac{^-108}{90} - \frac{^-110}{90} = \frac{2}{90}$, which is positive; therefore $-\frac{^-12}{10} > \frac{^-11}{9}$.

23. (a) $7^x = 343 \Rightarrow 7^x = 7^3 \Rightarrow \boldsymbol{x = 3}$.

(b) $2^{^-3x} = \frac{1}{512} \Rightarrow 2^{-3x} = 2^{-9}$
$\Rightarrow {}^-3x = {}^-9 \Rightarrow \boldsymbol{x = 3}$.

(c) $2x - \frac{5}{3} = \frac{5}{6} \Rightarrow 2x = \frac{5}{6} + \frac{5}{3} = \frac{15}{6} \Rightarrow$
$x = \frac{15}{6} \div 2 = \frac{15}{6} \cdot \frac{1}{2} \Rightarrow \boldsymbol{x = \frac{5}{4}}$.

(d) $x + 2\frac{1}{2} = 5\frac{2}{3} \Rightarrow x = 5\frac{2}{3} - 2\frac{1}{2} \Rightarrow$
$\boldsymbol{x = 3\frac{1}{6}}$.

(e) $\frac{20+x}{x} = \frac{4}{5} \Rightarrow 5(20 + x) = 4x \Rightarrow 100 +$
$5x = 4x \Rightarrow 100 = {}^-x \Rightarrow \boldsymbol{x = {}^-100}$.

(f) $2x + 4 = 3x - \frac{1}{3} \Rightarrow 4 + \frac{1}{3} =$
$3x - 2x \Rightarrow \boldsymbol{x = 4\frac{1}{3}}$.

24. **(a)** $\dfrac{(x^3a^{-1})^{-2}}{xa^{-1}} = \dfrac{x^{-6}a^2}{xa^{-1}} = \dfrac{\left(\frac{a^2}{x^6}\right)}{\left(\frac{x}{a}\right)} = \dfrac{\boldsymbol{a^3}}{\boldsymbol{x^7}}$.

(b) $\left(\dfrac{x^2y^{-2}}{x^{-3}y^2}\right)^{-2} = \left(\dfrac{x^{-4}y^4}{x^6y^{-4}}\right) = \dfrac{\left(\frac{y^4}{x^4}\right)}{\left(\frac{x^6}{y^4}\right)} = \dfrac{\boldsymbol{y^8}}{\boldsymbol{x^{10}}}$.

25. **(a)** $\dfrac{3a}{xy^2} + \dfrac{b}{x^2y^2} = \dfrac{3a}{xy^2}\cdot\dfrac{x}{x} + \dfrac{b}{x^2y^2} = \dfrac{\mathbf{3ax+b}}{\boldsymbol{x^2y^2}}$.

(b) $\dfrac{5}{xy^2} - \dfrac{2}{3x} = \dfrac{5}{xy^2}\cdot\dfrac{3}{3} - \dfrac{2}{3x}\cdot\dfrac{y^2}{y^2} = \dfrac{\mathbf{15-2y^2}}{\mathbf{3xy^2}}$.

(c) $\dfrac{a}{x^3y^2z} - \dfrac{b}{xyz} = \dfrac{a}{x^3y^2z} - \dfrac{b}{xyz}\cdot\dfrac{x^2y}{x^2y} =$
$\dfrac{\boldsymbol{a-bx^2y}}{\boldsymbol{x^3y^2z}}$.

(d) $\dfrac{7}{2^33^2} + \dfrac{5}{2^23^3} = \dfrac{7}{2^33^2}\cdot\dfrac{3}{3} + \dfrac{5}{2^23^3}\cdot\dfrac{2}{2} =$
$\dfrac{21+10}{2^33^3} = \dfrac{\mathbf{31}}{\mathbf{216}}$.

26. Answers may vary. The problem is to find how many $\frac{1}{2}$-yard pieces can be cut from a $1\frac{3}{4}$-yard ribbon. The method is to divide $1\frac{3}{4}$ by $\frac{1}{2} \Rightarrow \frac{7}{4} \div \frac{1}{2} = \frac{14}{4}$ or 3 pieces $\left(1\frac{1}{2}\text{ yards}\right)$ with $1\frac{3}{4} - 1\frac{1}{2} = \frac{1}{4}$ yard left over. This agrees with his answer obtained by his drawing. His algorithm concludes that he will have $3\frac{1}{2}$ half-yard pieces, and $\frac{1}{2}$ of a half-yard piece would be $\frac{1}{4}$ yard. He mistook a half-piece for a half-yard.

27. **(a)** $\dfrac{\text{Number of heads}}{\text{Total tosses}} = \dfrac{17}{30} = \mathbf{17{:}30}$.

(b) If 17 heads were recorded, there were 13 tails $\Rightarrow \dfrac{\text{Number of heads}}{\text{Number of tails}} = \dfrac{17}{13} = \mathbf{17{:}13}$.

(c) $\dfrac{\text{Number of tails}}{\text{Number of heads}} = \dfrac{13}{17} = \mathbf{13{:}17}$.

28. **(*i*)** 48 fl oz for \$3 $\Rightarrow \dfrac{300¢}{48\text{ fl oz}} \approx$ 6.25¢ per fl oz;

(*ii*) 64 fl oz for \$4 $\Rightarrow \dfrac{400¢}{64\text{ fl oz}} \approx$ 6.25¢ per fl oz;

Based on cost per ounce, neither is a better buy because they are the same price.

29. $\dfrac{18\text{ parts gold}}{6\text{ parts other metals}} \neq \dfrac{12\text{ parts gold}}{3\text{ parts other metals}}$ because $18\cdot 3 \neq 12\cdot 6$. Her ring may be more or less, but it is **not 18 karat gold**.

30.
4 people = 3 oranges
1 person = $\dfrac{3}{4}$ orange
11 people = $11\cdot\dfrac{3}{4}$ oranges
$=\dfrac{33}{4} = 8\dfrac{1}{4}$ orange.

4 people = 16 grapes
And 1 person = $\dfrac{16}{4}$=4 grapes
11 people = $11\cdot 4$ grapes
=44 grapes.

31. $\dfrac{1\text{ cm}}{2.5\text{ m}} = \dfrac{3\text{ cm}}{x\text{ m}} \Rightarrow x = 3\cdot 2.5 = \mathbf{7.5\,m}$.

32. If the ratio of O weight to H_2 weight is 8:1 then $8x + x = 16(\text{ounces})$. $9x = 16 \Rightarrow x = \frac{16}{9}$ and $1x = \frac{16}{9} = \mathbf{1\frac{7}{9}}$ **ounces**.

33. The ratio **cannot be determined exactly**, but it will always be between 12:100 and 15:100 as a ratio of defective to non-defective chips. 12:100 is $\frac{12}{112} = 10\frac{5}{7}\%$ defective; 15:100 is $\frac{15}{115} = 13\frac{1}{23}\%$ defective. If the observed defective percentage is closer to $13\frac{1}{23}\%$ then more chips came from the first plant. If the percentage is closer to $10\frac{5}{7}\%$, then more chips came from the second.

34. The area of a square is the square of the length of its side. If the ratio of the sides is $1{:}r$, then the ratio of their areas is $1^2{:}r^2$, or $\mathbf{1{:}\boldsymbol{r^2}}$.

35. **(a)** Games won to games lost was $\frac{18}{7}$ or **18:7**.

(b) 25 games were played. Games won to games played was $\frac{18}{25}$ or **18:25**.

36. **(a)** $\frac{1}{5}:1 \Rightarrow 5\left(\frac{1}{5}\right):5\left(1\right) = \mathbf{1:5}$.

(b) $\frac{2}{5}:\frac{3}{4} \Rightarrow 20\left(\frac{2}{5}\right):20\left(\frac{3}{4}\right) = \mathbf{8:15}$.

37. $\frac{\text{boys}}{\text{girls}} = \frac{3}{5} = \frac{x \text{ boys}}{15 \text{ girls}} \Rightarrow 5 \cdot x = 3 \cdot 15 \Rightarrow$
$x = \frac{3 \cdot 15}{5} = \mathbf{9\ boys}$.

38. If the ratio of states of the United States using the *Common Core* Standards to those not using it is 9:1, so 1 out of 10 are not using it. With 50 states, there are 45:5 states of the United States using the *Common Core* Standards to those not using it, so 5 states are not using the *Standards*.

39. $1-\frac{1}{20}-\frac{1}{3}-\frac{1}{4}-\frac{1}{4}-\frac{1}{20}$

$=\frac{60}{60}-\frac{3}{60}-\frac{20}{60}-\frac{15}{60}-\frac{15}{60}-\frac{3}{60}=\frac{4}{60}=\frac{1}{15}$
identified themselves with none of the labels.

40. The ratio of professors of other political persuasions to liberals is 1:7 to 1:9.

41. The ratio of perimeters is 6 + 6 + 6 = 18 to 10 + 10 + 10 = 30. The ration 18:30 is equal to 3:5.

42. The cup that has 1 oz cream because $1\text{oz} > \frac{9}{10}$ oz.

43. $\frac{45}{99} \bullet 121 = \frac{5}{11} \bullet 121 = 5 \bullet 11 = 55$ bulbs are expected to bloom and 121 – 55 = 66 are not expected to bloom.

44. No, the will is impossible to follow because the fractions of cats to be shared do not add to 1. If the woman had x cats then the number of cats her will directs to be distributed is
$\frac{1}{2}x+\frac{1}{3}x+\frac{1}{9}x=\frac{17}{18}x$, but the sum should be $1x$, or $(\frac{18}{18})x$.

45. If each guard demanded half the bags plus one more bag the problem can be worked backwards. Add a bag and double the amount. Starting with the 1 bag left, add one bag and double to get 4 bags (fourth bridge). Plus 1 and double is 10 bags (third bridge). Add one bag and double is 22 bags (second bridge). Add one bag and double is 46 bags (first bridge). Prince Juan stared with 46 bags of gold.

CHAPTER 7

RATIONAL NUMBERS AS DECIMALS AND PERCENT

Assessment 7-1A: Introduction to Finite Decimals

1. **(a)** $0.023 = \mathbf{0 \cdot 10^0 + 0 \cdot 10^{-1} + 2 \cdot 10^{-2} + 3 \cdot 10^{-3}}$.

   (b) $206.06 = \mathbf{2 \cdot 10^2 + 0 \cdot 10^1 + 6 \cdot 10^0 + 0 \cdot 10^{-1} + 6 \cdot 10^{-2}}$.

   (c) $312.0103 = \mathbf{3 \cdot 10^2 + 1 \cdot 10^1 + 2 \cdot 10^0 + 0 \cdot 10^{-1} + 1 \cdot 10^{-2} + 0 \cdot 10^{-3} + 3 \cdot 10^{-4}}$.

   (d) $0.000132 = \mathbf{0 \cdot 10^0 + 0 \cdot 10^{-1} + 0 \cdot 10^{-2} + 0 \cdot 10^{-3} + 1 \cdot 10^{-4} + 3 \cdot 10^{-5} + 2 \cdot 10^{-6}}$.

2. **(a)** $4000 + 300 + 50 + 6 + 0.7 + 0.08 = \mathbf{4356.78}$.

   (b) $4000 + 0.6 + 0.008 = \mathbf{4000.608}$.

   (c) $40{,}000 + 0.03 = \mathbf{40{,}000.03}$.

   (d) $0.2 + 0.0004 + 0.0000007 = \mathbf{0.2004007}$.

3. **(a) 536.0076**

   (b) 3.008

   (c) 0.000436

   (d) 5,000,000.2

4. **(a)** 0.34 = **thirty-four hundredths**.

   (b) 20.34 = **twenty and thirty-four hundredths**.

   (c) 2.034 = **two and thirty-four thousandths**.

   (d) 0.000034 = **thirty-four mllionths; i.e.**, $34 \cdot 10^{-6}$.

5. **(a)** $0.436 = \frac{436}{1000} = \frac{109 \cdot 4}{250 \cdot 4} = \mathbf{\frac{109}{250}}$.

   (b) $25.16 = 25\frac{16}{100} = 25 + \frac{16}{100} = \frac{2500}{100} + \frac{16}{100} = \frac{2516}{100} = \mathbf{\frac{629}{25}}$.

   (c) $^{-}316.027 = {^{-}316}\frac{27}{1000} = \mathbf{\frac{^{-}316{,}027}{1000}}$.

   (d) $28.1902 = 28\frac{1902}{10{,}000} = \frac{281{,}902}{10{,}000} = \mathbf{\frac{140{,}951}{5000}}$.

   (e) $^{-}4.3 = {^{-}4}\frac{3}{10} = \mathbf{\frac{^{-}43}{10}}$.

   (f) $^{-}62.01 = {^{-}62}\frac{1}{100} = \mathbf{\frac{^{-}6201}{100}}$.

6. A rational number in simplest form can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2 or 5:

   (a) **Terminating**. The denominator contains only 5 as a prime factor.

   (b) **Terminating**. The denominator contains no prime factors other than 2 and 5.

   (c) **Terminating**. The denominator in reduced form contains only 2 as a prime factor.

   (d) **Terminating**. The denominator contains only 2 as a prime factor.

   (e) **Not terminating**. The denominator contains 2 and 17 as prime factors.

   (f) **Terminating**. The denominator contains only 5 as a prime factor.

7. **(a)** $\frac{4}{5} = \frac{4}{5} \cdot \frac{2}{2} = \frac{8}{10} = \mathbf{0.8}$.

   (b) $\frac{61}{2^2 \cdot 5} = \frac{61}{2^2 \cdot 5} \cdot \frac{5}{5} = \frac{305}{2^2 \cdot 5^2} = \frac{305}{100} = \mathbf{3.05}$.

   (c) $\frac{3}{6} = \frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = \mathbf{0.5}$.

   (d) $\frac{1}{2^5} = \frac{1}{2^5} \cdot \frac{5^5}{5^5} = \frac{3125}{10^5} = \frac{3125}{100{,}000} = \mathbf{0.03125}$.

   (e) Not a terminating decimal.

   (f) $\frac{133}{625} = \frac{133}{5^4} \cdot \frac{2^4}{2^4} = \frac{2128}{10^4} = \frac{2128}{10{,}000} = \mathbf{0.2128}$.

8. One hour is 60 minutes, so 7 minutes would be $\frac{7}{60}$ of an hour. This is **nonterminating** decimal because $\frac{7}{60}$ is in simplest form and 3 is a factor of the denominator.

9. Answers may vary. Many values between 0 and 100 are composed of whole-number powers of 2 and 5, thus dividing 100 and being capable of being expressed as a two-digit decimal. These numbers include the coin designations of 1, 5, 10, 25, and 50 cents, but there are others which could have been used, such as 2¢ or 20¢.

10. **(a)** Line up by place value: 13.49190

13.49200

13.49183

13.49199

From greatest to least is:

$\mathbf{13.492 > 13.49199 > 13.4919 > 13.49183}$.

(b) Line up by place value: $^{-}1.45300$

$^{-}1.45000$

$^{-}1.40530$

$^{-}1.49300$

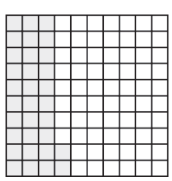
From greatest (i.e., closest to 0) to least is:

$\mathbf{^{-}1.4053 > {}^{-}1.45 > {}^{-}1.453 > {}^{-}1.493}$.

11. **(a)** Fourteen thousandths inch $= \frac{14}{1000}$ inch $=$ **0.014 inch**.

(b) Twenty-four hundredths $= \frac{24}{100} = 0.24$ days. The rotational period is thus **365.24 days**.

12. There are 32 of 100 squares shaded, representing $\frac{32}{100}$ of the whole grid $= 0.32$ of the grid.



13. The largest number is furthest to the right on the number line. Thus $0.804 < 0.8399 <$ **0.84**.

14. Answers may vary. A decimal carried to the ten-thousandths place would have four digits to the right of the decimal point, and a number between 8.3400 and 8.3410 might be 8.3401 or 8.3405.

15. **(a)** Answers may vary. One method could be to find the difference, not matter how slight, between the two decimal numbers and add some fraction of that to the smaller.

(b) Part (a) is a recursive process; no matter how small the difference between the terminating decimals, that differcence can be divided.

16. A meaning could be: $\mathbf{3 \cdot 6^0 + 1 \cdot 6^{-1} + 4 \cdot 6^{-2} + 5 \cdot 6^{-3}}$ in base ten. In base 6, 10_{six} is used instead of 6, so the numeral would be $3 \cdot (10_{six})^0 + 1 \cdot (10_{six})^{-1} + 4 \cdot (10_{six})^{-2} + 5 \cdot (10_{six})^{-3}$.

17. **Rhonda**, **Martha**, **Kathy**, **Molly**, **Emily**, because 63.54 (Rhonda) $<$ 63.59 (Martha) $<$ 64.02 (Kathy) $<$ 64.46 (Molly) $<$ 64.54 (Emily).

18. Using our knowledge of mixed number from the previous chapters, we can write: $\frac{1\frac{1}{2}}{16} = \frac{1.5}{16} = \frac{1.5 \cdot 2}{16 \cdot 2} = \frac{3}{32}.\ \frac{3}{32} = \frac{3 \cdot 5^5}{2^5 \cdot 5^5} = \mathbf{0.09375}$.

Assessment 7-1B

1. **(a)** $0.045 = \mathbf{0 \cdot 10^0 + 0 \cdot 10^{-1} + 4 \cdot 10^{-2} + 5 \cdot 10^{-3}}$.

(b) $103.03 = \mathbf{1 \cdot 10^2 + 0 \cdot 10^1 + 3 \cdot 10^0 + 0 \cdot 10^{-1} + 3 \cdot 10^{-2}}$.

(c) $245.6701 = \mathbf{2 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0 + 6 \cdot 10^{-1} + 7 \cdot 10^{-2} + 0 \cdot 10^{-3} + 1 \cdot 10^{-4}}$.

(d) $0.00034 = \mathbf{0 \cdot 10^0 + 0 \cdot 10^{-1} + 0 \cdot 10^{-2} + 0 \cdot 10^{-3} + 3 \cdot 10^{-4} + 4 \cdot 10^{-5}}$.

2. **(a)** $5000 + 200 + 0.4 = \mathbf{5200.4}$.

(b) $0.004 + 20{,}000 = \mathbf{20{,}000.004}$.

(c) $200 + 30{,}000 = \mathbf{30{,}200}$.

(d) $10^{-3} + 10^{-5} = 0.001 + 0.00001 = 0.00101$.

3. **(a)** **2.027**

(b) **2000.027**

(c) **2020.007**

(d) **0.00004**

4. **(a)** $0.45 =$ **forty-five hundredths**.

(b) $2.035 =$ **two and thirty-five thousandths**.

(c) $45.0006 =$ **forty-five and six ten-thousandths**. Six ten-thousandths $= 6 \cdot 10^{-4}$.

(d) $0.0000445 =$ **four hundred forty-five ten-millionths**. Four hundred forty five ten-millionths $= 4 \cdot 10^{-5} + 4 \cdot 10^{-6} + 5 \cdot 10^{-7}$.

5. **(a)** $28.32 = 28 + \frac{32}{100} = \frac{2800}{100} + \frac{32}{100} = \frac{2832}{100} = \frac{708 \cdot 4}{25 \cdot 4} = \mathbf{\frac{708}{25}}$.

(b) $34.1736 = 34 + \frac{1736}{10{,}000} = \frac{340{,}000}{10{,}000} + \frac{1736}{10{,}000} = \frac{341{,}736}{10{,}000} = \frac{42{,}717 \cdot 8}{1250 \cdot 8} = \mathbf{\frac{42{,}717}{1250}}$.

(c) $^{-}27.32 = {}^{-}\left(27 + \frac{32}{100}\right) = {}^{-}\left(\frac{2700}{100} + \frac{32}{100}\right) = {}^{-}\left(\frac{2732}{100}\right) = {}^{-}\left(\mathbf{\frac{683}{25}}\right)$.

6. A rational number in simplest form can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2 or 5:

(a) **Terminating**. The denominator in reduced form contains only 2 as a prime factor.

(b) **Terminating**. The denominator contains only 2 as a prime factor.

(c) **Terminating**. The denominator contains only 5 as a prime factor.

(d) **Nonterminating**. The denominator contains 17 as a prime factor.

(e) **Terminating**. The denominator contains only 5 as a prime factor.

(f) **Terminating**. $\frac{14}{35}$ reduces to $\frac{2}{5}$. Since the reduced fraction has only a five in its denominator, the fraction terminates.

7. **(a)** $\frac{4}{8} = \frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = \mathbf{0.5}$.

(b) $\frac{1}{2^6} = \frac{1}{2^6} \cdot \frac{5^6}{5^6} = \frac{15{,}625}{10^6} = \frac{15{,}625}{1{,}000{,}000} = \mathbf{0.015625}$.

(c) $\frac{137}{625} = \frac{137}{5^4} = \frac{137}{5^4} \cdot \frac{2^4}{2^4} = \frac{2192}{10^4} = \frac{2192}{10{,}000} = \mathbf{0.2192}$.

(d) Nonterminating.

(e) $\frac{3}{25} = \frac{3}{5^2} = \frac{3}{5^2} \cdot \frac{2^2}{2^2} = \frac{12}{10^2} = \frac{12}{100} = \mathbf{0.12}$.

(f) $\frac{14}{35} = \frac{2}{5} = \frac{4}{10} = \mathbf{0.4}$.

8. All parts of an hour having a reduced fraction without 3 as a factor in the denominator; i.e., having a numerator divisible by 3: **0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57**.

9. **(a)** If a block is represented by $\frac{1}{10}$, there are 1000 cubes in a base-ten block, so each cube would have a value of $\frac{1}{10} \div 1000 = \frac{1}{10{,}000}$, or **one ten-thousandth**.

(b) If each cube has a value of $\frac{1}{10{,}000}$, and there are 100 cubes in a flat, each flat would have a value of $100 \cdot \frac{1}{10{,}000} = \frac{1}{100}$, or **one hundredth**.

(c) One long is composed of 10 cubes, thus a long would have a value of $10 \cdot \frac{1}{10{,}000} = \frac{1}{1000}$, or **one thousandth**.

(d) Three blocks, 1 long, and 4 cubes $= 3 \cdot \frac{1}{10} + 1 \cdot \frac{1}{1000} + 4 \cdot \frac{1}{10{,}000} = 0.3 + 0.001 + 0.0004 = \mathbf{0.3014}$, or **three thousand fourteen ten-thousandths**.

10. In each part, insert following zeros to make each value have the same number of digits.

(a) Line up by place value:

24.94190
24.94200
24.94189
24.94199

From least to greatest is:
$\mathbf{24.94189 < 24.9419 < 24.94199 < 24.942}$.

(b) Line up by place value:

$^{-}34.2500$
$^{-}34.2510$
$^{-}34.2050$
$^{-}34.2519$

From least to greatest is:

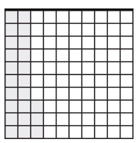
$\mathbf{^{-}34.2519 < {}^{-}34.251 < {}^{-}34.25 < {}^{-}34.205}$.

(The least numbers are those furthest from 0.)

11. **(a)** $\frac{1}{16} = \frac{1}{2^4} = \frac{5^4}{2^4 \cdot 5^4} = \frac{625}{10{,}000} = \mathbf{0.0625}$.

(b) $224 + \frac{7006}{10{,}000} = 224.7006$.

12. There are 23 of 100 squares shaded, representing $\frac{23}{100}$ of the whole grid $= 0.23$ of the grid.



13. $0.8114 = \frac{8114}{10{,}000}$, $0.8119 = \frac{8119}{10{,}000}$, and $0.82 = \frac{8200}{10{,}000}$. Since $8114 < 8119 < 8200$, $\frac{8200}{10{,}000} =$ **0.82** is furthest to the right. Alternetively, .82 has two hundredths while the other two fractions have one hundredth.

14. Answers may vary. Write the decimals as 8.3450 and 8.3456; a decimal between them could be **8.34553**.

15. Answers may vary, but for example just as there are infinitely many rational numbers of the form $\frac{a}{b}$ where $a, b \in I$ and $b \neq 0$, there are infinitely many terminating decimals. E.g., just as between 0.0625 and 0.125. (representing $\frac{1}{16}$ and $\frac{1}{8}$, respectively), $\frac{1}{16} = \frac{2}{32}$ and $\frac{1}{8} = \frac{2}{16} = \frac{4}{32}$, so $\frac{3}{32} = 0.09375$ is between 0.0625 and 0.125. This process can continue indefinitely.

16. $0.00334_{seven} = \mathbf{0 \cdot 7^1 + 0 \cdot 7^{-1} + 0 \cdot 7^{-2} + 3 \cdot 7^{-3} + 3 \cdot 7^{-4} + 4 \cdot 7^{-5}}$. This is the number in base ten. To remain in base-seven, replace 7 with 10_{seven}.

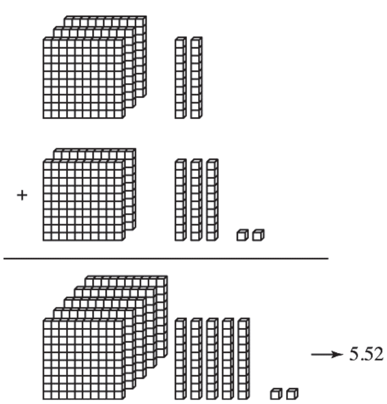
17. Sort first by the tens values, then by the ones values, then by the tenths, and finally by the hundredths. **Ricky**, **Michael**, **Karl**, **Marius**, **Eddie**.

18. Change each sixteenth to a decimal by dividing the numerators, 1 through 16, by the denominator, 16. (Each will be a terminating decimal.)

Rule Mark	Decimal Equivalent
0	0
$\frac{1}{16}$	0.0625
$\frac{2}{16}\left(=\frac{1}{8}\right)$	0.125
$\frac{3}{16}$	0.1875
$\frac{4}{16}\left(=\frac{1}{4}\right)$	0.25
$\frac{5}{16}$	0.3125
$\frac{6}{16}\left(=\frac{3}{8}\right)$	0.375
$\frac{7}{16}$	0.4375
$\frac{8}{16}\left(=\frac{1}{2}\right)$	0.5
$\frac{9}{16}$	0.5625
$\frac{10}{16}\left(=\frac{5}{8}\right)$	0.625
$\frac{11}{16}$	0.6875
$\frac{12}{16}\left(=\frac{3}{4}\right)$	0.75
$\frac{13}{16}$	0.8125
$\frac{14}{16}\left(=\frac{7}{8}\right)$	0.875
$\frac{15}{16}$	0.9375
$\frac{16}{16}$	1.0

Assessment 7-2A: Operations on Decimals

1.



2. **(a)** $\frac{86}{10}+\frac{231}{10}+\frac{92}{100}=\frac{860}{100}+\frac{2310}{100}+\frac{92}{100}=\frac{3262}{100}$
$= 32.62.$

(b) $\frac{232}{100}+\frac{21008}{1000}=\frac{2320}{1000}+\frac{21008}{1000}$.
$=\frac{23328}{1000}=23.328$

3. **(a)** $0.5 \bullet 0.6=\frac{5}{10}\bullet\frac{6}{10}=\frac{30}{100}=0.30.$

(b) $203 \bullet 0.03=203 \bullet \frac{3}{100}=\frac{609}{100}=6.09.$

(c) $0.003 \bullet 0.006=\frac{3}{1000}\bullet\frac{6}{1000}=\frac{18}{1,000,000}$
$=0.000018.$

4. Answers vary.

(a) 934.23 has 2 decimal places, and 3.2 has 1 decimal place so we move 2 + 1 = 3 decimal places: 2989.536.

(b) The product is 900 • 3 = 2700, so the decimal point is placed as 2989.436.

5. **(a)** $6 \bullet 854.14 (\approx 6 \bullet 850 \approx 5100) = 5124.84$

(b) $81.6 \bullet 212.34 (\approx 80 \bullet 200 \approx 16,000)$
$= 17,326.944$

(c) $\frac{137.025}{1.75}(\approx \frac{140}{2} \approx 70) = 78.3$.

6. Dividing each number by 0.01 is equal to multiplying each number by $\frac{1}{100}$ which is equal to dividing each number by 100.

(a) $\frac{2586}{100}=25.86$

(b) $\frac{34.79}{100}=0.3479$

(c) $\frac{0.24}{100}=0.0024$

(d) $\frac{0.0037}{100}=0.000037$.

7. **(a)** When you multiply a natural number by a decimal the result is less than the natural number if the decimal is less than 1. For example $5 \bullet 0.5 = 2.5$ and $2.5 < 5$.

(b) When you multiply a natural number by a decimal the result is greater than the natural number when the decimal is greater than 1. For example $5 \bullet 1.5 = 7.5$ and $7.5 > 5$.

8. She bought a total of:

$17.95
13.59
14.86
179.98
2.43
2.43
$231.24

in her shopping (excluding sales tax)

9. **(a)** Sum along the diagonal, yielding 16.5. In rows and columns with two figures subtract their sum from 16.5 to obtain the missing element:

8.2	**1.9**	**6.4**
3.7	5.5	**7.3**
4.6	9.1	2.8

(b) **Yes**.

(c) Each row, column, and diagonal will have $3 \cdot 0.85 = 2.55$ added to it, for a sum of $16.5 + 2.55 = \mathbf{19.05}$.

10. Let $x =$ the price per pound of the other type of fruit. The total cost of the fruit is $25 \cdot 4 + 15 \cdot 2 + 10x = 130 + 10x$ dollars. The total weight of the fruit is $25 + 15 + 10 = 50$ lbs.

To find the average cost per pound divide the total cost by the total weight. This will equal $3.50 per lb.

$\frac{130+10x}{50} = 3.50 \Rightarrow \frac{130+10x}{50} = \frac{350}{100}$
$\Rightarrow 130 + 10x = \frac{350}{100} \cdot 50 \Rightarrow 130 + 10x = 175$
$\Rightarrow 10x = 45 \Rightarrow x = \mathbf{4.5\ lbs}.$

11. The ratio $\dfrac{2.54\,\text{cm}}{1\,\text{in}}$ is equal to 1. Thus,

$$\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)^3 = 1\cdot\left(\frac{2.54\,\text{cm}}{\text{in}}\right)^3$$
$$= \frac{254}{100}\cdot\frac{254}{100}\cdot\frac{254}{100}\,\frac{\text{cm}^3}{\text{in}^3}$$
$$= \frac{16{,}387{,}064}{1{,}000{,}000}\,\frac{\text{cm}^3}{\text{in}^3}$$
$$= 16.387064\,\frac{\text{cm}^3}{\text{in}^3}$$
$$390\ \text{in}^3\cdot 16.387064\,\frac{\text{cm}^3}{\text{in}^3} \approx \mathbf{6{,}391\ cm^3}.$$

12. Let p be the price of the stock. Then $p + 0.24 = 73.245 \Rightarrow p = 73.245 - 0.24$, or

$$\begin{array}{r} 73.245 \\ -0.240 \\ \hline \mathbf{73.005} \end{array}$$

13. If one U.S. dollar is valued at 1.046 Canadian, then \$27.32 U.S. would be valued at $27.32 \times 1.046 = \$28.57672.$ which rounds to **\$28.58** Canadian.

14. **(a)** $\dfrac{3\text{ heaters}}{1}\cdot\dfrac{1200\text{ w}}{1\text{ heater}}\cdot\dfrac{24\text{ hrs}}{1\text{ day}}\cdot\dfrac{1\text{ kw}}{1000\text{ w}}\cdot\dfrac{\$0.06715}{\text{kw-hr}} = \$5.80176$ which rounds to **\$5.80**. (The actual bill would be lumped with other appliance usage.)

(b) 75 watts $= 0.075$ kilowatts. 0.075 kw $\times$ 1 hour $\times$ \$0.06715 per kw-hr $= \$0.00503625$ to operate one bulb for one hour. $\$1.00 \div \$0.00503625 =$ **199 hours** (rounded to the nearest hour).

15. If one liter is 4.224 cups, then the number in 36.5 cups is $36.5 \div 4.224 \approx$ **8.64 liters**.

16. **(a)** There is a difference of 0.9 between each element of the sequence, thus it is arithmetic. The next three elements are: $4.5 + 0.9 =$ **5.4**, $5.4 + 0.9 =$ **6.3**, $6.3 + 0.9 =$ **7.2**.

(b) There is a difference of 0.2 between each element of the sequence, thus it is arithmetic. The next three elements are: $1.1 + 0.2 =$ **1.3**, $1.3 + 0.2 =$ **1.5**, $1.5 + 0.2 =$ **1.7**.

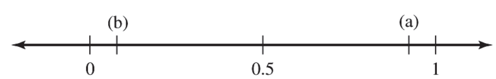
17. If: $a_1 = 0.9$;
$a_2 = 0.9\cdot 0.2 = 0.18$;
$a_3 = 0.18\cdot 0.2 = 0.036$;
$a_4 = 0.036\cdot 0.2 = 0.0072$; and
$a_5 = 0.0072\cdot 0.2 = 0.00144$:
Their sum is **1.12464**.

18. A finite geometric sequence is one with a constant ratio between terms and a finite number of terms. 0.2222 can be expressed as a sum: $\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10{,}000}$; note that the denominators are powers of ten. Thus to make it a geometric sequence with constant ratio $\frac{1}{10}$, it could be written as $\frac{2}{10} + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right)^2 + \left(\frac{2}{10}\right)\left(\frac{1}{10}\right)^3$, or $\mathbf{0.2 + (0.2)(0.1) + (0.2)(0.1)^2 + (0.2)(0.1)^3}$.

19. **(a)** $0.3 \div 0.31$ is close to 1.

(b) $0.3 \cdot 0.31$ is close to 0.09.

Thus on the numbers line:



20. The bank is **not correct**. The total of outstanding checks is:

$$\begin{array}{r} \$3.21 \\ 14.56 \\ 12.44 \\ 6.98 \\ 9.51 \\ 7.49 \\ \hline \$54.19 \end{array}$$

Adding the total of outstanding checks to the checkbook balance gives $\$54.19 + 21.69 = \$75.88$, which differs from the bank statement by \$7.74. The bank balance is too high.

21. $1776\,ft \bullet 0.005 = 8.88\,ft$ The height of the model will be 8.88 ft.

22. $102.3\,cm \bullet 1.5 = 153.45\,cm$ If Allie is 102.3 cm now she will be 153.45 cm when she is twelve years old.

23. **(a)** $\$150 \cdot 99.925 = 14988.75 yen$. If \$1 is worth 99.925 yen then \$150 is worth 14,988.75 yen.

(b) If \$1 is worth 99.925 yen then 1 yen is worth \$0.010007506 and 3500 yen is worth approximately \$35.03.

24. If there are 2.54 cm in 1 inch then there are 0.393700787 inches in 1 cm.

(a) 35 cm would roughly equal 14 inches.

(b) 35 cm • 0.393700787 = 13.78 inches.

(c) 154.5 cm • 0.393700787 = 60.83 inches.

25. If 1 km is approximately 0.62 miles then 1 mile is approximately 1.612903226 km.

(a) 30 km • 0.62 = 18.6 miles.

(b) 90 km/hr • 0.62 = 55.8 mi/hr.

(c) 75 mi/hr • 1.612903226 = 120.97 km/hr.

26. **(a)** $3.2 \div 10^9 = \mathbf{0.0000000032}$; i.e., move the decimal point 9 places to the left.

(b) $3.2 \cdot 10^9 = \mathbf{3,200,000,000}$; i.e., move the decimal point 9 places to the right.

(c) $4.2 \div 10^1 = \mathbf{0.42}$; i.e., move the decimal point 1 place to the left.

(d) $6.2 \cdot 10^5 = \mathbf{620,000}$; i.e., move the decimal point 5 places to the right.

27. **(a)** Move the decimal point 7 places to the left to obtain the product of a number between 1 and 10 and an integer power of 10. I.e., 1.27 multiplied by 10^7, or about $\mathbf{1.27 \cdot 10^7\,m}$.

(b) Move the decimal point 9 places to the left and multiply the result by 10^9 to obtain about $\mathbf{4.486 \cdot 10^9\,km}$.

(c) Move the decimal point 7 places to the left and multiply the result by 10^7 to obtain about $\mathbf{5 \cdot 10^7\,cans}$.

28. **(a)** $8.56 = 3 - 2x \Rightarrow 8.56 - 3 = 2x \Rightarrow$

$5.56 = {}^-2x \Rightarrow x = \frac{5.56}{{}^-2} = \frac{{}^-556}{100} \cdot \frac{1}{2} =$

$\frac{{}^-278}{100} = \mathbf{{}^-2.78}$.

(b) $2.3x - 2 = x + 2.55 \Rightarrow 2.3x - x = 2 + 2.55 \Rightarrow 1.3x = 4.55 \Rightarrow x = \frac{4.55}{1.3} = \frac{455}{130} = 3.5.$

29. **(a)** Move the decimal point ten places to the left to obtain **0.000000000753 g**.

(b) Move the decimal point five places to the right to obtain **298,000 km per sec**.

(c) Move the decimal point eight places to the right to obtain **778,570,000 km**.

30. **(a)** $(8 \cdot 10^{12}) \cdot (6 \cdot 10^{15}) = 8 \cdot 6 \cdot 10^{12} \cdot 10^{15} = 48 \cdot 10^{27} = \mathbf{4.8 \cdot 10^{28}}$.

(b) $(16 \cdot 10^{12}) \div (4 \cdot 10^5) = \frac{16}{4} \cdot \frac{10^{12}}{10^5} = 4 \cdot 10^{12} \cdot 10^{{}^-5} = \mathbf{4 \cdot 10^7}$.

(c) $(5 \cdot 10^8) \cdot (6 \cdot 10^9) \div (15 \cdot 10^{15}) = \frac{5 \cdot 6}{15} \cdot 10^8 \cdot 10^9 \cdot 10^{{}^-15} = \mathbf{2 \cdot 10^2}$.

31. **(a)** $\underline{2}$03.651 is between 200 and 300 and is closer to 200, so round down to **200**.

(b) 2$\underline{0}$3.651 is between 200 and 210 and is closer to 200, so round down to **200**.

(c) 20$\underline{3}$.651 is between 203 and 204 and is closer to 204, so round up to **204**.

(d) 203.$\underline{6}$51 is between 203.6 and 203.7 and is closer to 203.7, so round up to **203.7**.

(e) 203.6$\underline{5}$1 is between 203.65 and 203.66 and is closer to 203.65, so round down to **203.65**.

32. $\frac{243\text{ miles}}{12\text{ gallons}} = 20.25$ **mpg**.

33. Camera rounds to \$55, film rounds to \$5, and case rounds to \$18. Total estimated cost is $55 + 5 + 18 = \mathbf{\$78}$.

34. Answers may vary:

(a) **(*i*)** 65.84 rounds to 66;
24.29 rounds to 24;
12.18 rounds to 12;
19.75 rounds to 20;
Rounded estimate $= 66 + 24 + 12 + 20 = \mathbf{122}$.

(*ii*) Lead digits sum to $60 + 20 + 10 + 10 = 100$; adjustments are about $6 + 4 + 2 + 10 = 22$. Front-end estimate $= 100 + 22 =$ **122**.

(*iii*) Actual sum $=$ **122.06**.

(b) (*i*) 89.47 rounds to 89;
32.16 rounds to 32.
Rounded estimate $= 89 - 32 =$ **57**.

(*ii*) Lead digit difference is $80 - 30 = 50$; adjustments are about $9 - 2 = 7$. Front-end estimate $= 50 + 7 =$ **57**.

(*iii*) Actual difference $=$ **57.31**.

(c) (*i*) 5.85 rounds to 6;
6.13 rounds to 6;
9.10 rounds to 9;
4.32 rounds to 4.
Rounded estimate $= 6 + 6 + 9 + 4 =$ **25**.

(*ii*) Lead digits sum to $5 + 6 + 9 + 4 = 24$; adjustments are about $1 + 0 + 0 + 0 = 1$.
Front-end estimate $= 24 + 1 =$ **25**.

(*iii*) Actual sum $=$ **25.40**.

(d) (*i*) 223.75 rounds to 224;
87.60 rounds to 88.
Rounded estimate $= 224 - 88 =$ **136**.

(*ii*) Lead digit difference $= 200 - 80 = 120$; adjustments are about $24 - 8 = 16$. Front-end estimate $= 120 + 16 =$ **136**.

(*iii*) Actual difference $=$ **136.15**.

35. To find the greatest products use the largest digits in the highest place values:

(*i*) Least: $\boxed{2}.\boxed{3} \times \boxed{1} = \mathbf{2.3}$.

(*ii*) Greatest: $\boxed{8}.\boxed{7} \times \boxed{9} = \mathbf{78.3}$.

36. To find the greatest possible number use the largest digits in the highest place values:

$$4\boxed{9}\,\boxed{7}\,3\boxed{6}\cdot\boxed{5}\,\boxed{2}\,8\boxed{1}.$$

37. Answers may vary. E.g., $40 \cdot \$8 + 40 \cdot \$\left(\frac{1}{4}\right) = \$320 + \$10 = \mathbf{\$330}$.

38. **(a)** $8.4 \cdot 6 = 4.2 \cdot \underline{\mathbf{12}} \left[\frac{8.4}{\cancel{2}} \cdot (6 \cdot \cancel{2}) = 8.4 \cdot 6\right]$.

(b) $10.2 \div 0.3 = 20.4 \div \underline{\mathbf{0.6}}\left(\frac{10.2\cdot\cancel{2}}{0.3\cdot\cancel{2}} = \frac{10.2}{0.3}\right)$.

(c) $a \cdot b = \frac{a}{2} \cdot \underline{\mathbf{2b}}\left(\frac{a}{\cancel{2}} \cdot \frac{\cancel{2}b}{1} = a \cdot b\right)$.

(d) $a \div b = 2a \div \underline{(\mathbf{2b})}\left(\frac{\cancel{2}a}{\cancel{2}b} = \frac{a}{b}\right)$.

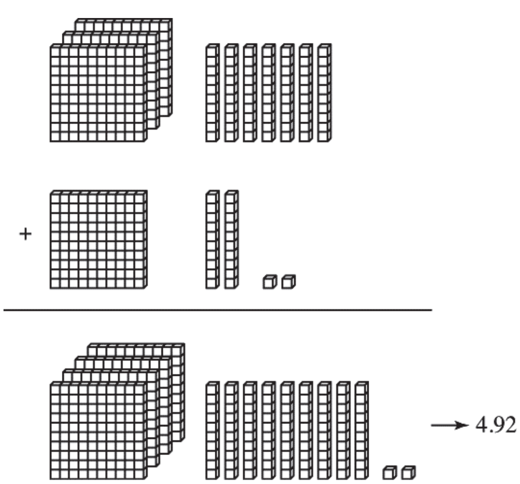
39. $\frac{7}{0.25} = \frac{70}{2.5} = \frac{700}{25}$. Thus, ***a*, *b*, and *d*** have equal quotients.

40. **(a)** $2 \cdot 1 + 0.25 = 2.25 = 1.5^2$
$3 \cdot 2 + 0.25 = 6.25 = 2.5^2$
A conjecture is that the next two are:
$4 \cdot 3 + 0.25 = (3.5)^2$ and $5 \cdot 4 + 0.25 = (4.5)^2$.

(b) Since $1.5 = 2 - .5$, $2.5 = 3 - .5$, $3.5 = 4 - .5$, and $4.5 = 5 - .5$,
We conjecture that for each natural numbers, $n\,(n - 1) + 0.25 = (n - .5)^2$.
This is true since
$(n - .5)^2 = (n - .5)(n - .5)$
$= n^2 - .5n - .5n + .25 = n^2 - n + .25 = n(n - 1) + .25$.

Assessment 7-2B

1.



2. **(a)** $\frac{53}{10}+\frac{132}{10}+\frac{86}{100}=\frac{530}{100}+\frac{1320}{100}+\frac{86}{100}=\frac{1936}{100}$ $=19.36.$

(b) $\frac{542}{100}+\frac{32005}{1000}=\frac{5420}{1000}+\frac{32005}{1000}$.
$=\frac{37425}{1000}=37.425$

3. **(a)** $0.3 \cdot 0.8=\frac{3}{10} \cdot \frac{8}{10}=\frac{24}{100}=0.24.$

(b) $502 \cdot 0.04=502 \cdot \frac{4}{100}=\frac{2008}{100}=20.08.$

(c) $0.004 \cdot 0.06=\frac{4}{1000} \cdot \frac{6}{100}=\frac{24}{100{,}000}$
$=0.00024.$

4. Answers vary.

(a) 564.23 has 2 decimal places, and 6.7 has 1 decimal place so we move 2 + 1 = 3 decimal places: 3780.341.

(b) The product is 600 • 7 = 4200, so the decimal point is placed as 3780.341.

5. **(a)** $5 \cdot 254.15(\approx 5 \cdot 250 \approx 1250)=1270.75$

(b) $31.6 \cdot 122.34(\approx 30 \cdot 120 \approx 3600)=3865.944$

(c) $\frac{813.45}{4.25}(\approx \frac{800}{4} \approx 200)=191.4$.

6. Dividing each number by 0.001 is equal to multiplying each number by $\frac{1}{1000}$ which is equal to dividing each number by 1000.

(a) $\frac{5280}{1000}=5.280$

(b) $\frac{54.56}{1000}=0.05456$

(c) $\frac{0.275}{1000}=0.000275$

(d) $\frac{0.78}{1000}=0.00078$ **.**

7. When you multiply a positive decimal less than 1 by a decimal greater than 1 the results is not always greater than 1. For example $0.5 \cdot 1.5=0.75$ and 0.75 < 1.

8. He bought a total of:

$$\begin{array}{r} \$4.99 \\ 0.79 \\ 49.99 \\ \underline{1.49} \\ \$57.26 \end{array}$$

in his shopping (excluding sales tex).

9. **(a)** Sum along the diagonal, yielding 14.4. In rows and columns with two figures subtract their sum from 14.4 to obtain the missing element:

7.5	**1.2**	**5.7**
3.0	4.8	**6.6**
3.9	8.4	2.1

(b) **Yes**, the square is still magic. For example, the sum of the first row is $.5 \cdot 7.5 + .5 \cdot 1.2 + .5 \cdot 5.7 = .5(7.5 + 1.2 + 5.7) = .5(14.4) = \mathbf{7.2}$.

This key idea example demonstrates that if we multiply all the cells by .5 the sum of any row, column, or diagonal is halved.

10. There would be a total of $30 + 20 + 10 =$ 60 pounds of nuts; at an average price per pound of \$4.50 he would pay a total of $60 \cdot 4.50 = \$270$. He has already paid $\$3.00 \cdot 30 + \$5.00 \cdot 20 = \$190.00$, so he has $\$270.00 - 190.00 =$ **\$80.00** to pay for 10 pounds.

11. $3000 \text{ cm}^3 \div 16.387064 \text{ cm}^3 \text{ per in}^3 \approx \mathbf{183\ in.}^3$ (to the nearest in^3).

(*i*) Kluft versus Sokolova: $\begin{array}{r} 48.89 \\ \underline{-47.86} \\ \mathbf{1.03\ m} \end{array}$

(*ii*) Kluft versus Burrell: $\begin{array}{r} 48.89 \\ \underline{-47.69} \\ \mathbf{1.20\ m} \end{array}$

12. $\begin{array}{r} \$63.28 \\ \underline{-27.45} \end{array}$

\$35.83 was the loss.

13. The ratio of Canadian dollars to U.S. dollars is $\frac{1.046}{1}$ Thus. 28.43 U.S. dollars $= 28.43 \cdot \frac{1.046}{1}$ $= 29.74$ **Canadian dollars**.

14. 18.5 cups $=$ 18.5 cups $\cdot \frac{1\text{ quarts}}{4\text{ cups}} = \frac{185}{40}$ quarts $=$ $\frac{37}{8}$ quarts $=$ **4.625 quarts**.

15. **(a)** Each element of the sequence is 0.5 times the previous element, thus it is geometric. The next three elements are: $0.125 \cdot 0.5 = \mathbf{0.0625}$, $0.0625 \cdot 0.5 = \mathbf{0.03125}$, $0.03125 \cdot 0.5 =$ $\mathbf{0.015625}$.

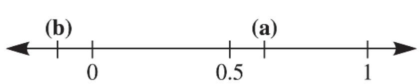
(b) There is a difference of 1.3 between each element of the sequence, thus it is arithmetic. The next three elements are: $5.4 + 1.3 =$ $\mathbf{6.7}$, $6.7 + 1.3 = \mathbf{8.0}$, $8.0 + 1.3 = \mathbf{9.3}$.

16. If $a_1 = 0.4$;
$a_2 = 0.4 \cdot 0.3 = 0.12$;
$a_3 = 0.12 \cdot 0.3 = 0.036$;
$a_4 = 0.036 \cdot 0.3 = 0.0108$;
$a_5 = 0.0108 \cdot 0.3 = 0.00324$;
Their sum is **0.57004**.

17. $0.3333333 = 0.3 + 0.03 + 0.003 + \cdots +$ 0.0000003 or $\frac{3}{10} + \frac{3}{10}\left(\frac{1}{10}\right) + \frac{3}{10}\left(\frac{1}{10}\right)^2 + \cdots +$ $\frac{3}{10}\left(\frac{1}{10}\right)^6$. This is the finite sum of a geometric sequence with $a_1 = 0.3$, $n = 7$, $a_7 = 0.0000003$, and $r = \frac{0.03}{0.3} = \frac{1}{10} = 0.1$.

18. **(a)** $0.3 + 0.31$ is close to 0.6.

(b) $0.3 - 0.31$ is close to 0. Thus on the number line:



19. 18 shares $\cdot$ \$61.48 $=$ \$1106.64 net return on first transaction.
350 shares $\cdot$ \$85.35 $-$ \$495 commission $=$ \$29,377.50 net return on second transaction.
Total net return $=$ \$1106.64 $+$ 29377.50 $=$ \$30,484.14.
Total cost $=$ \$964.00 $+$ 27,422.50 $=$ \$28,386.50.
Profit $=$ \$30,484.14 $-$ 28,386.50 $=$ \$2097.64, which rounds to **\$2098**.

20. $1451\,ft \cdot 0.005 = 7.255\,ft$ The height of the model will be 7.255 ft.

21. $105.2\,cm \cdot 1.5 = 157.8\,cm$ If Allie is 105.2 cm now she will be 157.8 cm when she is twelve years old.

22. **(a)** $\$200 \cdot 99.925 = 19,985\,yen$. If \$1 is worth 99.925 yen then \$200 is worth 19.985 yen.

(b) If \$1 is worth 99.925 yen then 1 yen is worth \$0.010007506 and 7000 yen is worth approximately \$70.05.

23. If there are 2.54 cm in 1 inch then there are 0.393700787 inches in 1 cm.

(a) 160 cm would roughly equal 60 inches.

(b) 160 cm • 0.393700787 = 62.99 inches.

(c) 182 cm • 0.393700787 = 71.65 inches.

24. If 1 km is approximately 0.62 miles then 1 mile is approximately 1.612903226 km.

(a) 50 km • 0.62 = 31 miles.

(b) 120 km/hr • 0.62 = 74.4 mi/hr.

(c) 50 mi/hr • 1.612903226 = 80.65 km/hr.

25. **(a)** $3.5 \cdot 10^7 = \mathbf{35{,}000{,}000}$; i.e., move the decimal point seven places to the right.

(b) $3.5 \cdot 10^{-7} = \mathbf{0.00000035}$; i.e., move the decimal point seven places to the left.

(c) $^-2.4 \cdot 10^{-3} = {}^-\mathbf{0.0024}$; i.e., move the decimal point three places to the left.

26. **(a)** Move the decimal point five places to the left to obtain the product of a number between l and 10 and an integer power of 10, i.e., 9.98199 multiplied by 10^5, or $9.98199 \cdot 10^5$ people.

(b) Move the decimal point seven places to the left and multiply the result by 10^7 to obtain $\mathbf{2.449 \cdot 10^7\ mi^2}$.

27. **(a)** $2x + 1.3 = 4.1 \Rightarrow 2x = 2.8 \Rightarrow$ $\boldsymbol{x = 1.4}$.

(b) $4.2 - 3x = 10.2 \Rightarrow {}^{-}3x = 6.0 \Rightarrow$

$\boldsymbol{x = {}^{-}2.0}$.

28. **(a)** Move the decimal point 6 places to the left to obtain **0.0000044 sec**.

(b) Move the decimal point 4 places to the right to obtain about **19,900 km**.

(c) Move the decimal point 9 places to the right to obtain approximately **3,000,000,000 years**.

29. **(a)** $(5 \cdot 10^7)(7 \cdot 10^{12}) = 5 \cdot 7 \cdot 10^7 \cdot 10^{12}$

$= 35 \cdot 10^{19} = \mathbf{3.5 \cdot 10^{20}}$.

(b) $({}^{-}13 \cdot 10^4) \div 65 = \frac{{}^{-}13}{65} \cdot 10^4 =$

$\frac{{}^{-}1}{5} \cdot 10^4 = {}^{-}0.2 \cdot 10^4 = \mathbf{{}^{-}2.0 \cdot 10^3}$.

(c) $(3 \cdot 10^7)(4 \cdot 10^5) \div (6 \cdot 10^{-7}) =$

$\frac{3 \cdot 4}{6} \cdot 10^7 \cdot 10^5 \cdot 10^7 = \mathbf{2.0 \cdot 10^{19}}$.

30. **(a)** $\underline{7}15.04$ is between 700 and 800 and is closer to 700, so round down to **700**.

(b) $715.\underline{0}4$ is between 715.0 and 715.1 and is closer to 715.0, so round down to **715.0**.

(c) $71\underline{5}.04$ is between 715 and 716 and is closer to 715, so round down to **715**.

(d) $7\underline{1}5.04$ is between 710 and 720 and is closer to 720, so round up to **720**.

(e) $\underline{0}715.04$ is between 0 and 1000, and is closer to 1000, so round up to **1000**.

31. Distance = speed × time = 55.5 mph × 0.75 hr = **41.625 miles**.

32. Answers may vary. E.g., **10.3** because $9 \cdot 10 = 90$ and $9 \cdot 0.3$ is greater than 2 and less than 4.

33. **(a)** **(*i*)** 47.62 rounds to 48;

27.99 rounds to 28;

13.14 rounds to 13;

7.61 rounds to 8;

Rounded estimate $= 48 + 28 + 13 + 8 =$ **97**.

(*ii*) Lead digits sum to $40 + 20 + 10 + 0 = 70$; adjustments are about $8 + 8 + 3 + 8 = 27$. Front-end estimate $= 70 + 27 =$ **97**.

(*iii*) Actual sum = **96.36**.

(b) **(*i*)** 79.86 rounds to 80;

27.37 rounds to 27.

Rounded estimate $= 80 - 27 =$ **53**.

(*ii*) Lead digit difference is $70 - 20 = 50$; adjustments are about $10 - 7 = 3$. Front-end estimate $= 50 + 3 =$ **53**.

(*iii*) Actual difference = **52.49**.

(c) **(*i*)** 5.85 rounds to 6;

6.17 rounds to 6;

9.10 rounds to 9;

4.23 rounds to 4.

Rounded estimate $= 6 + 6 + 9 + 4 =$ **25**.

(*ii*) Lead digits sum to $5 + 6 + 9 + 4 = 24$; adjustments are about $1 + 0 + 0 + 0 = 1$. Front-end estimate $= 24 + 1 =$ **25**.

(*iii*) Actual sum = **25.35**.

(d) **(*i*)** 232.65 rounds to 230;

78.92. rounds to 80.

Rounded estimate $= 230 - 80 =$ **150**.

(*ii*) Lead digit difference $= 200 - 70 = 130$; adjustments are about $33 - 9 = 24$. Front-end estimate $= 130 + 24 =$ **154**.

(*iii*) Actual difference = **153.73**.

34. To find the least products use the smallest digits in the highest place values. To find the greatest products use the largest digits in the highest place values:

(*i*) Least: $\boxed{1}\,.\,\boxed{3} \times \boxed{2}\,.\,\boxed{4} =$ **3.12**.

(*ii*) Greatest: $\boxed{8}\,.\,\boxed{7} \times \boxed{9}\,.\,\boxed{6} =$ **83.52**.

35. To find the least possible number use the smallest digits in the highest place values:

$4\boxed{1}\boxed{2}3\boxed{5}\,.\,\boxed{6}\boxed{7}8\boxed{9}$.

36. Answers may vary. E.g., $40 \cdot \$6 + 40 \cdot \$\left(\frac{1}{4}\right) =$ $\$240 + \$10 = \mathbf{\$250}$.

37. **(a)** $12.4 \cdot 7 = 6.2 \cdot 2 \cdot 7 = 6.2 \cdot \mathbf{14}$.

(b) $12.4 \div 0.2 = \dfrac{12.4}{2 \cdot 0.1} = \dfrac{6.2}{0.1} = \mathbf{6.2} \div 0.1$.

(c) $ab = a \cdot 10^{-1} \cdot \mathbf{10 \cdot b}$

(d) $12.3 = \dfrac{12.3}{1} = \dfrac{1230}{10^2} = 10^{-2} \cdot \mathbf{1230}$.

38. $9 \div 0.35 = \dfrac{9}{0.35} = \dfrac{90}{3.5} = \dfrac{900}{35} =$ $\dfrac{.9 \cdot 1000}{.035 \cdot 1000} = \dfrac{.9}{.035}$.

Thus, *i*, *ii*, *iii* and *iv* have the same quotient.

39. **(a)** **(*i*)** $1 \cdot 2 + 0.25 = \mathbf{(1.5)^2}$;

(*ii*) $2 \cdot 3 + 0.25 = \mathbf{(2.5)}^2$;

(*iii*) The next two equations might be $3 \cdot 4 + 0.25 = (3.5)^2$ and $4 \cdot 5 + 0.25 = (4.5)^2$.

(b) The above equations are correct.

(c) $n(n+1) + 0.25 = \mathbf{(n + 0.5)^2}$.

Mathematical Connections 7-2: Review Problems

19. $14.0479 = 1 \cdot 10^1 + 4 \cdot 10^0 + 0 \cdot 10^{-1} + 4 \cdot 10^{-2} + 7 \cdot 10^{-3} + 9 \cdot 10^{-4}$.

20. **(a)** **Nonterminating**. The reduced denominator has only 3 as a factor.

(b) **Terminating**. The simplified fraction has a denominator with only 2 as a factor.

21. **Yes**, if the numerator is a multiple of 13. E.g., $\frac{13}{26} = \frac{1}{2}$ in simplest form, and $\frac{1}{2}$ can be written as a terminating decimal.

22. $\frac{35}{56} = \frac{5}{8}$ in simplest form, and 8 in the denominator may be written as 2^3.

Assessment 7-3A: Repeating Decimals

1. The fraction is in simplest form and if the prime factorization of the denominator contains only 2 and/or 5's then the fraction will be a terminating decimal. Since the denominator ends on 9 it is not divisible by 2 or by 5 it will be a non-terminating decimal.

2. **(a)** $0.45\overline{777}, 0.45\overline{77}$, and $0.45\overline{7}$ are all the same number. The only difference is that the repeating block of 7's is indicated in different ways.

(b) The preferred part is the shortest repeating part: $0.45\overline{7}$.

(c) The shortest period in $0.45\overline{7}$ is 1.

3. **(a)** The first ten decimal places of $0.\overline{123} = 0.1231231231$

(b) The first ten decimal places of $0.1\overline{23} = 0.1232323232$

(c) The first ten decimal places of $0.12\overline{3} = 0.1233333333$

(d) The first ten decimal places of $0.123\overline{43} = 0.1234343434$

4. We can write out the first ten decimal to compare:

$0.36\overline{25} = 0.3625252525$

$0.3\overline{625} = 0.3625625625$

and in the fifth decimal place 6>2 so $0.3\overline{625}$ is greater.

5. **(a)** $0.4444... = 0.\overline{4}$ and the period is 1.

(b) $0.36454545... = 0.36\overline{45}$ and the period is 2.

(c) $0.18273273273... = 0.18\overline{273}$ and the period is 3.

6. In each of the following, divide the numerator by the denominator either with a calculator or by use of the division algorithm. The overline in the quotient indicates that the block of digits underneath is repeated an infinite number of times.

(a) $\frac{4}{9} = 4 \div 9 = \mathbf{0.\overline{4}}$.

(b) $\frac{2}{7} = 2 \div 7 = \mathbf{0.\overline{285714}}$.

(c) $\frac{3}{11} = 3 \div 11 = \mathbf{0.\overline{27}}$.

(d) $\frac{1}{15} = 1 \div 15 = \mathbf{0.0\overline{6}}$.

(e) $\frac{2}{75} = 2 \div 75 = \mathbf{0.02\overline{6}}$.

(f) $\frac{1}{99} = 1 \div 99 = \mathbf{0.\overline{01}}$.

(g) $\frac{5}{6} = 5 \div 6 = \mathbf{0.8\overline{3}}$.

(h) $\frac{1}{13} = 1 \div 13 = \mathbf{0.\overline{076923}}$.

(i) $\frac{1}{21} = 1 \div 21 = \mathbf{0.\overline{047619}}$.

(j) $\frac{3}{19} = 3 \div 19 = \mathbf{0.\overline{157894736842105263}}$.

The repetend has more digits than most calculator's display capability. Use the division algorithm, assuming an 8-digit display:

$$
\begin{array}{r}
0.157894736842105263 \\
19\overline{)3.000000000000000000} \\
\underline{2.999986} \\
14000000 \\
\underline{13999998} \\
2000000 \\
\underline{1999997} \\
3
\end{array}
$$

where the last remainder, 3, again begins the repeating pattern. Note that since the denominator is 19, the most digits that can be in the repetend is 18.

7. **(a)** $n = 0.\overline{4}$; a one-digit repetend:

$$
\begin{array}{rcl}
10n & = & 4.4444\ldots \\
{}^{-}n & = & {}^{-}0.4444\ldots \\
\hline
9n & = & 4
\end{array}
$$

$\Rightarrow n = \mathbf{\frac{4}{9}}$.

(b) $n = 0.\overline{61}$; a two-digit repetend:

$$
\begin{array}{rcl}
100n & = & 61.6161\ldots \\
{}^{-}n & = & {}^{-}0.6161\ldots \\
\hline
99n & = & 61
\end{array}
$$

$\Rightarrow n = \mathbf{\frac{61}{99}}$.

(c) $n = 1.3\overline{96}$; a two-digit repetend:

$$
\begin{array}{rcl}
1000n & = & 1396.9696\ldots \\
{}^{-}10n & = & {}^{-}13.9696\ldots \\
\hline
990n & = & 1383
\end{array}
$$

$\Rightarrow n = \frac{1383}{990} = \mathbf{\frac{461}{330}}$.

(d) $n = 0.\overline{55} = 0.\overline{5}$; a one-digit repetend:

$$
\begin{array}{rcl}
10n & = & 5.5555\ldots \\
{}^{-}n & = & {}^{-}0.5555\ldots \\
\hline
9n & = & 5
\end{array}
$$

$\Rightarrow n = \mathbf{\frac{5}{9}}$.

(e) $n = {}^{-}2.3\overline{4}$; $10n = {}^{-}23.\overline{4}$; now a one-digit repetend:

$$
\begin{array}{rcl}
100n & = & {}^{-}234.4444\ldots \\
{}^{-}10n & = & 23.4444\ldots \\
\hline
90n & = & {}^{-}211
\end{array}
$$

$\Rightarrow n = \mathbf{\frac{{}^{-}211}{90}}$.

(f) $n = {}^{-}0.0\overline{2}$; a one-digit repetend:

$$
\begin{array}{rcl}
100n & = & {}^{-}2.2222\ldots \\
{}^{-}10n & = & 0.2222\ldots \\
\hline
90n & = & {}^{-}2
\end{array}
$$

$\Rightarrow n = \frac{{}^{-}2}{90} = \mathbf{\frac{{}^{-}1}{45}}$.

8. 1 minute $= \frac{1}{60}$ hour, and $1 \div 60 = \mathbf{0.01\overline{6}}$ **hour**.

9. Line up the decimal points:

$$
\begin{array}{rcl}
{}^{-}1.4\overline{54} & = & {}^{-}1.4545454\ldots \\
{}^{-}1.\overline{454} & = & {}^{-}1.4544544\ldots \\
{}^{-}1.\overline{45} & = & {}^{-}1.4545454\ldots \\
{}^{-}1.45\overline{4} & = & {}^{-}1.4544444\ldots \\
{}^{-}1.454 & = & {}^{-}1.4540000\ldots
\end{array}
$$

Ordering from greatest (i.e., closest to 0) yields:
$\mathbf{{}^{-}1.454 > {}^{-}1.45\overline{4} > {}^{-}1.\overline{454} > {}^{-}1.4\overline{54} = {}^{-}1.\overline{45}}$.

10. Each element could be a decimal representation of $\frac{n}{n+1}$, beginning at $n = 0$. If so, then the next three elements would be:

$\frac{6}{7} = \mathbf{0.\overline{857142}}$,

$\frac{7}{8} = \mathbf{0.875}$, and

$\frac{8}{9} = \mathbf{0.\overline{8}}$.

11. $0.454545\ldots$ may be written as $0.45 + (0.45)(0.01) + (0.45)(0.01)(0.01) + \cdots$. This is a geometric sequence with common ratio $\boldsymbol{r = 0.01}$.

Mulitply $.\overline{5} \cdot .\overline{5}$ using the decimal expansion to obtain a precise answer. Instead, write $.\overline{5}$ in the form $\frac{a}{b}$.

$$\begin{array}{rcl} 10(.\overline{5}) & = & 5.\overline{5} \\ -\ .\overline{5} & = & -\ .\overline{5} \\ \hline 9(.\overline{5}) & = & 5 \Rightarrow .\overline{5} = \dfrac{5}{9}. \end{array}$$

Thus, $(.\overline{5})^2 = \left(\dfrac{5}{9}\right)^2 = \dfrac{25}{81} = .\overline{308641975}$

12. The repeating decimal part is determined by the 7 in the denominator of $3\frac{1}{7} = \frac{22}{7}$. Seven has no prime factors of 2 or 5 and so is a repeating decimal.

13. ***(i)*** $a + b =$

$$\begin{array}{r} 0.323232323232\ldots \\ +\ 0.123123123123\ldots \\ \hline \mathbf{0.446355446355\ldots} \end{array}$$

(ii) there are six digits in the repetend.

14. **Yes**. Zeros after the last non-zero digit of the decimal can be repeated.

15. Answers may vary; e.g.,

(a) Write $3.\overline{2}$ as $3.222\ldots$ and write 3.22 as 3.220. $3.220 < 3.221 < 3.2211 < 3.22111 < 3.222\ldots$

(b) Write $462.\overline{24}$ as $464.2424\ldots$and write 462.243 as 462.2430. $462.2424\ldots < 462.2425 < 462.2426 < 462.2427 < 462.2430$.

16. To find a number halfway between any two others, find their average; i.e., add the original numbers and divide by 2.

$(0.5 + 0.\overline{4}) \div 2 = (0.9444\ldots) \div 2 =$
$\mathbf{0.47222}\cdots = \mathbf{0.47\overline{2}}$.

17. Answers may vary; e.g.,

(a) Write $\frac{3}{4}$ as 0.750 and write $0.\overline{75}$ as $0.757\ldots$; $\frac{3}{4} < 0.751 < 0.752 < 0.753 < 0.\overline{75}$.

(b) Write $\frac{1}{3}$ as $0.333\ldots$ and write $0.\overline{34}$ as $0.343\ldots$; $\frac{1}{3} < 0.334 < 0.335 < 0.336 < 0.\overline{34}$.

18. **(a)** $\frac{3}{7} = 0.\overline{428571}$, a six-digit repetend. 21 when divided by 6 has a remainder of 3. Thus the 21st digit is the third in the repetend $= \mathbf{8}$.

(b) $17^{-1} = 0.\overline{0588235294117647}$, a sixteen-digit repetend. 5280 when divided by 16 has a remainder of 0. Thus the 5280th digit is the sixteenth in the repetend $= \mathbf{7}$.

19. **(a)** ***(i)*** Use the repeating block strategy:
$n = 0.\overline{1}$, a one-digit repetend.

$$\begin{array}{rcl} 10n & = & 1.1111\ldots \\ {}^{-}n & = & {}^{-}0.1111\ldots \\ \hline 9n & = & 1 \end{array}$$

$\Rightarrow n = \mathbf{\frac{1}{9}}$.

Observe that when the repetend is immediately to the right of the decimal point in the repeating decimal, a in the fraction $\frac{a}{b}$ will be the digit(s) in the repetend and b will be one or more 9's. The number of 9's will be the same as the number of digits in the repetend.

(ii) $0.\overline{01} = \mathbf{\frac{1}{99}}$.

(iii) $0.\overline{001} = \mathbf{\frac{1}{999}}$.

(b) $0.\overline{0001} = \mathbf{\frac{1}{9999}}$.

(c) $0.\overline{1} = \frac{1}{9}$ is true, so $(0.\overline{1})\left(\frac{1}{10}\right) = \frac{1}{9}\left(\frac{1}{10}\right) \Rightarrow$ $\mathbf{(0.0\overline{1})} = \frac{1}{90}$.

20. **(a)** Since $0.\overline{9} = 1, 0.0\overline{9} = \frac{1}{10} \cdot (0.\overline{9}) = \frac{1}{10}(1) = \mathbf{\frac{1}{10}} = \mathbf{0.1}$.

(b) $0.3\overline{9} = 0.3 + 0.0\overline{9} = 0.3 + 0.1 = \mathbf{0.4}$.

(c) $9.\overline{9} = 9 + 0.\overline{9} = 9 + 1 = \mathbf{10}$.

21. See problem 14(a)(*i*) above and observe that when the repetend is immediately to the right of the decimal point in the repeating decimal, *a* in the fraction $\frac{a}{b}$ will be the digit(s) in the repetend and *b* will be one or more 9's. The number of 9's will be the same as the number of digits in the repetend.

(a) $0.\overline{2} = \mathbf{\frac{2}{9}}$.

(b) $0.\overline{3} = \frac{3}{9} = \mathbf{\frac{1}{3}}$.

(c) $0.\overline{5} = \mathbf{\frac{5}{9}}$.

(d) $2.\overline{7} = 2\frac{7}{9} = \mathbf{\frac{25}{9}}$.

(e) $9.\overline{9} = 9\frac{9}{9} = \mathbf{10}$.

22. See problem 14(a)(*i*) above:

(a) $0.\overline{05} = \mathbf{\frac{5}{99}}$. In the fraction $\frac{a}{b}, a = 5$ and *b* is two 9's.

(b) $0.\overline{003} = \frac{3}{999} = \mathbf{\frac{1}{333}}$. In the fraction $\frac{a}{b}$, $a = 3$ and *b* is three 9's.

23. The sum is $0.4 + 0.4(0.5) + 0.4(0.5)^2 + 0.4(0.5)^3 + 0.4(0.5)^4 = 0.4 + 0.2 + 0.1 + 0.05 + 0.025 = \mathbf{0.775}$.

Another method by which to attack this problem would be to find the sum of a finite geometric sequence:

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} \\ rS_n &= a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^n \\ rS_n - S_n &= a_1 r^n - a_1 \\ (r-1)S_n &= a_1(r^n - 1) \\ S_n &= \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1(1 - r^n)}{1 - r} \end{aligned}$$

Thus if $a_1 = 0.4, r = 0.5$, and $n = 5$,

$S_5 = \frac{0.4[(1-0.5^5)]}{1-0.5} = \frac{0.4 \cdot 0.96875}{0.5} = \mathbf{0.775}$.

24. To find the sum of an infinite geometric series with ratio $0 < r < 1$ (if *r* were greater than 1, the series could not have a finite sum; if $r = 0$ the series would be just the first term):

$$\begin{aligned} S_\infty &= a + ar + ar^2 + ar^3 + \cdots \\ {}^-rS_\infty &= {}^-(ar + ar^2 + ar^3 + \cdots) \\ S_\infty - rS_\infty &= a \\ S_\infty(1 - r) &= a \\ S_\infty &= \frac{a}{1 - r} \end{aligned}$$

(a) $0.2\overline{9} = 0.29999\ldots = 0.2 + 0.9999\ldots = 0.2 + 0.09 + 0.09(0.1) + 0.09(0.01) + \cdots$.

Thus $0.2\overline{9}$ is 0.2 plus an infinite geometric sequence where $a_1 = 0.09$ and $r = 0.1$.

$S_\infty = 0.2 + \frac{0.09}{1-0.1} = 0.2 + \frac{0.09}{0.9} = 0.2 + 0.1 = \frac{3}{10}$.

(b) $2.0\overline{29} = 2 + 0.029 + 0.00029 + \cdots$.

Thus $2.0\overline{29}$ is 2 plus an infinite geometric sequence where $a_1 = 0.029$ and $r = 0.01$.

$S_\infty = 2 + \frac{0.029}{1-0.01} = 2\frac{29}{990} = \mathbf{\frac{2009}{990}}$.

25. If we sum $.\overline{235}$ and $.\overline{2356}$, the digits in each place value will not realign until we reach the least common multiple of the length of each number's repetend. Thus, we expect the length of the repetend to be $LCM(3,4) = 12$. To see this, consider the follwing.

$$\begin{array}{r} 0.235\,235\,235\,235\,235\ldots \\ +\,0.235\,623\,562\,356\,2356\ldots \\ \hline \end{array}$$

At the 10^{-13} place the digits realign.

26. **(a)** $1 - 3x = 8 \Rightarrow -3x = 7 \Rightarrow x = \frac{^-7}{3} = {}^-2.\overline{3}$.

(b) $1 = 3x + 8 \Rightarrow {}^-7 = 3x \Rightarrow 3x = {}^-7 \Rightarrow x = \frac{^-7}{3} = {}^-2.\overline{3}$.

(c) $1 = 8 - 3x \Rightarrow {}^-7 = -3x \Rightarrow 3x = 7 \Rightarrow x = \frac{7}{3} = 2.\overline{3}$.

27. $n = 0.\overline{abc}$; a three-digit repetend:

$$\begin{array}{rcl} 1000n &=& abc.abcabc\ldots \\ \underline{-n} &=& \underline{-0.abcabc\ldots} \\ 999n &=& abc \end{array}$$

$\Rightarrow n = \frac{abc}{999}$.

$0.\overline{abc}$ written as a fraction is $\dfrac{abc}{999}$.

28. $\dfrac{1}{333,333} = 0.\overline{000003}$ so the repetend is 0.000003.

29. If $\dfrac{9}{23} = 0.\overline{3913043478260869565217}$ and there are 22 repetends, then the digits repeat after 22, 44, 66, and 88 digits. The 100$^{\text{th}}$ digit in decimal form is the 12$^{\text{th}}$ digit of the reptend (88 + 12 = 100) which is 6.

Assessment 7-3B

1. The fraction is in simplest form because $3 \nmid 1234567$and if the prime factorization of the denominator contains only 2 and/or 5's then the fraction will be a terminating decimal. Since the denominator ends on 7 it is not divisible by 2 or by 5 it will be a non-terminating decimal.

2. **(a)** $0.34\overline{666}, 0.34\overline{66},$ and $0.34\overline{6}$ are all the same number. The only difference is that the repeating block of 6's is indicated in different ways.

(b) The preferred part is the shortest repeating part: $0.34\overline{6}$.

(c) The shortest period in $0.34\overline{6}$ is 1.

3. **(a)** The first ten decimal places of $0.\overline{246} = 0.2462462462$

(b) The first ten decimal places of $0.2\overline{46} = 0.2464646464$

(c) The first ten decimal places of $0.24\overline{6} = 0.2466666666$

(d) The first ten decimal places of $0.24\overline{68} = 0.2468686868$

4. We can write out the first ten decimal to compare:

$0.57\overline{89} = 0.5789898989$

$0.5\overline{789} = 0.5789789789$

and in the fifth decimal place 8>7 so $0.57\overline{89}$ is greater.

5. **(a)** $0.9999\ldots = 0.\overline{9}$ and the period is 1.

(b) $0.567323232\ldots = 0.567\overline{32}$ and the period is 2.

(c) $0.1827482748274\ldots = 0.1\overline{8274}$ and the period is 4.

6. In each of the following, divide the numerator by the denominator either with a calculator or by use of the division algorithm. The overline in the quotient indicates that the block of digits underneath is repeated an infinite number of times.

(a) $\frac{2}{3} = 2 \div 3 = \mathbf{0.\overline{6}}$.

(b) $\frac{7}{9} = 7 \div 9 = \mathbf{0.\overline{7}}$.

(c) $\frac{1}{24} = 1 \div 24 = \mathbf{0.041\overline{6}}$.

(d) $\frac{3}{60} = \frac{1}{20} = 1 \div 20 = \mathbf{0.05}$.

(e) $\frac{2}{99} = 2 \div 99 = \mathbf{0.\overline{02}}$.

(f) $\frac{7}{6} = 7 \div 6 = \mathbf{1.1\overline{6}}$.

(g) $\frac{2}{21} = 2 \div 21 = \mathbf{0.\overline{095238}}$.

(h) $\frac{4}{19} = 4 \div 19 = \mathbf{0.\overline{210526315789473684}}$.

The repetend has more digits than most calculators' display capability. Use the division algorithm, assuming an 8-digit display:

$$\begin{array}{r|l} & 0.\,2\,1\,0\,5\,2\,6\,3\,1\,5\,7\,8\,9\,4\,7\,3\,6\,8\,4 \\ \hline 19 & 4.0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ & \underline{3.9\,9\,9\,9\,9\,4} \\ & \qquad\qquad 6\,0\,0\,0\,0\,0\,0 \\ & \qquad\qquad \underline{5\,9\,9\,9\,9\,9\,1} \\ & \qquad\qquad\qquad\qquad 9\,0\,0\,0\,0\,0\,0 \\ & \qquad\qquad\qquad\qquad \underline{8\,9\,9\,9\,9\,9\,6} \\ & \qquad\qquad\qquad\qquad\qquad\qquad 4 \end{array}$$

where the last remainder, 4, again begins the repeating pattern. Note that since the denominator is 19, the most digits that can be in the repetend is 18.

7. **(a)** $n = 0.\overline{7}$; a one-digit repetend:

$$\begin{array}{rcl} 10n &=& 7.7777\ldots \\ {}^-n &=& {}^-0.7777\ldots \\ \hline 9n &=& 7 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{7}{9}}.$$

(b) $n = 0.\overline{46}$; a two-digit repetend:

$$\begin{array}{rcl} 100n &=& 46.4646\ldots \\ {}^-n &=& {}^-0.4646\ldots \\ \hline 99n &=& 46 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{46}{99}}.$$

(c) $n = 2.\overline{37}$; a two-digit repetend:

$$\begin{array}{rcl} 100n &=& 237.3737\ldots \\ {}^-n &=& {}^-2.3737\ldots \\ \hline 99n &=& 235 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{235}{99}}.$$

(d) $n = 2.3\overline{4}$; a one-digit repetend:

$$\begin{array}{rcl} 100n &=& 234.4444\ldots \\ {}^-10n &=& {}^-23.4444\ldots \\ \hline 90n &=& 211 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{211}{90}}.$$

(e) $n = {}^-4.3\overline{4}$; a one-digit repetend:

$$\begin{array}{rcl} 100n &=& {}^-434.4444\ldots \\ {}^-10n &=& 43.4444\ldots \\ \hline 90n &=& {}^-391 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{{}^-391}{90}}.$$

(f) $n = {}^-0.0\overline{3}$; a one-digit repetend:

$$\begin{array}{rcl} 100n &=& {}^-3.3333\ldots \\ {}^-10n &=& 0.3333\ldots \\ \hline 90n &=& {}^-3 \end{array}$$

$$\Rightarrow \boldsymbol{n = \tfrac{{}^-3}{90} = \tfrac{{}^-1}{30}}.$$

8. One second $= \frac{1}{3600}$ hour; $1 \div 3600 =$ $\mathbf{0.0002\overline{7}}$ **hour**.

9. Line up the decimal points:

$${}^-4.340000\ldots$$
$${}^-4.343434\ldots$$
$${}^-4.344444\ldots$$
$${}^-4.343434\ldots$$
$${}^-4.434343\ldots$$

Ordering from least (i.e., furthest from 0) yields: ${}^-4.4\overline{34} < {}^-4.3\overline{4} < {}^-4.3\overline{43} = {}^-4.\overline{34} < {}^-4.34$.

10. If this is an arithmetic sequence, then add 0.3 to each element to obtain the following element. The next three elements thus could be:

$$1.\overline{3} + 0.\overline{3} = \mathbf{1.\overline{6}};$$
$$1.\overline{6} + 0.\overline{3} = \mathbf{1.\overline{9}} = \mathbf{2.0};$$
$$2.0 + 0.\overline{3} = \mathbf{2.\overline{3}}.$$

11. We can write the decimals in the product as a fraction and perform the operation. $0.4\overline{9} \cdot 0.\overline{62} = 0.\overline{5} \cdot 0.\overline{62} = \frac{5}{9} \cdot \frac{62}{99} = \frac{310}{891} = \mathbf{0.\overline{347923681257014590}}$.

12. $\frac{2}{26} = \frac{1}{13}$. Since the denominator has a prime factor other than 2 or 5, the decimal will not terminate. Because $\frac{2}{26}$ is equal to the reduced fraction $\frac{1}{13}$ and the number of digits in the repetend for a proper fraction cannot exceed the denominator minus one, the maximum possible length of the repetend is **12**.

13.

$$\begin{array}{rcl} a &=& 1.23\ 43\ 43\ 43\ 43\ldots \\ +\ b &=& +\ 0.12\ 34\ 12\ 34\ 12\ldots \\ \hline a+b &=& \mathbf{1.35\ 77\ \ 55\ 77\ 55\ldots} \end{array}$$

Since $a + b$ is a repeating decimal, it is a rational number. Alternatively, a and b are rational numbers and the sum of two rational numbers is also rational. In this case, the length of the repetend is four.

14. **Yes**. If, for example, the repeating digits are zeros, then the repeating decimal could be written as a terminating decimal.

15. Answers may vary; e.g.,

(a) Write $4.\overline{3}$ as $4.3333\ldots$ and 4.3 as $4.3000\ldots$. Then $4.3 < 4.310 < 4.320 < 4.330 < 4.\overline{3}$.

(b) Write $203.\overline{76}$ as $203.7676\ldots$ and $203.\overline{7}$ as $203.7777\ldots$. Then $203.\overline{76} < 203.7680 < 203.7690 < 203.7691 < 203.\overline{7}$.

16. To find a number halfway between any two others, find their average; i.e., add the original numbers and divide by 2. (Note that $0.\overline{9} = 1.0$.)

$(0.\overline{9} + 1.1) \div 2 = (2.1) \div 2 = \mathbf{1.05}$.

17. Answers may vary; e.g.,

(a) Write $\frac{2}{3}$ as $0.6666\ldots$ and 0.67 as 0.6700.

Then $\frac{2}{3} < 0.6671 < 0.06672 < 0.06673 < 0.67$.

(b) Write $\frac{2}{3}$ as $0.6666\ldots$ and $0.6\overline{7}$ as $0.6777\ldots$.

Then $\frac{2}{3} < 0.6667 < 0.6668 < 0.6669 < 0.6\overline{7}$.

18. $17^{-1} = 0.\overline{0588235294117647}$, a sixteen-digit repetend. 23 when divided by 16 has a remainder of 7, so the 23rd digit is the 7th in the repetend, or **5**.

19. **(a)** ***(i)*** Use the repeating block strategy:

$n = 0.\overline{2}$, a one-digit repetend.

$$\begin{array}{rcl} 10n & = & 2.2222\ldots \\ \underline{{}^{-}n} & \underline{=} & \underline{{}^{-}0.2222\ldots} \\ 9n & = & 2 \end{array}$$

$\Rightarrow n = \mathbf{\frac{2}{9}}$.

(ii) $n = 0.0\overline{2}$, a one-digit repetend.

$$\begin{array}{rcl} 100n & = & 2.2222\ldots \\ \underline{{}^{-}10n} & \underline{=} & \underline{0.2222\ldots} \\ 90n & = & 2 \end{array}$$

$\Rightarrow n = \mathbf{\frac{2}{90}}$.

(iii) $n = 0.00\overline{2}$, a one-digit repetend.

$$\begin{array}{rcl} 1000n & = & 2.2222\ldots \\ \underline{{}^{-}100n} & \underline{=} & \underline{0.2222\ldots} \\ 900n & = & 2 \end{array}$$

$\Rightarrow n = \mathbf{\frac{2}{900}}$.

(b) Based on (a) (*i*) and (*ii*), $0.000\overline{2} = \mathbf{\frac{2}{9000}}$.

(c) $\frac{4}{90} = 2\left(\frac{2}{90}\right) = 2(0.0\overline{2}) = \mathbf{0.0\overline{4}}$.

20. **(a)** $1.\overline{9} = 1 + 0.\overline{9} = 1 + 1 = 2$.

(b) $0.00\overline{9} = \frac{1}{100} \cdot 0.\overline{9} = \frac{1}{100} \cdot 1 = \mathbf{0.01}$.

(c) $0.3\overline{9} = 0.3 + 0.0\overline{9} = 0.3 + \frac{1}{10} \cdot 0.\overline{9}$
$= 0.3 + 0.1 = \mathbf{0.4}$.

21. Observe that when the repetend is immediatly to the right of the decimal point in the repeating decimal, a in the fraction $\frac{a}{b}$ will be the digit(s) in the repetend and b will be one or more 9's. The number of 9's will be the same as the number of digits in the repetend.

(a) $0.\overline{4} = \mathbf{\frac{4}{9}}$.

(b) $0.\overline{12} = \frac{12}{99} = \mathbf{\frac{4}{33}}$.

(c) $0.\overline{111} = \frac{111}{999} = \mathbf{\frac{1}{9}}$.

22. **(a)** First observe that $25(0.010101\ldots) = 0.252525\ldots$. Therefore, $3.\overline{25} = 3 + 25(0.\overline{01}) = 3 + \frac{25}{99} = \mathbf{\frac{322}{99}}$.

(b) $3.\overline{125} = 3 + 125(0.\overline{001}) = 3 + \frac{125}{999} = \mathbf{\frac{3122}{999}}$.

23. The sum is $0.1 + 0.1(0.3) + 0.1(0.3)^2 + 0.1(0.3)^3 = 0.1 + 0.03 + 0.009 + 0.0027 = \mathbf{0.1417}$.

Another method by which to attack this problem would be to find the sum of a finite geometric sequence:

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} \\ rS_n &= a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n \\ rS_n - S_n &= a_1r^n - a_1 \\ (r-1)S_n &= a_1(r^n - 1) \\ S_n &= \frac{a_1(r^n - 1)}{r - 1} = \frac{a_1(1 - r^n)}{1 - r} \end{aligned}$$

Thus if $a_1 = 0.1$, $r = 0.3$, and $n = 4$,

$S_4 = \frac{0.1[(1-0.3^4)]}{1-0.3} = \frac{0.1 \cdot 0.9919}{0.7} = \mathbf{0.1417}$.

24. To find the sum of an infinite geometric series with ratio $0 < r < 1$ (if r were greater than l, the series could not have a finite sum; if $r = 0$ the series would be just the first term):

$$S\infty = a + ar + ar^2 + ar^3 + \cdots$$
$$^{-}rS_\infty = {}^{-}(ar + ar^2 + ar^3 + \cdots)$$
$$S_\infty - rS\infty = a$$
$$S_\infty(1 - r) = a$$
$$S\infty = \frac{a}{1 - r}$$

(a) $0.\overline{29} = 0.29 + 0.0029 + \cdots = 0.29 + 0.29(0.01) + 0.29(0.0001) + \cdots$.
Thus $a_1 = 0.29$ and $r = 0.01$.
$S_\infty = \frac{0.29}{1-0.01} = \frac{0.29}{0.99} = \mathbf{\frac{29}{99}}$.

(b) $0.00\overline{29} = 0.00029 + 0.0000029 + \cdots$.
Thus $0.00\overline{29}$ is an infinite geometric sequence where $a_1 = 0.00029$ and $r = 0.01$.
$S = \frac{0.00029}{1-0.01} = \frac{0.00029}{0.99} = \mathbf{\frac{29}{99{,}000}}$.

25.
$$\begin{aligned} & 0.\overline{23} = 0.2323232323\ldots \\ + & 0.\overline{235} = \underline{0.2352352352\ldots} \\ & = 0.4675584675\ldots \end{aligned}$$
So there are six places in the repetend of the sum.

26. **(a)** $3x = 8 \Rightarrow x = \frac{8}{3} = \mathbf{2\frac{2}{3}} = \mathbf{2.\overline{6}}$.

(b) $3x + 1 = 8 \Rightarrow 3x = 7 \Rightarrow x = \frac{7}{3} = \mathbf{2\frac{1}{3}} = \mathbf{2.\overline{3}}$.

(c) $3x - 1 = 8 \Rightarrow 3x = 9 \Rightarrow x = \mathbf{3.0}$.

27. $n = 0.\overline{ab}$; a two-digit repetend:

$$\begin{aligned} 100n &= ab.abab\ldots \\ {}^{-}n &= {}^{-}0.abab\ldots \\ \hline 99n &= ab \end{aligned}$$
$\Rightarrow n = \frac{ab}{99}$.

$0.\overline{ab}$ written as a fraction is $\frac{ab}{99}$.

28. $\frac{1}{333{,}333{,}333} = 0.\overline{000000003}$ so the repetend is 0.000000003.

29. If $\frac{9}{23} = 0.\overline{3913043478260869565217}$ and there are 22 repetends, then the digits repeat after 22, 44, 66, 88, etc. digits. The 900th digit in decimal form is the 20th digit of the repetend (880 + 20 = 900) which is 2.

Mathematical Connections 7-3: Review Problems

16. Total deductions were $\$1520.63 + \$723.30 + \$2843.62 = \$5087.55$. Gross pay less deductions was $\$27{,}849.50 - \$5087.55 = \mathbf{\$22{,}761.95}$.

17. Distance at the speed of light would be $1.86 \cdot 10^5$ mi/second $\cdot$ $3.1536 \cdot 10^7$ sec/yr $\cdot$ 4 yrs $\approx 23.5 \cdot 10^{12} = \mathbf{2.35 \cdot 10^{13}}$ **miles**.

18. The rule states that decimal point placement should be four places, or 0.0770. Because 0.077 as found on the calculator equals 0.0770; i.e., trailing zeros add no value; the rule still applies.

19. Answers may vary

(a) $\mathbf{3.024} + {}^{-}0.023 = 3.001$, which is between 3 and 4.

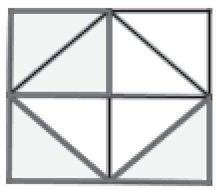
(b) ${}^{-}0.023 - {}^{-}\mathbf{3.024} = 3.001$, which is between 3 and 4.

(c) Pick 3.5 as a number between 3 and 4.
$0.023 \cdot n = 3.5 \Rightarrow n = \frac{3.5}{0.023} \approx \mathbf{152}$.

(d) Pick 3.5 as a number between 3 and 4.
$\frac{0.023}{n} = 3.5 \Rightarrow n = \frac{0.023}{3.5} \approx \mathbf{0.00657}$.

Assessment 7-4A: Percents and Interest

1. Cut the shape into equal size pieces. If squares are created and then triangles inside the squares, there are 8 triangles and 4 are shaded. $\frac{4}{8} = 50\%$.



2.

0% 50% 100% 150% 200% 250% 300% 350%

0 1/2 1 3/2 2 5/2 3 7/2

0 .5 1 1.5 2 2.5 3 3.5

3. **(a)** $7.89 = 100\left[\frac{7.89}{100}\right] = \frac{789}{100} = \mathbf{789\%}$.

(b) $193.1 = 100\left[\frac{193.1}{100}\right] = \frac{19310}{100} = \mathbf{19310\%}$.

(c) $\frac{5}{6} = 100\left[\frac{\frac{5}{6}}{100}\right] = \frac{\mathbf{100}\left[\frac{5}{6}\right]}{100}$.

$= \frac{\frac{50}{1}\cdot\frac{5}{3}}{100} = \frac{\frac{250}{3}}{100} = \frac{83\frac{1}{3}}{100} = \mathbf{83\frac{1}{3}\%}$.

(d) $\frac{1}{8} = 100\left[\frac{\left(\frac{1}{8}\right)}{100}\right] = \frac{\left(\frac{100}{8}\right)}{100} = \frac{12.5}{100} = \mathbf{12.5\%}$.

(e) $\frac{5}{8} = 100\left[\frac{\left(\frac{5}{8}\right)}{100}\right] = \frac{\left(\frac{500}{8}\right)}{100} = \frac{62.5}{100} = \mathbf{62.5\%}$.

(f) $\frac{4}{5} = 100\left[\frac{\left(\frac{4}{5}\right)}{100}\right] = \frac{\left(\frac{400}{5}\right)}{100} = \frac{80}{100} = \mathbf{80\%}$.

4. **(a)** $16\% = \frac{16}{100} = \mathbf{0.16}$.

(b) $\frac{1}{5}\% = 0.2\% = \frac{0.2}{100} = \frac{2}{1000} = \mathbf{0.002}$.

(c) $13\frac{2}{3}\% = 13.\overline{6}\% = \frac{13.\overline{6}}{100} = \mathbf{0.13\overline{6}}$.

(d) $\frac{1}{3}\% = 0.\overline{3}\% = \frac{0.\overline{3}}{100} = \mathbf{0.00\overline{3}}$.

5. **(a)** $\underline{\mathbf{4}}$ for every 100. $4\% = \frac{4}{100}$.

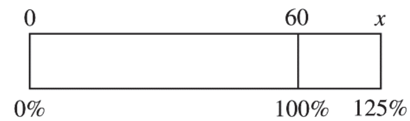
(b) $\underline{\mathbf{2}}$ for every 50. $4\% = \frac{4}{100} = \frac{2}{50}$.

(c) 1 for every $\underline{\mathbf{25}}$. $4\% = \frac{4}{100} = \frac{1}{25}$.

(d) 8 for every $\underline{\mathbf{200}}$. $4\% = \frac{4}{100} = \frac{8}{200}$.

(e) 0.5 for every $\underline{\mathbf{12.5}}$. $4\% = \frac{4}{100} = \frac{0.5}{12.5}$.

6.



$$\frac{60}{100} = \frac{x}{125} \Rightarrow 100x = 6000 \Rightarrow x = 75.$$

7. **(a)** 6% of $34 \Rightarrow \frac{6}{100}\cdot 34 = \frac{204}{100} = \mathbf{2.04}$.

(b) 17 is $n\%$ of $34 \Rightarrow 17 = \frac{n}{100}\cdot 34$

$\Rightarrow 34n = 1700 \Rightarrow n = \frac{1700}{34} =$

50, or 17 is **50%** of 34.

(c) 18 is 30% of $n \Rightarrow 18 = \frac{30}{100}\cdot n$

$\Rightarrow 30n = 1800 \Rightarrow n = \frac{1800}{30} = \mathbf{60}$.

(d) 7% of $49 \Rightarrow \frac{7}{100}\cdot 49 = \frac{343}{100} = \mathbf{3.43}$.

8. **(a)** 5% of $x \Rightarrow \frac{5}{100}x$, or $\frac{5x}{100} = \frac{x}{20}$.

(b) 10% of amount $= a \Rightarrow \frac{10}{100}\cdot$ amount $= a$

$\Rightarrow$ amount $= \mathbf{10a}$.

9. 75% of 84 boxes $\Rightarrow \frac{75}{100}\cdot 84 = \mathbf{63}$ **boxes**.

10. Let s be her last salary. 7% of $s + 100\%$ of $s = \$27{,}285 \Rightarrow 107\%\cdot s = \$27{,}285 \Rightarrow 1.07\cdot s = 27{,}285$. Thus $s = \frac{27{,}285}{1.07} = \mathbf{\$25{,}500}$.

11. **(a)** **Bill**. Joe sold 180 papers; Bill sold $\frac{85}{100}\cdot 260 = 221$ papers; Ron sold 212 papers.

(b) **Joe**. Joe sold $100\left[\frac{\left(\frac{180}{200}\right)}{100}\right] = 90\%$ of his papers; Bill sold 85% of his papers; Ron sold 80% of his papers.

(c) **Ron**. Joe started with 200 papers; Bill started with 260 papers; Ron started with $\frac{212}{0.80} = 265$ papers.

12. The amount of the discount is $\$35 - \$28 = \$7$. The percent of the discount is \$7 as percent of $\$35 \Rightarrow \frac{7}{35} = 0.2 = 100\left(\frac{0.2}{100}\right) = \frac{20}{100} = \mathbf{20\%}$.

13. The amount of decrease in value was $\$359{,}000 - \$195{,}000 = \$164{,}000$. The decrease as a percentage of the original value was $\frac{164{,}000}{359{,}000} \approx 0.46 = 100\left(\frac{0.46}{100}\right) = \frac{46}{100}$, or **about 46%.**

14. Sale price was regular price $-$ 20% of regular price. $\$28.00 - 20\%$ of $\$28.00 = \$28 - (0.2 \cdot \$28) = \$28.00 - \$5.60 = \mathbf{\$22.40}$.

15. The amount of the tax is 5% of \$320, or $0.05 \cdot 320 = \$16$. The total cost is $\$320 + \$16 = \mathbf{\$336}$.

16. Bill answered $80 - 52 = 28$ questions incorrectly. 28 as percentage of 80 is $\frac{28}{80} = 0.35 = 100\left(\frac{0.35}{100}\right) = \frac{35}{100} = \mathbf{35\%}$.

17. $66\frac{2}{3}\%$ of 1800 employees means $\frac{66\frac{2}{3}}{100} = \frac{\left(\frac{200}{3}\right)}{100} = \frac{200}{300} = \frac{2}{3}$ of 1800. $\frac{2}{3} \cdot 1800 =$ **1200 employees**.

18. $\frac{325}{500} \cdot \frac{325}{500} = \frac{650}{1000} = \frac{65}{100} = 65\%$; $\frac{600}{1000} = \frac{60}{100} = 60\%$.

19. **(a)** **(*i*)** $8 \cdot \$9.50 = \mathbf{\$76.00}$ for 8 items.

(*ii*) 10 items at \$9.50 each $= \$95$.
$\$95 - 20\%$ of $\$95 = 95 - 0.2 \cdot 95$
$= \$95.00 - \$19.00 = \mathbf{\$76}$ for 10 items.

(b) **10 items**. You get two more items for the same price.

20. The amount of John's 20% profit is $0.20 \cdot \$330 = \$66$. Sale price of the bike after a 10% discount must be $\$330 + \$66 = \$396$.

Let p be the list price; then $p - 10\%$ of p is the sale price. $p - 10\%$ of $p = \$396 \Rightarrow 0.9p = 396 \Rightarrow p = \frac{396}{0.9} = \$440$. Thus if John prices the bike at **\$440** he can offer a 10% discount of \$44 and still realize his \$66 profit.

21. Techniques may vary:

(a) 10% of $\$22 = \$2.20$. 5% of \$22 is half 10%, or \$1.10. $\$2.20 + \$1.10 = \mathbf{\$3.30}$.

(b) 10% of $\$120 = \$12$. 20% is twice 10%, or twice $\$12 = \mathbf{\$24}$.

(c) 10% of $\$38 = \$3.80$. 5% is half of 10%, or half of $\$3.80 = \mathbf{\$1.90}$.

(d) 25% is $\frac{1}{4}$; $\frac{1}{4}$ of $\$98 = \$98 \div 4$, or **\$24.50**. (To divide 98 by 4, think of it as $100 \div 4 = 25$. Then subtract $2 \div 4 = 0.50$.)

22. A journeyman makes 200% of an apprentice's pay and a master makes 150% of a journeyman's pay. 150% of $200\% = 1.5 \cdot 2 = 3 = 300\%$ of an apprentice's pay.

The \$4200 must have $1 + 2 + 3 = 6$ shares, or $\frac{\$4200}{6} = \$700$ per share. The **apprentice earns \$700**. The **journeyman** earns 200% of $\$700 = 2 \cdot \$700 = \mathbf{\$1400}$. The **master** earns 300% of $\$700 = 3 \cdot \$700 = \mathbf{\$2100}$.

23. **(a)** $\frac{\text{20 math majors}}{\text{500 incoming students}} = \frac{4}{100} = \mathbf{4\%}$ math majors.

(b) **(*i*)** There were 480 non-math-major students originally. 5% of $480 = 0.05 \cdot 480 = 24$ who switched. 20 original + 24 switches = **44** math majors now.

(*ii*) $\frac{\text{44 math majors}}{\text{500 students}} = 0.088 = 100\left(\frac{0.088}{100}\right) = \frac{8.8}{100} =$ **8.8%** math majors.

24. Let s be the salary of the previous year. 100% of s + 10% of s = 110% of $s = 1.1 \cdot s$ is current salary. $1.1s = \$100{,}000$ (this year), so $s = \frac{\$100.000}{1.1} \approx \$90{,}909.09$ (last year).

$1.1s = \$90{,}909.09$ (last year), so $s = \frac{\$90{,}909{,}09}{1.1}$, or about **\$82,644.63** (two years ago).

25. **(a)** There are $815 \cdot 365 \cdot 24 \cdot 60^2 \approx$ 25.7 billion seconds in 815 years. \$100 per second means about \$2.57 trillion, so the report is essentially **true**.

(b) $100\left[\dfrac{\left(\frac{1}{815}\right)}{100}\right] \approx$ **0.12%** of the money each year.

26. **No**. Since 56% is more than double 25%, \$950 should be more than double \$500, but it is not.

27. **(a)** Since there is no quorum requirement stated, **3** members; i.e., the number of members required to obtain a majority vote; can make a change.

(b) The vote of two non-chair members, or $100\left[\dfrac{\left(\frac{2}{100{,}000}\right)}{100}\right] = \mathbf{0.002\%}$, can change the bylaws, even though 3 members including one chair, or 0.003%, must attend the meeting.

28. **(a)** 15% of \$30 $= 0.15 \cdot 30 =$ **\$4.50**.

(b) $100\left[\dfrac{\left(\frac{1}{2}\right)}{100}\right] = \mathbf{50\%}$.

(c) $100\left(\frac{1}{100}\right) = \mathbf{100\%}$.

29. According to the percent bar, 374 students represent 68% of the total. Let t be the total number of students. Then 68% of $t = 374 \Rightarrow$ $0.68t = 374 \Rightarrow t = \frac{374}{0.68} = \mathbf{550\ students}$.

30. **(a)** There are 9 single digit numbers in the numbers 1 through 50: 1, 2, 3, 4, 5, 6, 7, 8, and, 9. This is $\dfrac{9}{50} = 0.18$ or 18%. There are 100% - 18% = 82% two-digit numbers.

(b) The multiples of 9 are 9, 18, 27, 36, and 45. That is 5 out of 50 or 10%.

(c) Half of the numbers are even. That is 25 out of 50 or 50%.

(d) The prime numbers between 1 and 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47. That is 15 out of 50 or 30%.

(e) The numbers that are even and prime is only 2. That is 1 out of 50 or 2%.

31. **(a)** $\dfrac{14}{15} = 0.9\overline{3} = 93\dfrac{1}{3}\%$.

(b) $80\% = \dfrac{80}{100} = \dfrac{4}{5}$ of the sun's UV rays are blocked.

(c) The SPF is 5.

(d) **Yes.** A sunblock with SPF 30 blocks $\dfrac{29}{30} = 0.9\overline{6} = 96\dfrac{2}{3}\%$. The label is correct.

32. In the table below:

(i) Interest rate per period is found by dividing the annual rate by the number of compounding periods per year.

(ii) Number of periods: Semiannually is twice per year, quarterly is four times per year, monthly is twelve times per year, daily is 365 times per year.

(iii) Amount of interest paid: Where
$A =$ compound amount;
$P =$ principal (\$1000 in this example);
$i =$ interest rate per period expressed as a decimal;
$n =$ number of periods per year;
$I =$ compound interest;
then $A = P(1 + i)^n$ and $I = A - P$.

	Interest Rate per Period (i)	Number of Periods (n)	Amount of Interest Paid (I)	Compound Amount (A)
(a)	$\frac{6\%}{2} = 3\% = 0.03$	4	\$125.51	\$1125.51
(b)	$\frac{8\%}{4} = 2\% = 0.02$	12	\$268.24	\$1268.24
(c)	$\frac{10\%}{12} = 0.8\overline{3}\%$ $= 0.008\overline{3}$	60	\$645.31	\$1645.31
(d)	$\frac{12\%}{365} = \frac{0.12}{365}$	1460	\$615.95	\$1615.95

33. Where I is interest, P is principal, r is rate expressed as a decimal, and t is time in years: $I = Prt = \$42{,}000 \cdot 0.0875 \cdot 1 = \mathbf{\$3675.00}$.

34. Look for the principal to be invested.
If $A = P(1+i)^n$ then $P = \frac{A}{(1+i)^n}$, where
$A = \$50,000, i = \frac{3\%}{4} = 0.75\% = 0.0075,$
and $n = 5$ years $\cdot$ 4 periods per year $= 20$.
$P = \frac{\$50,000}{(1+0.0075)^{20}} \approx \mathbf{\$43,059.49}$ rounded to the nearest penny.

35. $A = P(1+i)^n$ where $P = \$4000,\ i = \frac{5.9\%}{4}$ $= 1.475\% = 0.01475, n = 20 \cdot 4 = 80$ periods.
$A = 4000(1 + 0.01475)^{80} \approx \mathbf{\$12,905.80}$, the value of the investment after 20 years.

36. \$300 compounded monthly at $1.1\% = 0.011$ per month for 11 months will yield:
$A = P(1+i)^n = \$300(1+0.011)^{11} \approx \$338.36$.
Interest earned is $A - P = \$338.36 - \$300 = \$38.36$. The *annual* rate, though, is based on annual investment, or $300(1+0.011)^{12} \approx \$342.09 \Rightarrow \$42.09$ interest earned. Interest as a percentage of investment would be $\frac{\$12.09}{\$300}$ $= 0.1403 = \mathbf{14.03\%}$.

37. First three years: $P = \$3000, i = \frac{2\%}{4} = 0.005,$ and $n = 4 \cdot 3 = 12$. $A = \$3000(1+0.005)^{12}$ $\approx \$3185.03$.
Second three years: $P = \$3185.03,\ i = \frac{3\%}{4} = 0.0075$, and $n = 4 \cdot 3 = 12$.
$A = \$3185.03(1+0.0075)^{12} \approx \mathbf{\$3483.81}$ after six years.

38. Suppose \$1 is invested in each bank for 1 year.

(*i*) At New Age, $i = \frac{0.04}{365}$ and $n = 365$.
$A = \$1\left(1 + \frac{0.04}{365}\right)^{365} \approx \$1.041.$
Subtracting the \$1 invested, interest is $4.1¢$ which corresponds to an effective annual rate of $\frac{4.1¢}{100¢} = 4.1\%$.

(*ii*) At Pay More, $i = \frac{0.052}{1}$ and $n = 1 \cdot 1 = 1$. $A = \$1(1+0.052)^1 = \$1.052$. Subtracting the \$1 invested, interest is $5.2¢$ which corresponds to an effective annual rate of 5.2%. The **Pay More bank offers the better rate**.

39. Observe that Amy's monthly interest is $12\% \div 12 = 1\%$ and that over two years, she has $12 \cdot 2 = 24$ compoundings. Thus, she owes $2000(1+0.01)^{24} = 2000(1.01)^{24} = \$2539.47$. The answer is **no**, the amount she owes has not doubled.

Assessment 7-4B: Percents and Interests

1. Cut the shape into equal size pieces. There are 8 rectangles and 6 are shaded. $\frac{6}{8} = 75\%$.

2.

3. **(a)** $0.032 = 100\left[\frac{0.032}{100}\right] = \frac{3.2}{100} = \mathbf{3.2\%}.$

(b) $0.2 = 100\left[\frac{0.2}{100}\right] = \frac{20}{100} = \mathbf{20\%}.$

(c) $\frac{3}{20} = 100\left[\frac{\left(\frac{3}{20}\right)}{100}\right] = \frac{\left(\frac{300}{20}\right)}{100} = \frac{15}{100} = \mathbf{15\%}.$

(d) $\frac{13}{8} = 100\left[\frac{\left(\frac{13}{8}\right)}{100}\right] = \frac{\left(\frac{1300}{8}\right)}{100} = \frac{162.5}{100}$
$= \mathbf{162.5\%}.$

(e) $\frac{1}{6} = 100\left[\frac{\left(\frac{1}{6}\right)}{100}\right] = \frac{\left(\frac{100}{6}\right)}{100} = \frac{16.\overline{6}}{100} = \mathbf{16.\overline{6}\%}$
$= \mathbf{16\tfrac{2}{3}\%}.$

(f) $\frac{1}{40} = 100\left[\frac{\left(\frac{1}{40}\right)}{100}\right] = \frac{\left(\frac{100}{40}\right)}{100} = \frac{2.5}{100} = \mathbf{2.5\%}.$

4. **(a)** $4\frac{1}{2}\% = 4.5\% = \frac{4.5}{100} = \frac{45}{1000} = \mathbf{0.045}$.

(b) $\frac{2}{7}\% = 0.\overline{285714}\% = \frac{0.\overline{285714}}{100}$
$= \mathbf{0.00\overline{285714}}$.

(c) $125\% = \frac{125}{100} = \mathbf{1.25}$.

(d) $\frac{1}{4}\% = 0.25\% = \frac{0.25}{100} = \mathbf{0.0025}$.

5. **(a)** $\underline{\mathbf{5}}$ for every 100. $5\% = \frac{5}{100}$.

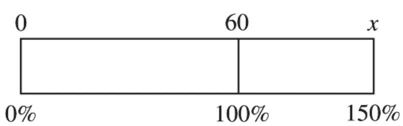
(b) $\underline{\mathbf{2.5}}$ for every 50. $5\% = \frac{5}{100} = \frac{2.5}{50}$.

(c) 1 for every $\underline{\mathbf{20}}$. $5\% = \frac{5}{100} = \frac{1}{20}$.

(d) 8 for every $\underline{\mathbf{160}}$. $5\% = \frac{5}{100} \cdot \frac{1.6}{1.6} = \frac{8}{160}$.

(e) 0.5 for every $\underline{\mathbf{10}}$. $5\% = \frac{5}{100} = \frac{0.5}{10}$.

6.



$$\frac{60}{100} = \frac{x}{150} \Rightarrow 100x = 9000 \Rightarrow x = 90$$

7. **(a)** 63 is 30% of $n \Rightarrow 63 = \frac{30}{100} \cdot n \Rightarrow 30n$
$= 6300 \Rightarrow n = \frac{6300}{30} = \mathbf{210}$.

(b) 7% of 150 $\Rightarrow \frac{7}{100} \cdot 150 = \frac{1050}{100} = \mathbf{10.5}$.

(c) 61.5 is n% of 20.5 $\Rightarrow 61.5 = \frac{n}{100} \cdot 20.5$
$\Rightarrow 20.5n = 6150 \Rightarrow n = \frac{6150}{20.5} = 300$,
or 61.5 is **300%** of 20.5.

(d) 16 is 40% of $n \Rightarrow 16 = \frac{40}{100} \cdot n \Rightarrow 40n =$
$1600 \Rightarrow n = \frac{1600}{40} = \mathbf{40}$.

8. **(a)** $.005x = \frac{5}{1000}x = \frac{\boldsymbol{x}}{\mathbf{200}}$.

(b) One percent of x (an amount) is $0.001x$. If this equals a, then $0.001x = a \Rightarrow x = \frac{a}{0.001} \Rightarrow x = \mathbf{1000}\boldsymbol{a}$.

9. The amount of the discount is 25% of \$6.80, or $0.25 \cdot \$6.80 = \$1.70$. The sale price is $\$6.80 - \$1.70 = \mathbf{\$5.10}$.

10. Let x represent her previous salary.
$x + .1x = 60{,}000 \Rightarrow 1.1x = 60{,}000 \Rightarrow$
$x = \frac{60{,}000}{1.1} \approx \$54{,}545.45$.

11. A line segment 3 in long would be 50%, or half of, a 6 in segment.

(a) 100% of $6 = \frac{100}{100} \cdot 6 = \mathbf{6}$ **in**.

(b) 25% of $6 = \frac{25}{100} \cdot 6 = \mathbf{1\frac{1}{2}}$ **in**.

(c) 150% of $6 = \frac{150}{100} \cdot 6 = \mathbf{9}$ **in**.

12. The amount of depreciation was $\$1700 - \$1400 = \$300$. The depreciation as a percentage of \$1700 is given by $\frac{300}{1700} \approx 0.1765 = 100\left[\frac{0.1765}{100}\right] = \frac{17.65}{100} = \mathbf{17.65\%}$.

13. The number of eagles by which the population decreased was $728 - 594 = 134$. The decrease as a percentage of the original population was $\frac{134}{728} \approx 0.184 = 100\left(\frac{0.184}{100}\right) = \frac{18.4}{100}$,
or **about 18.4%**.

14. Let t be the total income for the month. Then $t = \$900 + 4\%$ of $\$1800 = 900 + \frac{4}{100} \cdot 1800 = \mathbf{\$972}$.

15. Let t be the total bill. Then $t = \$380 +$ 9% of $\$380 \Rightarrow t = 380 + \frac{9}{100} \cdot 380 = \mathbf{\$414.20}$.

16. $\frac{4500 \text{ mammals}}{1{,}700{,}000 \text{ species}} \approx 0.0026 = 100\left(\frac{0.0026}{100}\right)$
$= \frac{0.26}{100}$, or **about 0.26%**.

17. Shirt a cost \$40 − 20% of $\$40 = 40 - \frac{20}{100} \cdot 40$
$= \$32$. Shirt b cost \$40 − 25% of $\$40 =$
$40 - \frac{25}{100} \cdot 40 - \$30$. Sales tax was 4.5% of the total, or $\frac{4.5}{100}(32 + 30) = \$2.79$. Jim's total amount spent was thus $\$32.00 + \$30.00 + \$2.79 = \mathbf{\$64.79}$.

18. **0.625%** is greater. $0.625\% = \frac{62.5}{100}$. On the other hand $(0.625\%)^2 = \frac{62.5}{100} \cdot \frac{62.5}{100} = \frac{62.5 \cdot \frac{62.5}{100}}{100} = \frac{.625(62.5)}{100}$. Notice that 62.5 is multiplied by a number less than one and thus, it decreases.

19. **(a)** 30% of $2400 = \frac{30}{100} \cdot 2400 = 720$. No more than **720 calories** should be from fat.

(b) **Yes**. Three cookies would have 210 calories from fat, $720 - 210 = 510$ calories less than the recommended maximum.

20. The reduced price of the suit is $\mathbf{200 - .25(200) = 150}$. Let x be the percent increase. Then $150 + x(150) = 200 \Rightarrow x = \frac{50}{150} \Rightarrow$ $x = \frac{1}{3} = \mathbf{33\frac{1}{3}\%}$.

21. Techniques may vary:

(a) 10% of \$42 = \$4.20. 5% of \$42 is half of 10%, or \$2.10. \$4.20 + \$2.10 = **\$6.30**.

(b) 10% of \$280 = \$28. 20% is twice 10%, or twice \$28 = **\$56**.

(c) 10% of \$28 = \$2.80. 5% is half of 10%, or half of \$2.80 = **\$1.40**.

(d) 25% is $\frac{1}{4}$; $\frac{1}{4}$ of \$84 = 84 ÷ 4, or **\$21**.

22. Set up a proportion: $\frac{\frac{1}{4}\text{cup}}{x \text{ cups}} = \frac{0.5\%}{100\%} \Rightarrow$ $\frac{0.25 \text{ cup}}{x \text{ cups}} = \frac{0.005}{1} \Rightarrow x \cdot 0.005 = 0.25 \cdot 1 \Rightarrow$ $x = \frac{0.25}{0.005}$. $x = \mathbf{50\ cups}$.

23. 12.13% of price = \$1116.88, or price = $\frac{1116.88}{0.1213}$, or **about \$9207.58** or \$9208 to the nearest dollar.

24. Let n be the original number and let N be 20% more than $n \Rightarrow 120\%$ of $n = 1.2n$. Then n as a percentage of N is $\frac{n}{N} = \frac{n}{1.2n} = \frac{1}{1.2} = 0.83\frac{1}{3}$; i.e., $n = 83\frac{1}{3}\%$ of N. 100% of $N - 83\frac{1}{3}\%$ of $N = \mathbf{16\frac{2}{3}\%}$ of N.

25. **(a)** Only the four corner blocks would have four faces painted, so there would be $\frac{4}{100} = \mathbf{4\%}$ painted.

(b) The blocks along each edge, exclusive of the corner blocks, would have three faces painted. There are eight of these along each edge, or 32 blocks. Thus there would be $\frac{32}{100} = \mathbf{32\%}$ painted.

(c) Only the interior blocks would have two faces painted. There are $100 - 4 - 32 = 64$ of these, so there would be $\frac{64}{100} = \mathbf{64\%}$ painted. Alternatively, the interior can be seen as an 8 by 8 square, which has 64 blocks.

26. If 2% milk has 5g of fat per cup and whole milk has 8g of fat per cup then the reduction is 3 g of fat. $\frac{3}{8} = .375 = 37.5\%$ The reduction is 37.5%.

27. 20% of 80 coins $= \frac{20}{100} \cdot 80 = 16$ quarters, worth $16 \cdot 25 = 400¢$. If the remaining $80 - 16 = 64$ coins are all pennies (the least possible amount), there would be $400¢ + 64¢ = 464¢$, or **\$4.64** in the piggy bank.

28. According to the percent bar, 100% of the population is 302 million. 46% of 302 million = $\frac{46}{100} \cdot 302$ million = 138.92 million ≈ **139 million**.

29. **(a)** There are 9 single digit numbers in the numbers 1 through 50: 1, 2, 3, 4, 5, 6, 7, 8, and, 9. This is $\frac{9}{50} = 0.18$ or 18%. So 82% of the numbers are greater than 9.

(b) The factors of 12 are 1, 2, 3, 4, 6, and 12. That is 6 out of 50 or 12%.

(c) The numbers with even digits are 2, 4, 6, 8, 20, 22, 24, 26, 28, 40, 42, 44, 46, and 48. That is 14 out of 50 or 28%.

(d) The numbers with odd digits are 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 31, 33, 35, 37, and 39. That is 15 out of 50 or 30%.

30. **(a)** 20% of \$30 = .20(30) = \$6 . The clock increased in value by \$6.

(b) The clock is now worth \$30 + \$ 6 = \$36.

(c) \$30 + (.20)\$30 = (1.20)\$30 = \$36 . The original price times 1.20 would be the way to calculate part (b) in one step.

31. In the table below:

(*i*) Interest rate per period is found by dividing the annual rate by the number of compounding periods per year.

(*ii*) Number of periods: Semiannually is twice per year, quarterly is four times per year, monthly is twelve times per year, daily is 365 times per year.

(*iii*) Amount of interest paid: Where

$A =$ compound amount;
$P =$ principal (\$1000 in this example);
$i =$ interest rate per period expressed as a decimal;
$n =$ number of periods per year;
$I =$ compound interest;

then $A = P(1+i)^n$ and $I = A - P$.

	Interest Rate per Period (i)	Number of Periods (n)	Amount of Interest Paid (I)	Compound Amount (A)
(a)	$\frac{4\%}{2} = 2\% = 0.02$	4	\$82.43	\$1082.43
(b)	$\frac{6\%}{4} = 1.5\% = 0.015$	12	\$195.62	\$1195.62
(c)	$\frac{18\%}{12} = 1.5\% = 0.015$	60	\$1443.22	\$2443.22
(d)	$\frac{18\%}{365} = \frac{0.18}{365}$	1460	\$1054.07	\$2054.07

32. Interest charged was $\$28{,}500 - \$25{,}000 = \$3500$. If $I = Prt$ then $r = \frac{I}{Pt} = \frac{\$3500}{\$25{,}000 \cdot 4} = 0.035 = \mathbf{3.5\%}$.

33. An interest rate of 3% annually is a rate of $\frac{3}{4}\%$ each quarter. Over 5 years, there will be $5 \cdot 4 = 20$ compoundings.

$100{,}000 = P(1+.0075)^{20} \Rightarrow 100{,}000 = P(1.161184142) \Rightarrow P \approx \mathbf{\$86{,}119}$.

34. $A = P(1+i)^n \Rightarrow P = \frac{A}{(1+i)^n}$, where $A = \$4650$, $i = \frac{3.5\%}{4} = 0.875\% = 0.00875$, and $n = 4 \cdot 4 = 16$ periods. $P = \frac{\$4650}{(1+0.00875)^{16}} \approx$ **\$4044.98**.

35. \$10,000 compounded daily at 2% annual interest for 15 years is calculated as $A = \$10{,}000\left(1 + \frac{0.02}{365}\right)^{365 \cdot 15} \approx \mathbf{\$13{,}498.48}$.

36. $(1+0.005)^{12} - 1 \approx$ **43% per year**.

37. The expression for compound growth, where the growth rate is 10% compounded annually, the end amount is 1000, the beginning amount is 500, and the number of editions is n gives:

$$1000 = 500(1+0.10)^n$$
$$\frac{1000}{500} = 2 = 1.1^n$$

By trial and error, 1.1^8 is slightly more than 2, so only **7 editions** could be published if the limit is 1000 pages. $500(1+0.10)^7 =$ would result in about 974 pages.

38. Al's yearly return is given by $P\left(1 + \frac{.06}{365}\right)^{365}$. Betty's is given by $1000(1+.07)$.

Al's Investment		Betty's Investment
Year 1	1061.83	1070
Year 2	1127.49	1140
Year 3	1197.20	1210
Year 4	1271.22	1280
Year 5	1349.83	1350
Year 6	1433.29	1420

39. To develop a formula for compound decay, let A be the amount remaining after one period, r be the rate of decay, and B be the beginning amount. $A = B - \text{decay} = B - rB = B(1-r)$. After n periods, $A = B(1-r)^n$.

In the rain forest, where $B = 2.34 \cdot 10^9$, $r = 0.5\% = 0.005$, and $n = 12 \cdot 20 = 240$, $A = 2.34 \cdot 10^9(1-0.005)^{240} \approx \mathbf{7.03 \cdot 10^8}$ **trees** remaining after 20 years.

Mathematical Connections 7-4: Review Problems

31. **(a)** $0.18(120) = \mathbf{21.6\ lb}$.

(b) $0.4(120) = \mathbf{48\ lb}$.

32. **(a)** $16.72 = \frac{16.72 \cdot 100}{100} = \frac{1672}{100} = \mathbf{\frac{418}{25}}$.

(b) $0.003 = \mathbf{\frac{3}{1000}}$.

(c) $^{-}5.07 = \mathbf{\frac{^{-}507}{100}}$.

(d) $0.123 = \frac{.123 \cdot 1000}{1000} = \mathbf{\frac{123}{1000}}$.

33. **(a)** $5 = 4 + 1 = 4 + .\overline{9} = \mathbf{4.\overline{9}}$.

(b) $5.1 = 5 + .1 = 5 + \frac{1}{10}.\overline{9} = \mathbf{5.0\overline{9}}$.

(c) $\frac{1}{2} = .5 = 0.\mathbf{4\overline{9}}$.

34. $0.00024 = \frac{0.00024 \cdot 100{,}000}{100{,}000} = \frac{24}{100{,}000}$

$= \frac{2^3 \cdot 3}{10^5} = \frac{2^3 \cdot 3}{2^5 \cdot 5^5} = \frac{3}{2^2 5^5} = \mathbf{\frac{3}{12500}}$.

35. $0.\overline{24} = \frac{24}{99} = \frac{8.3}{3.33} = \mathbf{\frac{8}{33}}$.

36. **(a)** $2.08 \cdot 10^5 = \mathbf{208{,}000}$.

(b) $3.8 \cdot 10^{^{-}4} = \mathbf{0.00038}$.

Chapter 7 Review

1. Each division on the number line corresponds to 0.01.

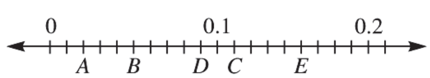
(a) **(*i*)** Point *A* is two divisions to the right of point 0, or **0.02**.

(*ii*) Point *B* is five divisions to the right of point 0, or **0.05**.

(*iii*) Point *C* is one division to the right of point 0.1, or **0.11**.

(b) **(*i*)** Point *D* (0.09) is one division to the left of point 0.1; i.e., $0.1 - 0.01 = 0.09$.

(*ii*) Point *E* (0.15) is five divisions to the right of point 0.1; i.e., $0.1 + 0.05 = 0.15$.



2. **(a)** $32.012 = 32\frac{12}{1000} = \frac{32.012}{1000} = \mathbf{\frac{8003}{250}}$.

(b) $0.00103 = \mathbf{\frac{103}{100{,}000}}$.

3. A fraction in simplest from, $\frac{a}{b}$, can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2 or 5. If only 2's and 5's are present, then a power of 10 can be obtained using the Fundamental Law of Fractions (which shows that the fraction can be written directly as a terminating decimal to the same number of decimal places as the smallest power of 10 in the denominator).

4. $442.4\text{ cm} \div 55.3\text{ cm} = $ **8 shelves**.

5. **(a)** $\frac{4}{7} = 4 \div 7 = \mathbf{0.\overline{571428}}$.

(b) $\frac{1}{8} = 1 \div 8 = \mathbf{0.125}$.

(c) $\frac{2}{3} = 2 \div 3 = \mathbf{0.\overline{6}}$.

(d) $\frac{5}{8} = 5 \div 8 = \mathbf{0.625}$.

6. **(a)** $0.28 = \frac{28}{100} = \mathbf{\frac{7}{25}}$.

(b) $^{-}6.07 = {}^{-}6\frac{7}{100} = \mathbf{\frac{^{-}607}{100}}$.

(c) $0.\overline{3} = \frac{3}{9} = \mathbf{\frac{1}{3}}$.

(d) $2.0\overline{8} = 2\frac{80}{900} = 2\frac{4}{45} = \mathbf{\frac{94}{45}}$.

7. **(a)** **307.63**. $307.6\underline{2}5$ is between 307.62 and 307.63; when the number to be rounded is at the mid-point, by convention it is commonly rounded up.

(b) **307.6**. $307.\underline{6}25$ is between 307.6 and 307.7 and is closer to 307.6.

(c) **308**. $30\underline{7}.625$ is between 307 and 308 and is closer to 308.

(d) **300**. $\underline{3}07.625$ is between 300 and 400 and is closer to 300.

8. **(a)** Move the decimal point five units to the left and multiply by 10^5, or $\mathbf{4.26 \cdot 10^5}$.

(b) $324 \cdot 10^{^{-}6} = 3.24 \cdot 10^2 \cdot 10^{^{-}6} = \mathbf{3.24 \cdot 10^{^{-}4}}$.

(c) Move the decimal point six units to the right and multiply by 10^{-6}, or $0.00000237 = \mathbf{2.37 \cdot 10^{-6}}$.

(d) $^{-}0.325 = {}^{-}\mathbf{3.25 \cdot 10^{-1}}$.

9. Line up the decimal points:

$$1.4\overline{519} = 1.4519519519...$$
$$1.451\overline{9} = 1.4519999999...$$
$$1.4519 = 1.4519000000...$$
$$1.45\overline{19} = 1.4519191919...$$
$$^{-}0.134 = {}^{-}0.1340000000...$$
$$^{-}0.13\overline{401} = {}^{-}0.1340140140...$$
$$0.13\overline{401} = 0.1340140140...$$

Ordering from greatest to least:

$\mathbf{1.451\overline{9}} > \mathbf{1.4\overline{519}} > \mathbf{1.45\overline{19}} > \mathbf{1.4519} > \mathbf{0.13\overline{401}} > {}^{-}\mathbf{0.134} > {}^{-}\mathbf{0.13\overline{401}}$.

10. **(a)** Move the decimal point six units to the right and multiply by 10^6, or $\mathbf{1.78341156 \cdot 10^6}$.

(b) $\frac{347}{10^8} = 347 \cdot 10^{-8} = 3.47 \cdot 10^2 \cdot 10^{-8} = \mathbf{3.47 \cdot 10^{-6}}$.

(c) $49.3 \cdot 10^8 = 4.93 \cdot 10^1 \cdot 10^8 = \mathbf{4.93 \cdot 10^9}$.

(d) $29.4 \cdot \frac{10^{12}}{10^1} = 2.94 \cdot 10^1 \cdot 10^{12} \cdot 10^4 = \mathbf{2.94 \cdot 10^{17}}$.

(e) $0.47 \cdot 1000^{12} = 4.7 \cdot 10^{-1} \cdot (10^3)^{12} = 4.7 \cdot 10^{-1} \cdot 10^{36} = \mathbf{4.7 \cdot 10^{35}}$.

(f) $\frac{3}{5^9} = \frac{3 \cdot 2^9}{5^9 \cdot 2^9} = \frac{1536}{10^9} = 1536 \cdot 10^{-9} = 1.536 \cdot 10^3 \cdot 10^{-9} = \mathbf{1.536 \cdot 10^{-6}}$.

11. **(a)** Answers may vary. E.g., $0.11 > 0.105 > 0.104 > 0.103 > 0.102 > 0.101 > 0.1$.

(b) Begin with 0.1 and find half of each succeeding element, or $0 < 0.00625 < 0.0125 < 0.025 < 0.05 < 0.1$.

(c) $0.1 < 0.15 < 0.175 < 0.1875 < 0.19375 < 0.2$.

12. **(a)** $6 = n\%$ of $24 \Rightarrow n\% = \frac{6}{24} \Rightarrow \frac{n}{100} = \frac{1}{4} \Rightarrow 4n = 100 \Rightarrow n = 25$, or $6 = \mathbf{25\%}$ of 24.

(b) $n = 320\%$ of $60 \Rightarrow n = \frac{320}{100} \cdot 60 = \mathbf{192}$.

(c) $17 = 30\%$ of $n \Rightarrow n = \frac{17}{\left(\frac{30}{100}\right)} = \frac{1700}{30} = \mathbf{56.\overline{6}}$.

(d) $0.2 = n\%$ of $1 \Rightarrow \frac{n}{100} = \frac{0.2}{1} \Rightarrow n = 20$, or $0.2 = \mathbf{20\%}$ of 1.

13. **(a)** $\frac{1}{8} = 0.125 = 100\left(\frac{0.125}{100}\right) = \frac{12.5}{100} = \mathbf{12.5\%}$.

(b) $\frac{3}{40} = 0.075 = 100\left(\frac{0.075}{100}\right) = \frac{7.5}{100} = \mathbf{7.5\%}$.

(c) $6.27 = 100\left(\frac{6.27}{100}\right) = \frac{627}{100} = \mathbf{627\%}$.

(d) $0.0123 = 100\left(\frac{0.0123}{100}\right) = \frac{1.23}{100} = \mathbf{1.23\%}$.

(e) $\frac{3}{2} = 1.5 = 100\left(\frac{1.5}{100}\right) = \frac{150}{100} = \mathbf{150\%}$.

14. **(a)** $60\% = \frac{60}{10} = \mathbf{0.60}$.

(b) $\frac{2}{3}\% = \frac{\left(\frac{2}{3}\right)}{100} = \frac{2}{300} = \mathbf{0.00\overline{6}}$.

(c) $100\% = \frac{100}{100} = \mathbf{1}$.

15. $11\% \cdot \text{investment} = \$1020.80 \Rightarrow \text{investment} = \frac{\$1020.80}{0.11} = \mathbf{\$9280}$.

16. Percent defective $= 100\left[\frac{\left(\frac{5}{150}\right)}{100}\right] = \frac{\left(\frac{500}{150}\right)}{100} = \frac{3.\overline{3}}{100} = \mathbf{3.\overline{3}\%}$ or $\mathbf{3\frac{1}{3}\%}$.

17. Assume "nearest tenth" refers to the nearest tenth of a percent. Then percent correct $= 100\left(\frac{\frac{62}{70}}{100}\right) = \frac{\left(\frac{6200}{70}\right)}{100} \approx \frac{88.6}{100} = \mathbf{88.6\%}$.

18. Let C be the cost four years ago. $60\% \cdot C = \$3450 \Rightarrow C = \frac{\$3450}{0.60} = \mathbf{\$5750}$.

19. A discount of $d\%$ means the customer pays $1 - \frac{d}{100}$ for the purchase. Discounts of 5%, 10%, and 20% mean the customer pays 0.95, 0.90, and 0.80, respectively, of cost. Their product, 0.648, is the same in any order, therefore there is **no difference**. The customer would pay 0.684 times the purchase price, for a discount of $1 - 0.684 = 0.316$, or 31.6%.

20. Let c be the cost of the bicycle. 100% of $c + 30\%$ of $c = \$104 \Rightarrow c + 0.3c = \$104 \Rightarrow 1.3c = \$104$. $c = \frac{\$104}{1.3} = \mathbf{\$80}$.

21. The price difference was $\$89.95 - \$62.00 = \$27.95$. The percentage difference was $100\left[\frac{\left(\frac{27.95}{89.95}\right)}{100}\right] \approx \mathbf{31.1\%}$.

22. Answers may vary. If the dress was originally priced at \$100, 60% off would result in a sale price of \$40. Then the 40% off coupon would result in a final price of $\$40 - (0.40 \cdot \$40) = \$24$. The reasoning could be applied to the actual list price of the dress.

23. If the item is \$100 then the markup is \$130. The sales price is then $\$130 - (.30)\$130 = \$91$ which is \$9 less than the whole sale price. The store would be losing 9%.

24. $I = Prt$, where $P = \$30,000$, $r = 12.5\% = 0.125$, and $t = 4$ years. $I = \$30,000 \cdot 0.125 \cdot 4 = \mathbf{\$15,000}$.

25. \$10,000 compounded quarterly at 14% annual interest for 3 years will, where $A = P(1+i)^n$, yield $A = \$10,000(1+\frac{0.14}{4})^{3\cdot 4} \approx \mathbf{\$15,110.69}$.

26. To reduce the crust from 25% to 20% the crust is decreased by 5% which is 25% of 20%. So it should be reduced by 25%.

CHAPTER 8

Real Numbers and Algebraic Thinking

Assessment 8-1A: Real Numbers

1. Answers may vary. E.g., one such number could be 0.232233222333…, continuing the pattern of adding a 2 and a 3 to each succeeding group.

2. (a) $x^2 = 2^2 + (\sqrt{2})^2 = 4 + 2 = 6$.
$x = \mathbf{\sqrt{6}}$.

   (b) $x^2 = (2)^2 + (2)^2 = 4 + 4 = 8$.
$x = \sqrt{8} = 2\sqrt{2}$.

   (c) $5^2 = 2^2 + x^2 \Rightarrow x^2 = 5^2 - 2^2$
$= 25 - 4 = 21$
$x = \sqrt{21}$.

3. Line up the decimal points:
$$0.9 = 0.90000000\ldots$$
$$0.\overline{9} = 0.99999999\ldots$$
$$0.\overline{98} = 0.98989898\ldots$$
$$0.9\overline{88} = 0.98888888\ldots$$
$$0.\overline{998} = 0.99898989\ldots$$
$$0.\overline{898} = 0.89889889\ldots$$
$$\sqrt{0.98} = 0.98994949\ldots$$
Ordering from greatest to least:
$$0.\overline{9} > 0.\overline{998} > \sqrt{0.98} > 0.\overline{98} > 0.9\overline{88} > 0.9 > 0.\overline{898}.$$

4. (a) **Irrational**. There is no rational number s such that $s^2 = 51$.

   (b) **Rational**. $8^2 = 64$.

   (c) **Rational**. $18^2 = 324$.

   (d) **Irrational**. There is no rational number s such that $s^2 = 325$.

   (e) **Irrational**. The sum of a rational number and an irrational number is irrational.

   (f) **Irrational**. The quotient of any non-zero rational number and any irrational number is irrational.

5. (a) $15 \cdot 15 = 225 \Rightarrow \sqrt{225} = \mathbf{15}$.

   (b) $13 \cdot 13 = 169 \Rightarrow \sqrt{169} = \mathbf{13}$.

   (c) $^{-}1 \cdot 9 \cdot 9 = {}^{-}81 \Rightarrow {}^{-}\sqrt{81} = {}^{-}9$.

   (d) $25 \cdot 25 = 625 \Rightarrow \sqrt{625} = \mathbf{25}$.

   (e) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \Rightarrow \sqrt{\frac{1}{4}} = \frac{1}{2}$.

   (f) $0.01 \cdot 0.01 = 0.0001 \Rightarrow \sqrt{0.0001} = 0.01$.

6. (a) $2 < \sqrt{7} < 3$.
$(2.6)^2 = 6.76$ and $(2.7)^2 = 7.29$
$\Rightarrow 2.6 < \sqrt{7} < 2.7$.
$(2.64)^2 = 6.9696$ and $(2.65)^2 = 7.0225$
$\Rightarrow 2.64 < \sqrt{7} < 2.65$.
7 is closer to 7.0225 than to 6.9696 $\Rightarrow$
$\sqrt{7} \approx \mathbf{2.65}$.

   (b) $0.1 < \sqrt{0.0120} < 0.2$.
$(0.10)^2 = 0.0100$ and $(0.11)^2 = 0.0121$
$\Rightarrow 0.10 < \sqrt{0.0120} < 0.11$.
0.0120 is closer to 0.0121 than to 0.0100
$\Rightarrow \sqrt{0.0120} \approx \mathbf{0.11}$.

7. Answers may vary.

   (a) **False**. $0 + \sqrt{2}$ is irrational.

   (b) **False**. $^{-}\sqrt{2} + \sqrt{2} = 0$, which is rational.

   (c) **False**. $\sqrt{2} \cdot \sqrt{2} = 2$, which is rational.

   (d) **True**. $\sqrt{2} - \sqrt{2} = 0$, which is rational.

8. Answers may vary.

   (a) $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ are three such irrational numbers.

   (b) $0.\overline{54} = 0.54545454\ldots$
$< 0.546010010001\ldots$
$< 0.547010010001\ldots$
$< 0.548010010001\ldots$
$< 0.\overline{55} = 0.55555555\ldots$.

(c) $\frac{1}{3} = 0.\overline{3} = 0.333....$

$< 0.34010010001\ldots$

$< 0.35010010001...$

$< 0.36010010001... < 0.5.$

9. Consider 8(c). We form an irrational number between $0.\overline{3}$ and 0.5 by placing digits in lesser place-values so that the decimal does not terminate or repeat, e.g., 0.3434434443…. A similar approach can be taken for any two rational numbers. For example, 1.7171171117… is between 1.7 and 1.8.

10. **(a)** $Q \cup S = \boldsymbol{R}$. The set of real numbers contains any number which is either rational or irrational.

(b) $Q \cap S = \varnothing$. A rational number cannot be an irrational number.

(c) $Q \cap R = \boldsymbol{Q}$. $Q \subset R$.

(d) $S \cap W = \varnothing$. No whole number can be irrational.

(e) $W \cup R = \boldsymbol{R}$. $W \subset R$.

(f) $Q \cup R = \boldsymbol{R}$. $Q \subset R$.

11. In the tables of 11. and 12. below: N is the set of natural (or counting) numbers; I is the set of integers; Q is the set of rational numbers; R is the set of real numbers; and S is the set of irrational numbers. $N \subset I \subset Q \subset R$; $R = Q \cup S$.

		N	I	Q	S	R
(a)	6.7			√		√
(b)	5	√	√	√		√
(c)	$\sqrt{2}$				√	√
(d)	$^-5$		√	√		√
(e)	$3\frac{1}{7}$			√		√

12.

		$x =$	N	I	Q	S	R
(a)	$x^2 + 1 = 5$	$2, {}^-2$	√	√	√		√
(b)	$2x - 1 = 32$	$\frac{33}{2}$			√		√
(c)	$x^2 = 3$	$\sqrt{3}, {}^-\sqrt{3}$				√	√
(d)	$\sqrt{x} = {}^-1$	no solution					
(e)	$\frac{3}{4}x = 0.\overline{4}$	$\frac{16}{27}$			√		√

13. **(a)** $x = \mathbf{64}$. $\sqrt{64} = 8$.

(b) **No real values**. $\sqrt{x}$ is the principal square root of x.

(c) $x = {}^-\mathbf{64}$. $\sqrt{{}^-({}^-64)} = \sqrt{64} = 8$.

(d) **No real values**. $\sqrt{{}^-x}$ is the principal square root of x, if $x < 0$.

(e) All real numbers > 0. If $x = 0$ then $\sqrt{x} \not> 0$.

(f) **No real values**. $\sqrt{x}$ is the principal square root of x.

14. **(a)** $\sqrt{180} = \sqrt{36 \cdot 5} = \sqrt{36} \cdot \sqrt{5} = \mathbf{6\sqrt{5}}$.

(b) $\sqrt{363} = \sqrt{121 \cdot 3} = \sqrt{121} \cdot \sqrt{3} = \mathbf{11\sqrt{3}}$.

(c) $\sqrt{252} = \sqrt{36 \cdot 7} = \sqrt{36} \cdot \sqrt{7} = \mathbf{6\sqrt{7}}$.

15. **(a)** $\sqrt[3]{{}^-54} = \sqrt[3]{{}^-27 \cdot 2} = \sqrt[3]{{}^-27} \cdot \sqrt[3]{2} = {}^-\mathbf{3\sqrt[3]{2}}$.

(b) $\sqrt[5]{96} = \sqrt[5]{32 \cdot 3} = \mathbf{2\sqrt[5]{3}}$.

(c) $\sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = \mathbf{5\sqrt[3]{2}}$.

(d) $\sqrt[5]{{}^-243} = {}^-\mathbf{3\sqrt[5]{1} = {}^-3 \cdot 1 = {}^-3}$.

16. **(a)** $a_1 = 5; a_4 = 10; n = 4$. Thus $10 = 5(r)^{4-1}$ $\Rightarrow 2 = r^3 \Rightarrow r = \sqrt[3]{2}$. The terms of the sequence are $5, \underline{\mathbf{5\sqrt[3]{2}}}, \underline{\mathbf{5\sqrt[3]{4}}}, 5\sqrt[3]{8} = 10$.

(b) $a_1 = 2; a_5 = 1; n = 5$. Thus $1 = 2(r)^{5-1}$ $\Rightarrow r = \sqrt[4]{\frac{1}{2}}$ or $r = {}^-\sqrt[4]{\frac{1}{2}}$. The terms of the

sequence are then $2 = \sqrt[4]{16}, \underline{\sqrt[4]{8}}, \underline{\sqrt[4]{4}}, \underline{\sqrt[4]{2}},$
$\sqrt[4]{1} = 1$ or $2 = \sqrt[4]{16}, \underline{{}^{-}\sqrt[4]{8}}, \underline{\sqrt[4]{4}}, {}^{-}\sqrt[4]{2}, \sqrt[4]{1} = 1.$

(c) $a_2 = 5$ and $a_4 = 3$ Thus $5 = a_1(r)^{2-1}$ and $3 = a_1(r)^{4-1}$, so $5 = a_1 r$ and $3 = a_1 r^3$ Thus

$a_1 = \frac{5}{r}$ and $a_1 = \frac{3}{r^3} \Rightarrow \frac{5}{r} = \frac{3}{r^3} \Rightarrow$

$5r^3 = 3r \Rightarrow 5r^3 - 3r = 0 \Rightarrow r(5r^2 - 3) = 0$

$\Rightarrow r = 0$ and $5r^2 - 3 = 0 \Rightarrow 5r^2 = 3$

$\Rightarrow r^2 = \frac{3}{5} \Rightarrow r = \pm\sqrt{\frac{3}{5}} \Rightarrow r = \pm\frac{\sqrt{15}}{5}$

The terms of the sequence are

$\frac{5\sqrt{15}}{3}, 5, \sqrt{15}, 3$ or ${}^{-}\frac{5\sqrt{15}}{3}, 5, {}^{-}\sqrt{15}, 3$

(d) $a_2 = {}^{-}2$ and $a_4 = {}^{-}3$ Thus ${}^{-}2 = a_1(r)^{2-1}$ and ${}^{-}3 = a_1(r)^{4-1}$, so ${}^{-}2 = a_1 r$ and ${}^{-}3 = a_1 r^3$ Thus

$a_1 = \frac{{}^{-}2}{r}$ and $a_1 = \frac{{}^{-}3}{r^3} \Rightarrow \frac{{}^{-}2}{r} = \frac{{}^{-}3}{r^3} \Rightarrow {}^{-}2r^3 = {}^{-}3r$

$\Rightarrow {}^{-}2r^3 + 3r = 0 \Rightarrow {}^{-}r(2r^2 - 3) = 0 \Rightarrow {}^{-}r = 0$

and $2r^2 - 3 = 0 \Rightarrow 2r^2 = 3 \Rightarrow r^2 = \frac{3}{2}$

$\Rightarrow r = \pm\sqrt{\frac{3}{2}} \Rightarrow r = \pm\frac{\sqrt{6}}{2}$

The terms of the sequence when $r = \frac{\sqrt{6}}{2}$ are

${}^{-}\frac{2\sqrt{6}}{3}, {}^{-}2, {}^{-}\sqrt{6}, {}^{-}3$.

The terms of the sequence when $r = {}^{-}\frac{\sqrt{6}}{2}$ are

$\frac{2\sqrt{6}}{3}, {}^{-}2, \sqrt{6}, {}^{-}3$.

17. **(a)** $E(0) = 2^{10} \cdot 16^0 = \mathbf{2^{10}}$ **bacteria**.

(b) $E\left(\frac{1}{4}\right) = 2^{10} \cdot 16^{1/4} = 2^{10} \cdot 2$
$= \mathbf{2^{11}}$ **bacteria**.

(c) $E\left(\frac{1}{2}\right) = 2^{10} \cdot 16^{1/2} = 2^{10} \cdot 2^2$
$= \mathbf{2^{12}}$ **bacteria**.

18. **(a)** $(0.008)^{\frac{2}{3}} = (\frac{8}{1000})^{\frac{2}{3}} = (\frac{2^3}{10^3})^{\frac{2}{3}}$

$\frac{(2^3)^{\frac{2}{3}}}{(10^3)^{\frac{2}{3}}} = \frac{2^2}{10^2} = \frac{4}{100} = \frac{1}{25}$

(b) $(6.25)^{\frac{3}{2}} = (\frac{625}{100})^{\frac{3}{2}} = \left((\frac{25^2}{10^2})^{\left(\frac{1}{2}\right)}\right)^3$

$= \left(\frac{25}{10}\right)^3 = \left(\frac{5}{2}\right)^3 = \frac{125}{8}$

(c) $\left(\frac{25}{81}\right)^{{}^{-}\frac{3}{2}} = \left(\frac{81}{25}\right)^{\frac{3}{2}} = (\frac{9^2}{5^2})^{\frac{3}{2}} = \frac{9^3}{5^3} = \frac{729}{125}$

(d) $(0.0000128)^{\frac{3}{7}} = (\frac{128}{10000000})^{\frac{3}{7}} = (\frac{2^7}{10^7})^{\frac{3}{7}}$

$= \frac{2^3}{10^3} = \frac{8}{1000} = \frac{1}{125}$

(e) $({}^{-}27)^{{}^{-}\frac{4}{3}} = ({}^{-}\frac{1}{27})^{\frac{4}{3}} = ({}^{-}\frac{1^3}{3^3})^{\frac{4}{3}} = ({}^{-}\frac{1}{3})^4 = \frac{1}{81}$

(f) ${}^{-}(27)^{{}^{-}\frac{4}{3}} = {}^{-}(\frac{1}{27})^{\frac{4}{3}} = {}^{-}(\frac{1^3}{3^3})^{\frac{4}{3}} = {}^{-}(\frac{1}{3})^4 = {}^{-}\frac{1}{81}$

19. **(a)** $3^x = 243 \Rightarrow 3^x = 3^5 \Rightarrow x = 5.$

(b) $9^{{}^{-}x} = 27 \Rightarrow 3^{{}^{-}2x} = 3^3 \Rightarrow {}^{-}2x = 3 \Rightarrow x = {}^{-}\frac{3}{2}$

$\left(\frac{9}{4}\right)^{3x} = \frac{32}{243} \Rightarrow \left(\frac{3^2}{2^2}\right)^{3x} = \frac{2^5}{3^5}$

(c) $\Rightarrow \left(\frac{3}{2}\right)^{6x} = \left(\frac{2}{3}\right)^5 \Rightarrow \left(\frac{3}{2}\right)^{6x} = \left(\frac{3}{2}\right)^{{}^{-}5}$

$\Rightarrow 6x = {}^{-}5 \Rightarrow x = {}^{-}\frac{5}{6}$

(d) $\sqrt{{}^{-}x} = 3\sqrt{2} \Rightarrow \sqrt{{}^{-}x} = \sqrt{18} \Rightarrow$
${}^{-}x = 18 \Rightarrow x = {}^{-}18$

(e) $x^{-\frac{3}{4}} = 2 \Rightarrow (x^{-\frac{3}{4}})^{-\frac{4}{3}} = (2)^{-\frac{4}{3}} \Rightarrow x = (2)^{-\frac{4}{3}}$

(f) $(x-1)^2 = 2 \Rightarrow x-1 = \pm\sqrt{2} \Rightarrow x = 1 \pm \sqrt{2}$

20. Answers may vary.

(a) $\sqrt[5]{20} \Rightarrow x^5 - 20 = 0$

(b) $\sqrt[3]{^-2} \Rightarrow x^3 + 2 = 0$

(c) $\sqrt[3]{10} - 1 \Rightarrow (x+1)^3 - 10 = 0$

(d) $\dfrac{\sqrt{2}}{\sqrt{3}} \Rightarrow 3x^2 - 2 = 0$

21. Using "guess and check"

$4^3 = 64$

$4.6^3 = 97.3$

$4.7^3 = 103.8$

$5^3 = 125$

So, an integer approximation of $\sqrt[3]{103}$ is 5.

22. (a) $\dfrac{\sqrt{500}}{\sqrt{20}} = \dfrac{10\sqrt{5}}{2\sqrt{5}} = 5$ so it is rational.

(b) $8^{\frac{1}{3}} + 8^{-\frac{1}{3}} = (2^3)^{\frac{1}{3}} + (2^3)^{-\frac{1}{3}} = 2 + \dfrac{1}{2} = 2\dfrac{1}{2}$ so it is rational.

(c) $\dfrac{2}{\sqrt{2}} - \sqrt{2} = \dfrac{2}{\sqrt{2}} - \dfrac{2}{\sqrt{2}} = 0$ so it is rational.

(d) $\sqrt{1000} = 31.6227766...$ so it is irrational.

Assessment 8-1B

1. Answers may vary; one such could be 0.454455444555…, and continuing the pattern of adding a 4 and a 5 to each succeeding group.

2. (a) $x^2 = 3^2 + \left(\sqrt{3}\right)^2 = 9 + 3 = 12$

$\Rightarrow x = \sqrt{12} = \sqrt{4 \cdot 3} = \mathbf{2\sqrt{3}}$.

(b) $x^2 = (3)^2 + (3)^2 = 9 + 9 = 18$

$\Rightarrow x = \sqrt{18} = 3\sqrt{2}$

(c) $x^2 + 12^2 = 25^2 \Rightarrow x^2 = 25^2 - 12^2$

$x^2 = 481 \Rightarrow x = \sqrt{481}$

3. Line up the decimal points:

$$0.8 = 0.80000\ldots$$
$$0.\overline{8} = 0.88888\ldots$$
$$0.\overline{89} = 0.89898\ldots$$
$$0.\overline{889} = 0.88989\ldots$$
$$\sqrt{0.7744} = 0.88000\ldots$$

Ordering from greatest to least:
$0.\overline{89} > 0.\overline{889} > 0.\overline{8} > \sqrt{0.7744} > 0.8$.

4. (a) **Irrational**. There is no rational number n such that $n^2 = 78$.

(b) **Rational**. $9^2 = 81$.

(c) **Rational**. $7^3 = 343$.

(d) **Rational**. The sum of two rational numbers is rational, since the rationals are closed under addition.

(e) **Irrational**. The quotient of any non-zero rational number and any irrational number is irrational.

5. (a) $16 \cdot 16 = 256 \Rightarrow \sqrt{256} = \mathbf{16}$.

(b) $18 \cdot 18 = 324 \Rightarrow \sqrt{324} = \mathbf{18}$.

(c) **Impossible**. There is no real number n such that $n^4 = {}^-1$ (or any other negative number).

(d) $32 \cdot 32 = 1024 \Rightarrow \sqrt{1024} = \mathbf{32}$.

6. (a) $4 < \sqrt{20.3} < 5$

$(4.5)^2 = 20.25$ and $(4.6)^2 = 21.16 \Rightarrow$

$4.5 < \sqrt{20.3} < 4.6$

$(4.50)^2 = 20.25$ and $(4.51)^2 = 20.3401 \Rightarrow$

$4.50 < \sqrt{20.3} < 4.51$

20.3 is closer to 20.3401 than to 20.25 $\Rightarrow$

$\sqrt{20.3} \approx \mathbf{4.51}$.

(b) $1 < \sqrt{1.64} < 2$

$(1.2)^2 = 1.44$ and $(1.3)^2 = 1.69 \Rightarrow$

$1.2 < \sqrt{1.64} < 1.3$

$(1.28)^2 = 1.6384$ and $(1.29)^2 = 1.6641 \Rightarrow$

$1.28 < \sqrt{1.64} < 1.29$

1.64 is closer to 1.6384 than to 1.6641 $\Rightarrow$

$\sqrt{1.64} \approx \mathbf{1.28}$.

7. **(a)** **True**. The rationals are closed under addition.

(b) **False**. For example, $\sqrt{2} - \sqrt{2} = 0$, which is rational.

(c) **False**. If the rational number is 0, any product will be 0, which is rational.

8. Answers may vary.

(a) $\sqrt{9.1}, \sqrt{9.3}$, and $\sqrt{9.5}$ are three such numbers.

(b) $0.55555555\ldots = 0.\overline{55}$
$< 0.562010010001...$
$< 0.563010010001...$
$< 0.564010010001...$
$< 0.56565656\ldots. = 0.\overline{56}$

(c) 0.01
0.0101010010001…
0.0102010010001…
0.0103010010001…
0.011

9. Suppose $\frac{\sqrt{2}}{2}$ is a rational number which can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Then $\frac{\sqrt{2}}{2} = \frac{a}{b} \Rightarrow \sqrt{2} = \frac{2a}{b}$, which is a rational number. But $\sqrt{2}$ is an irrational number which cannot equal a rational number, so a contradiction exists and therefore $\frac{\sqrt{2}}{2}$ must be irrational.

10. **(a)** $Q \cap I = \boldsymbol{I}$. $I \subset Q$, since the set of rational numbers contains the set of integers.

(b) $S - Q = \boldsymbol{S}$. There are no irrational numbers which are also rational.

(c) $R \cup S = \boldsymbol{R}$. $S \subset R$, or irrational numbers are real.

(d) $\boldsymbol{R}$. Q, I, W, and S are all subsets of the set of real numbers.

(e) $\overline{S} = \boldsymbol{Q}$, where Q is the set of rational numbers. $Q \cup S = R; Q \cap S = \varnothing$.

11. In tables 11. and 12. below: N is the set of natural (or counting) numbers; I is the set of integers; Q is the set of rational numbers; R is the set of real numbers; and S is the set of irrational numbers.
$N \subset I \subset Q \subset R; R = Q \cup S$.

		N	I	Q	S	R
(a)	$\sqrt{3}$				√	√
(b)	$4\frac{1}{2}$			√		√
(c)	$^-3\frac{1}{7}$			√		√

12.

		$x =$	N	I	Q	S	R
(a)	$x^2 + 2 = 4$	$\sqrt{2}, {}^-\sqrt{2}$				√	√
(b)	$1 - 2x = 32$	$\frac{^-31}{2}$			√		√
(c)	$x^3 = 4$	$\sqrt[3]{4}$				√	√
(d)	$\sqrt{x} = {}^-2$	no solution					
(e)	$0.\overline{7}x = 5$	$\frac{45}{7}$			√		√

13. **(a)** $\sqrt{x} = 5 \Rightarrow \sqrt{25} = 5 \Rightarrow x = 25$

(b) No real numbers. $\sqrt{x}$ is the principal square root of x.

(c) $\sqrt{^-x} = 5 \Rightarrow \left(\sqrt{^-x}\right)^2 = 5^2$
$\Rightarrow {}^-x = 25 \Rightarrow x = {}^-25$

(d) No real numbers. $\sqrt{x}$ is always positive, so its additive inverse would be negative.

(e) ${}^-\sqrt{x} = {}^-5 \Rightarrow \sqrt{x} = 5 \Rightarrow x = 25$

14. **(a)** $\sqrt{360} = \sqrt{36 \cdot 10} = \sqrt{36} \cdot \sqrt{10} = \mathbf{6\sqrt{10}}$.

(b) $\sqrt{40} = \sqrt{4 \cdot 10} = \sqrt{4} \cdot \sqrt{10} = \mathbf{2\sqrt{10}}$.

(c) $\sqrt{240} = \sqrt{16 \cdot 15} = \sqrt{16} \cdot \sqrt{15} = \mathbf{4\sqrt{15}}$.

15. **(a)** $\sqrt[3]{^-102} = {}^-\mathbf{1}\sqrt[3]{\mathbf{102}}$. There is no number n such that $n^3 = 102$.

(b) $\sqrt[6]{64} = \sqrt[6]{2^6} = \mathbf{2}$.

(c) $\sqrt[3]{64} = \sqrt[3]{(2^2)^3} = 2^2 = \mathbf{4}$.

16. **(a)** $a_1 = 4, a_4 = 8$, and $n = 4$. Thus $8 = 4(r)^{4-1} \Rightarrow r^3 = 2 \Rightarrow r = \sqrt[3]{2}$.

Then $a_2 = 4(\sqrt[3]{2})$ and $a_3 = 4(\sqrt[3]{2})^2 = 4\sqrt[3]{4}$.

The terms of the sequence are 4, $\underline{\mathbf{4\sqrt[3]{2}}}$, $\underline{\mathbf{4\sqrt[3]{4}}}$, 8.

(b) $a_1 = 1, a_4 = 2,$ and $n = 4$. Thus $2 = 1(r)^{4-1} \Rightarrow r^3 = 2 \Rightarrow r = \sqrt[3]{2}$.

Then $a_2 = 1(\sqrt[3]{2}) = \sqrt[3]{2}$ and $a_3 = (\sqrt[3]{2})^2 = \sqrt[3]{4}$. The terms of the sequence are $1, \underline{\boldsymbol{\sqrt[3]{2}}}, \underline{\boldsymbol{\sqrt[3]{4}}}, 2$.

(c) $a_3 = 2$ and $a_4 = \sqrt{2}$ Thus $2 = a_1(r)^{3-1}$ and $\sqrt{2} = a_1(r)^{4-1}$, so $2 = a_1r^2$ and $\sqrt{2} = a_1r^3$ Thus

$$a_1 = \frac{2}{r^2} \text{ and } a_1 = \frac{\sqrt{2}}{r^3} \Rightarrow \frac{2}{r^2} = \frac{\sqrt{2}}{r^3}$$
$$\Rightarrow 2r^3 = \sqrt{2}r^2 \Rightarrow 2r^3 - \sqrt{2}r^2 = 0$$
$$\Rightarrow r^2(2r - \sqrt{2}) = 0 \Rightarrow r^2 = 0$$
$$\text{and } 2r - \sqrt{2} = 0 \Rightarrow r = \frac{\sqrt{2}}{2}$$

The terms of the sequence are $4, 2\sqrt{2}, 2, \sqrt{2}$

(d) $a_2 = 1$ and $a_4 = 2$ Thus $1 = a_1(r)^{2-1}$ and $2 = a_1(r)^{4-1}$, so $1 = a_1r$ and $2 = a_1r^3$

$$a_1 = \frac{1}{r} \text{ and } a_1 = \frac{2}{r^3} \Rightarrow \frac{1}{r} = \frac{2}{r^3} \Rightarrow r^3 = 2r$$
$$\text{Thus } \Rightarrow r^3 - 2r = 0 \Rightarrow r(r^2 - 2) = 0 \Rightarrow r = 0$$
$$\text{and } r^2 - 2 = 0 \Rightarrow r^2 = 2 \Rightarrow r = \pm\sqrt{2}$$

The terms of the sequence are

$$\frac{^-\sqrt{2}}{2}, 1, {}^-\sqrt{2}, 2 \text{ or } \frac{\sqrt{2}}{2}, 1, \sqrt{2}, 2$$

17. Given the exponential function $E(t) = 8^t$, where $t \geq 0$ in this application.

(a) $E(0) = 8^0 = \mathbf{1}$.

(b) $E\left(\frac{1}{3}\right) = 8^{\frac{1}{3}} = \sqrt[3]{8} = \mathbf{2}$.

18. (a) $(0.008)^{^-\frac{2}{3}} = (\frac{1000}{8})^{\frac{2}{3}} = (\frac{10^3}{2^3})^{\frac{2}{3}}$

$$\frac{(10^3)^{\frac{2}{3}}}{(2^3)^{\frac{2}{3}}} = \frac{10^2}{2^2} = \frac{100}{4} = 25$$

(b) $(0.04)^{\frac{3}{2}} = (\frac{4}{100})^{\frac{3}{2}} = (\frac{2^2}{10^2})^{\frac{3}{2}} = \frac{2^3}{10^3}$

$$= \frac{8}{1000} = \frac{1}{125}$$

(c) $\left(\frac{^-8}{27}\right)^{^-\frac{2}{3}} = \left(\frac{^-27}{8}\right)^{\frac{2}{3}} = \left(\frac{^-3^3}{2^3}\right)^{\frac{2}{3}} = \left(\frac{^-3}{2}\right)^2 = \frac{9}{4}$

(d) $(0.00000256)^{\frac{3}{8}} = (\frac{256}{100000000})^{\frac{3}{8}} = (\frac{2^8}{10^8})^{\frac{3}{8}}$

$$= \frac{2^3}{10^3} = \frac{8}{1000} = \frac{1}{125}$$

(e) $(^-32)^{^-\frac{4}{5}} = (^-\frac{1}{32})^{\frac{4}{5}} = (^-\frac{1^5}{2^5})^{\frac{4}{5}} = (^-\frac{1}{2})^4 = \frac{1}{16}$

(f) $^-(32)^{^-\frac{4}{5}} = {}^-\left(\frac{1}{32}\right)^{\frac{4}{5}} = {}^-\left(\frac{1^5}{2^5}\right)^{\frac{4}{5}}$

$$= {}^-\left(\frac{1}{2}\right)^4 = {}^-\frac{1}{16}$$

19. (a) $3^x = {}^-243 \Rightarrow 3^x = \left(^-3\right)^5 \Rightarrow$ no solution.

(b) $81^{^-x} = 27 \Rightarrow 3^{^-4x} = 3^3 \Rightarrow {}^-4x = 3 \Rightarrow x = {}^-\frac{3}{4}$

$$\left(\frac{4}{9}\right)^{^-3x} = \frac{32}{243} \Rightarrow \left(\frac{3^2}{2^2}\right)^{3x} = \frac{2^5}{3^5}$$

(c) $\Rightarrow \left(\frac{3}{2}\right)^{6x} = \left(\frac{2}{3}\right)^5 \Rightarrow \left(\frac{3}{2}\right)^{6x} = \left(\frac{3}{2}\right)^{^-5}$

$$\Rightarrow 6x = {}^-5 \Rightarrow x = {}^-\frac{5}{6}$$

(d) $\sqrt[3]{^-x} = 2\sqrt[3]{3} \Rightarrow \sqrt[3]{^-x} = \sqrt[3]{24} \Rightarrow$

$$^-x = 24 \Rightarrow x = {}^-24$$

(e) $x^{^-\frac{2}{5}} = 2 \Rightarrow (x^{^-\frac{2}{5}})^{^-\frac{5}{2}} = (2)^{^-\frac{5}{2}} \Rightarrow x = (2)^{^-\frac{5}{2}}$

(f) $(x+1)^2 = 2 \Rightarrow x + 1 = \pm\sqrt{2} \Rightarrow x = {}^-1 \pm \sqrt{2}$

20. Answers may vary.

(a) $\sqrt[3]{5} \Rightarrow x^3 - 5 = 0$

(b) $\sqrt[5]{^-2} \Rightarrow x^5 + 2 = 0$

(c) $\sqrt{2} - 1 \Rightarrow (x+1)^2 - 2 = 0$

(d) $\frac{\sqrt{2}}{\sqrt[3]{2}} = \frac{2^{\frac{1}{2}}}{2^{\frac{1}{3}}} = 2^{\frac{1}{2}-\frac{1}{3}} = 2^{\frac{3}{6}-\frac{2}{6}} = 2^{\frac{1}{6}} \Rightarrow x^6 - 2 = 0$

21. Using "guess and check"

$12^3 = 1728$

$12.6^3 = 2000.4$

$12.7^3 = 2048.4$

$13^3 = 2197$

So, an integer approximation of $\sqrt[3]{2001}$ is13.

22. (a) $\frac{\sqrt{320}}{\sqrt{20}} = \frac{8\sqrt{5}}{2\sqrt{5}} = 4$ is rational.

(b) $2\sqrt{50} - 8\sqrt{5} = 10\sqrt{2} - 8\sqrt{5}$ is irrational.

(c) $\frac{\sqrt{2}}{2} - \sqrt{2} = \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = \frac{\sqrt{2} - 2\sqrt{2}}{2}$ is irrational.

(d) $\sqrt{10^9} = 31622.7766...$ is irrational.

Assessment 8-2A: Variables

1. (a) Use the format for an arithmetic sequence: $a_n = a_1 + (n-1)d$, if $a_1 = 10$. Then $a_4 = 10 + (4-1)d = 10 + 3d$

(b) If n is the number, then twice the number is $2n$ and 15 less than that is $2n - 15$

(c) If n is the number, then its square is n^2 and 15 times that is $15n^2$

(d) If n is the number, its square is n^2 and twice the number is $2n$. Their difference is $\boldsymbol{n^2 - 2n}$.

2. (a) Let n be the number:

(*i*) Adding $\sqrt{3} \Rightarrow n + \sqrt{3}$

(*ii*) Multiplying the sum by $7 \Rightarrow 7(n + \sqrt{3})$

(*iii*) Subtracting $14 \Rightarrow 7(n + \sqrt{3}) - 14$

(*iv*) Dividing the difference by $7 \Rightarrow \frac{7(n+\sqrt{3})-14}{7}$

(*v*) Subtracting the original number $\Rightarrow \boldsymbol{\frac{7(n+\sqrt{3})-14}{7} - n}$.

(b) $\frac{7n+7\sqrt{3}-14}{7} - n = n + \sqrt{3} - 2 - n = \boldsymbol{\sqrt{3} - 2}$.

3. (a) There are 4, 6, 8, and 10 shaded tiles, respectively, in the four figures. Assume and arithmetic sequence with $a_1 = 4$ and $d = 2$. Thus $a_n = 4 + (n-1)2 = \boldsymbol{2n + 2}$ or $\boldsymbol{2(n + 1)}$.

(b) There are a total of $(n+2)^2$ squares in each figure. Assume the pattern continues; then the number of white squares is $(n+2)^2$ – the number of shaded squares, or $(n+2)^2 - (2n+2) = n^2 + 4n + 4 - 2n - 2 = \boldsymbol{n^2 + 2n + 2}$.

4. (a) Cost is $\$(50 + 60h)$

(b) Let n, d, and q be the number of nickels, dimes, and quarters, respectively. It is given that $n = 3d$ and $q = 3n = 9d$. Then the value of dimes + Value of nickels + value of quarters is $10d + 5(3d) + 25(9d) = 250d$ cents

(c) The sum of the three numbers is $x + (x+1) + (x+2) = \boldsymbol{3x + 3}$.

(d) Make a table of the number of bacteria in terms of n minutes:

n minutes	Number of bacteria
1	$q \cdot 2$
2	$(q \cdot 2) \cdot 2 = q \cdot 2^2$
3	$(q \cdot 2^2) \cdot 2 = q \cdot 2^3$
$\vdots$	$\vdots$
n	$\boldsymbol{q \cdot 2^n}$

(e) The temperature after t hours is $\boldsymbol{(40 - 3t)}$**°F**.

(f) Pawel's salary is $\$s$ the first years; $\$(s + 5000)$ the second year; and $\$2(s + 5000) = \$(2s + 10{,}000)$ the third year, for a total of $\boldsymbol{\$(4s + 15{,}000)}$.

(g) The sum of the three numbers is $x + (x+2) + (x+4) = \boldsymbol{3x + 6}$.

(h) The sum of the three numbers is $(m-1) + m + (m+1) = \boldsymbol{3m}$.

5. If the number of students is 45 times the number of professors, then $S = 45P$

6. If there are five more girls than boys, then $\boldsymbol{g = b + 5}$.

7. Let m be the number of matchsticks Ryan uses. Then $m_1 = 13, m_2 = 19, m_3 = 25, \ldots$. Assume an arithmetic sequence with $m_1 = 13$ and $d = 6$. Thus $m_n = 13 + (n-1)\cdot 6 = 6n + 7$ matchsticks.

8. **(a)** $P = \$8$ per hour $\times\ t$ hours, or $\boldsymbol{P = \$(8t)}$.

(b) Assume \$15 is paid as a flat fee for the first hour or fraction thereof. Afterwards $P = \$15 + \$10$ per hour $\times\ (t-1)$ hours, or

$$P = \begin{cases} \$15 & \text{when } 0 < t < 1 \\ \$[15 + 10(t-1)] & \text{when } t \geq 1. \end{cases}$$

9. Let r be total revenue. Then $r = 5x + 13(100) = \boldsymbol{\$(5x + 1300)}$.

10. **(a)** If the youngest receives $\$x$:
The eldest receives $\$3x$;
The middle receives $\frac{1}{2}(3x) = \$\frac{3}{2}x$.

(b) If the middle receives $\$y$:
The eldest receives $\$2y$;
The youngest receives $\frac{1}{3}(2y) = \$\frac{2}{3}y$.

(c) If the eldest receives $\$z$:
The middle receives $\$\frac{1}{2}z$;
The youngest receives $\$\frac{1}{3}z$.

11. **(a)** **(*i*)** The pattern appears to be subtracting 3 from each term to obtain its successor. At the 100th term 3 will be subtracted 99 times. So the 100th term is $0 - 3(99) = {}^-297$.

(*ii*) The nth term can be found using the expression $\boldsymbol{0 - 3(n-1) = {}^-3n + 3}$.

(b) **(*i*)** The pattern appears to be adding $2\sqrt{2}$ to each term to obtain its successor. At the 100th term $2\sqrt{2}$ will be added 99 times. The 100th term is $1 + 197\sqrt{2}$.

(*ii*) The nth term can be found using the expression $(1 - \sqrt{2}) + \boldsymbol{2}\sqrt{2}(n-1)$
$1 - \sqrt{2} + 2\sqrt{2}\,n - 2\sqrt{2}$
$= 2\sqrt{2}\boldsymbol{n + 1 - 3\sqrt{2}}$.

(c) **(*i*)** The pattern appears to be adding 2 to each term to obtain its successor. At the 100th term 2 will be added 99 times. The 100th term is

$$\sqrt{3} + 0.5 + 2(99) = \sqrt{3} + \mathbf{198.5}$$

(*ii*) $\boldsymbol{\sqrt{3} + 0.5 + 2(n-1) = 2n + \sqrt{3} - 1.5}$

12. **(a)** **(*i*)** Multiply each term by $\sqrt{3}$ to obtain its successor. The 17th term is given by

$$2\cdot\sqrt{3}^{16} = 2\cdot\left((\sqrt{3})^2\right)^8 = 2(3)^8 = 13{,}122$$

(*ii*) The nth term is given by $\boldsymbol{2(\sqrt{3})^{n-1}}$.

(b) **(*i*)** Multiply each term by ${}^-\sqrt{2}$ to obtain its successor. The 17th term is given by

$$2\cdot\left({}^-\sqrt{2}\right)^{16} = 2\cdot\left({}^-2^{\frac{1}{2}}\right)^{16}$$
$$= 2\cdot 256 = 512$$

(*ii*) The nth term is given by $2\left({}^-\sqrt{2}\right)^{n-1}$

(c) **(*i*)** Multiply each term by ${}^-\sqrt{5}$ to obtain its successor. The 17th term is given by

$${}^-\sqrt{5}\cdot\left({}^-\sqrt{5}\right)^{16} = {}^-\sqrt{5}\cdot\left({}^-5^{\frac{1}{2}}\right)^{16}$$
$$= {}^-\sqrt{5}\cdot 5^8$$

(*ii*) The nth is given by $({}^-\sqrt{5})({}^-\sqrt{5})^{\boldsymbol{n-1}} = \left({}^-\sqrt{5}\right)^n$

13. If the 100th term is $\sqrt{2}$ and the 200th term is $\sqrt{3}$ and the sequence is arithmetic, then

$$a_1 + d(100-1) = \sqrt{2} \Rightarrow a_1 = \sqrt{2} - 99d$$
$$a_1 + d(200-1) = \sqrt{3} \Rightarrow a_1 = \sqrt{3} - 199d$$
$$\Rightarrow \sqrt{2} - 99d = \sqrt{3} - 199d$$
$$\Rightarrow 100d = \sqrt{3} - \sqrt{2}$$
$$\Rightarrow d = \frac{\sqrt{3} - \sqrt{2}}{100}$$

and the first term is

$$a_1 + 99\left(\frac{\sqrt{3} - \sqrt{2}}{100}\right) = \sqrt{2}$$
$$a_1 = \sqrt{2} - 99\left(\frac{\sqrt{3} - \sqrt{2}}{100}\right).$$

14. If the 11^{th} term is $^-128$ and the ratio is $^-\sqrt{2}$ then the first term is

$$^-128 = a_1(^-\sqrt{2})^{10} \Rightarrow {}^-128 = a_1\left(^-2^{\frac{1}{2}}\right)^{10}$$

$$^-128 = a_1 2^5 \Rightarrow a_1 = \frac{^-128}{32} = {}^-4$$

15. $10 - .20(10) = 8; 8 - .20(8) = 6.4;$
$6.4 - .20(6.4) = 5.12$
$5.12 - .20(5.12) = 4.096;$
$4.096 - .20(4.096) = 3.2768$ or 3.3 ft.

16. Starting with two hours and increasing by 20 minutes each day can be represented with an arithmetic sequence with $a_1 = 2$ and $d = \frac{1}{3}$ the n^{th} term is $2 + \frac{1}{3}(n-1)$. Set the n^{th} term equal to 12 hours and solve for n.

$$2 + \frac{1}{3}(n-1) = 12 \Rightarrow 2 + \frac{1}{3}n - \frac{1}{3} = 12$$
$$\Rightarrow 6 + n - 1 = 36 \Rightarrow 5 + n = 36 \Rightarrow n = 31$$

After 31 days Jake will be able to wear his contacts for 12 hours.

17. **(a)** The first 10 terms in simplest form:

$\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}, \sqrt{2}+2\sqrt{3},$
$2\sqrt{2}+3\sqrt{3}, 3\sqrt{2}+5\sqrt{3},$
$5\sqrt{2}+8\sqrt{3}, 8\sqrt{2}+13\sqrt{3},$
$13\sqrt{2}+21\sqrt{3}, 21\sqrt{2}+34\sqrt{3}$

(b) **(i)** The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, … $F_5 = 5$ and $F_6 = 8$ and $L_7 = 5\sqrt{2}+8\sqrt{3}$ so

$L_7 = F_5\sqrt{2} + F_6\sqrt{3}$
$5\sqrt{2}+8\sqrt{3} = 5\sqrt{2}+8\sqrt{3}$
$L_8 = F_6\sqrt{2} + F_7\sqrt{3}$

(ii) $L_9 = F_7\sqrt{2} + F_8\sqrt{3}$
$L_{10} = F_8\sqrt{2} + F_9\sqrt{3}$

(iii) $L_n = F_{n-2}\sqrt{2} + F_{n-1}\sqrt{3}$

Assessment 8-2B

1. **(a)** If n is the number, then 10 more than the number is $\boldsymbol{n + 10}$.

(b) If n is the number, then 10 less than the number is $\boldsymbol{n - 10}$.

(c) If n is the number, then half times it is $\frac{1}{2}n$.

(d) If n is the number, then the sum of it and 10 is $\boldsymbol{n + 10}$.

(e) If n is the number, its cube is n^3 and the difference between its cube and its square is $n^3 - n^2$.

2. **(a)** Let n be the number. Then

(***i***) Adding 25 $\Rightarrow n + 25$

(***ii***) Multiplying by 4 $\Rightarrow 4(n + 25)$

(***iii***) Subtracting 60 $\Rightarrow 4(n + 25) - 60$

(***iv***) Dividing by 4 $\Rightarrow \frac{4(n+25)-60}{4}$

(***v***) Adding 5 $\Rightarrow \frac{4(n+25)-60}{4} + 5$

(b) $\frac{4(n+25)-60}{4} + 5 = n + 25 - \frac{60}{4} + 5 = n + 15$

3. **(a)** There are 10,13, and 16 shaded tiles, respectively, in the three figures. Assume an arithmetic sequence with $a_1 = 10$ and $d = 3$. Thus $a_n = 10 + (n-1)3 = \boldsymbol{3n + 7}$.

(b) There are a total of $(n + 3)^2$ squares in each figure. The number of white squares is then $(n + 3)^2 -$ the number of shaded squares, or $(n + 3)^2 - (3n + 7) = n^2 + 6n + 9 - 3n - 7 = \boldsymbol{n^2 + 3n + 2}$.

4. **(a)** Cost is **$(30 + *xh*)**, where x is the cost per hour for h hours.

(b) Let n, d, and q be the number of nickels, dimes, and quarters, respectively. It is given that $n = 4d$ and $q = 3n = 12d$. Then the value of dimes + value of nickels + value of quarters is
$10d + 5(4d) + 25(12d) = 330d$ cents.

(c) The sum of the three numbers is $x + (x-1) + (x-2) = \boldsymbol{3x - 3}$.

(d) Make a table of the number of bacteria in terms of n minutes:

n minutes	Number of bacteria
1	$q \cdot 3^3$
2	$(q \cdot 3^3) \cdot 3^3 = q \cdot 3^6$
3	$(q \cdot 3^6) \cdot 3^3 = q \cdot 3^9$
$\vdots$	$\vdots$
n	$\boldsymbol{q \cdot 3^{3n}}$

(e) The temperature t hours ago was $\mathbf{(40 + 3t)°F}$.

(f) Pawel's salary is \$$s$ the first year; \$$(s + 5000)$ the second year; and \$$(2s)$ the third year, for a total of $\mathbf{\$(4s + 5{,}000)}$.

(g) The sum of the three numbers is $x + (x - 2) + (x - 4) = \mathbf{3x - 6}$.

5. If there are 100 more men than women $\Rightarrow m = \mathbf{w + 100}$.

6. If there are 15 more chairs than tables $\Rightarrow \mathbf{c = t + 15}$.

7. Let m be the number of matchsticks Ryan uses. Then $m_1 = 3, m_2 = 5, m_3 = 7, \ldots$ Assume an arithmetic series with $a_1 = 3$ and $d = 2$.

(*i*) $a_n = 3 + 2 \cdot (n - 1) \Rightarrow 2n + 1$ matchsticks.

(*ii*) $a_{n-1} = 2(n - 1) + 1 = 2n - 1$ matchsticks.

8. **(a)** $P = \$d$ per hour $\times$ t hours, or $\mathbf{P = \$(dt)}$.

(b) Assume \$15 is paid as a flat fee for the first hour or fraction thereof. Afterwards $P = \$25 + \$k$ per hour $\times (t - 1)$ hours, or

$$P = \begin{cases} \$25 & \text{when } 0 < t < 1 \\ \$[25 + k(t - 1)] & \text{when } t \geq 1 \end{cases}$$

(c) $P = \$30 + \$10$ per hour $\times$ t hours, or $\mathbf{P = \$(30 + 10t)}$.

(d) $C = \$300 + \$4$ per day $\times$ n days, or $\mathbf{C = \$(300 + 4n)}$.

(e) $C = \$30 + \$0.35$ per mile $\times$ m miles, or $\mathbf{C = \$(30 + 0.35m)}$.

9. $dx + 2d \cdot 100 = dx + 200d$ dollars.

10. If Matt gives David 10 stickers, David would have $\mathbf{d + 10}$ stickers. Matt would then have $\mathbf{2d - 10}$ stickers.

11. **(a)** **(*i*)** The pattern appears to subtract 5 from each term to obtain its successor. The 100th term is given by $0 - 5(99) = {}^{-}495$

(*ii*) The nth term is given by $\mathbf{0 -}$ $5(n - 1) = 0 - 5n + 5 = {}^{-}5n + 5$.

(b) **(*i*)** The pattern appears to subtract $2\sqrt{2}$ from each term to obtain its successor. The 100th term is given by $1 + \sqrt{2} - 2\sqrt{2}(99) = 1 - 197\sqrt{2}$.

(*ii*) The nth term is given by
$1 + \sqrt{2} - 2\sqrt{2}(n - 1) =$
$1 + \sqrt{2} - 2\sqrt{2}n + 2\sqrt{2}$
$= 1 + 3\sqrt{2} - 2\sqrt{2}n$

(c) **(*i*)** The pattern appears to add $x + 2\sqrt{3}$ to each term to obtain its successor. The 100th term is given by
$x + \sqrt{3} + (x + 2\sqrt{3})(99)$
$= 199\sqrt{3} + 100x$.

(*ii*) The nth term is given by $x + \sqrt{3} + (x + 2\sqrt{3})(n - 1)$

12. **(a)** **(*i*)** The pattern is to multiply each term by $\sqrt{2}$ to obtain its successor. The 20th term is

$$\sqrt{\frac{3}{2}} \cdot \sqrt{2}^{19} = \frac{\sqrt{3}}{\sqrt{2}} \cdot 2^{\frac{19}{2}} = \frac{\sqrt{3} \cdot 2^{\frac{19}{2}}}{2^{\frac{1}{2}}}$$

$$= \sqrt{3} \cdot 2^{\frac{19}{2} - \frac{1}{2}} = \sqrt{3} \cdot 2^9 = 512\sqrt{3}$$

(*ii*) The nth term is given by $\sqrt{\frac{3}{2}}(\sqrt{2})^{n-1}$.

(b) **(*i*)** The pattern is to multiply each term by $\sqrt{2}$ to obtain its successor. The 20th term is given by $\sqrt{2}(\sqrt{2})^{19} = (\sqrt{2})^{20} = 2^{10}$.

(*ii*) The nth term is given by $\sqrt{2}(\sqrt{2})^{n-1} = \left(\sqrt{2}\right)^n$

(c) **(*i*)** The pattern is to multiply each term by $^-\sqrt{3}$ to obtain its successor. The 20th term is $(^-\sqrt{3})(^-\sqrt{3})^{19} = (^-\sqrt{3})^{20} = 3^{10}$.

(*ii*) The *n*th term is given by $^-\sqrt{3}\cdot\left(^-\sqrt{3}\right)^{n-1}$ $(^-\sqrt{3})^n$.

(d) **(*i*)** The pattern is to multiply each term by $^-\sqrt{3}$ to obtain its successor. The 20th term is $3(^-\sqrt{3})^{19} = 3^1 \cdot {}^-3^{\frac{19}{2}} =$

$(^-1)\cdot 3^{1+\frac{19}{2}} = (^-1)\cdot 3^{\frac{21}{2}} =$.

$(^-1)\cdot 3^{\frac{1}{2}+\frac{20}{1}} = {}^-\sqrt{3}\cdot 3^{10}$

(*ii*) The *n*th term is given by $3\cdot\left(^-\sqrt{3}\right)^{n-1}$

13. If the 10th term is 25 and the 20th term is 100 and the sequence is geometric, then

$$25 = a_1(r)^9 \text{ and } 100 = a_1(r)^{19}$$

$$a_1 = \frac{25}{r^9} \text{ and } a_1 = \frac{100}{r^{19}}$$

$$\Rightarrow \frac{25}{r^9} = \frac{100}{r^{19}} \Rightarrow 25r^{19} = 100r^9$$

$$25r^{19} - 100r^9 = 0 \Rightarrow 25r^9(r^{10} - 4) = 0$$

$$25r^9 = 0 \text{ and } r^{10} - 4 = 0$$

$$r = 0 \text{ and } r^{10} = 4 \Rightarrow r = \sqrt[10]{4}$$

The first term is

$$a_1 = \frac{25}{r^9} \Rightarrow a_1 = \frac{25}{\left(\sqrt[10]{4}\right)^9}$$

$$\Rightarrow a_1 = \frac{25}{4^{\frac{9}{10}}} = \frac{25}{4^{\frac{5}{10}+\frac{4}{10}}} = \frac{25}{4^{\frac{1}{2}}.4^{\frac{2}{5}}}$$

$$= \frac{25}{\sqrt{4}\cdot(2^2)^{\frac{2}{5}}} = \frac{25}{2\sqrt[5]{2^4}}$$

14. **(a)** The common difference in the arithmetic sequence is 3.

(b) The common ratio in the geometric sequence is $3^{\frac{3}{2}}$.

15.
$20 - .10(20) = 18;$
$18 - .10(18) = 16.2;$
$16.2 - .10(16.2) = 14.58;$
$14.58 - .10(14.58) = 13.122$ or 13.122 m.

16. Create a table:

0.01	100
0.11	100.01
0.21	100.02
0.31	100.03
0.41	100.04
0.51	100.05
…	…
0.01 + (0.1)(*n* - 1)	100 + (0.01)(*n* - 1)

When is 0.01 + (0.1)(*n* - 1) > 100 + (0.01)(*n* - 1)?

$$0.01 + (0.1)(n-1) > 100 + (0.01)(n-1)$$
$$0.01 + 0.1n - 0.1 > 100 + 0.01n - 0.01$$
$$0.1n - 0.09 > 0.01n + 99.99$$
$$0.09n > 100.08$$
$$n > 1112$$

At 1112 terms they are equal so at the 1113th term the number in the first sequence be greater than the number in the second sequence.

17. (a) 1, 2, 3, 5, 8, 13, 21

(b) 1 + 2 + 3 = 6

(c) 1 + 2 + 3 +5 = 11

(d) 1 + 2 + 3 + 5 + 8= 19

(e) 1 + 2 + 3 +5 + 8 + 13= 32

(f) 1 + 2 + 3 +5 + 8 + 13 + 21 = 53

(g) The sum of the first *n* terms is 2 less than two terms later in the sequence.

(h) If the *n*th term is F_n then

$$F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 2.$$

Mathematical Connections 8-2: Review Problems

11. Answers may vary. For example, $\sqrt{2} \approx$ 1.414213…, so rational numbers between 1.41 and $\sqrt{2}$ include **1.411** and **1.412**. Irrationals between 1.41 and $\sqrt{2}$ include **1.411011101111…** and **1.412022022202222…**. Alternatively,

irrational numbers can be obtained by adding an irrational number to 1.41, e.g., $1.41 + \frac{1}{1000\cdot\sqrt{2}}$ is an irrational number between 1.41 and $\sqrt{2}$.

12. (a) $\sqrt[3]{728} = 2\sqrt[3]{91}$ is irrational.

(b) $\frac{2}{\sqrt{2}} = \sqrt{2}$ is irrational.

(c) $\sqrt[3]{2^9 \cdot 3^{12}} = 2^3 \cdot 3^4 = 8 \cdot 81 = 648$ is rational.

(d) $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$ is rational.

(e) $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right) = \sqrt{9}+\sqrt{6}-\sqrt{6}-\sqrt{4}$
$= 3 - 2 = 1$
is rational

(f) $\left(\sqrt{3}-\sqrt{2}\right)^2 = \left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$
$= \sqrt{9}-\sqrt{6}-\sqrt{6}+\sqrt{4} = 5-2\sqrt{6}$
is irrational.

13. **(a)** $\left(\frac{^-8}{27}\right)^{^-\frac{2}{3}} = \left(\frac{^-27}{8}\right)^{\frac{2}{3}} = \left(\frac{^-3^3}{2^3}\right)^{\frac{2}{3}} = \left(\frac{^-3}{2}\right)^2 = \frac{9}{4}$

(b) $\left(\frac{^-9}{4}\right)^{\frac{1}{2}} = \sqrt{\frac{^-9}{4}}$ is not real.

(c) $^-\left(\frac{9}{4}\right)^{\frac{3}{2}} = {^-\left(\frac{3^2}{2^2}\right)^{\frac{3}{2}}} = {^-\frac{3^3}{2^3}} = {^-\frac{27}{8}}$

14. **(a)** $\sqrt{^-x} = 5 \Rightarrow {^-x} = 25 \Rightarrow x = {^-25}$

(b) $\sqrt{x^2} = {^-x} \Rightarrow$ true for all $x \le 0$.

(c)

$\left(^-x\right)^{\frac{3}{2}} = 125 \Rightarrow {^-x} = \left(5^3\right)^{\frac{2}{3}} \Rightarrow {^-x} = 25 \Rightarrow x = {^-25}$

15. $n < \sqrt[3]{50} < n+1$
$n < 3.684031499... < n+1$
$3 < 3.684031499... < 4$

If n is an integer, then n is 3.

Assessment 8-3A: Equations

1. The balance scales imply addition, so:
$\Delta + \square = 12$.
$O + O + \Delta + \square = 18$;
$2O + 12 = 18$; so
$\mathbf{O = 3}$.
$2O + \square = 10$;
$6 + \square = 10$; so
$\square = \mathbf{4}$.
$\Delta + \square = 12$;
$\Delta + 4 = 12$; so
$\Delta = 8$.

2. **(a)** $x + \sqrt{3} = 2\sqrt{3} - x \Rightarrow 2x = \sqrt{3}$

$\Rightarrow x = \frac{\sqrt{3}}{2}$

(b) $\frac{3}{2}x - 3 = x + \sqrt{2} \Rightarrow \frac{3}{2}x - x = 3 + \sqrt{2}$

$\Rightarrow \frac{1}{2}x = 3 + \sqrt{2} \Rightarrow x = 6 + 2\sqrt{2}$

(c) $5\left(2x + \sqrt{2}\right) + 7\left(2x + \sqrt{2}\right) = 12 \Rightarrow$

$12(2x + \sqrt{2}) = 12 \Rightarrow 2x + \sqrt{2} = 1$

$\Rightarrow x = \frac{1 - \sqrt{2}}{2}$

(d) $3(\sqrt{x} - 3) = 5(\sqrt{x} - 3)$

$3(\sqrt{x} - 3) - 5(\sqrt{x} - 3) = 0$
$^-2(\sqrt{x} - 3) = 0$
$\sqrt{x} - 3 = 0 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$

(e) $\left(\sqrt{x} - 5\right)^2 = 9$

$\sqrt{x} - 5 = 3$ and $\sqrt{x} - 5 = {^-3}$
$x = 64$ and $x = 4$

3. Let m be the number of matchsticks Ryan uses. Then $m_1 = 4, m_2 = 7, m_3 = 10, ..., m_n = 67$. Assume an arithmetic series with $m_1 = 4, d = 3$, and $m_n = 67$.

Thus $67 = 4 + (n - 1) \cdot 3 \Rightarrow 67 = 4 + 3n - 3 \Rightarrow 66 = 3n \Rightarrow n =$ **22 squares**.

4. If s is the number of student tickets sold, then $812 - s$ is the number of nonstudent tickets. Adding their values gives the total amount taken in:

$$\$2 \cdot s + \$3 \cdot (812 - s) = \$1912$$
$$2s + 2436 - 3s = 1912$$
$$\Rightarrow s = \textbf{524 student tickets sold}.$$

5. Let e be the amount the eldest receives,
m be the amount the middle sibling receives,
y be the amount the youngest receives.

Then $e = 3y$ and $m = y + 14{,}000$

So
$$\begin{aligned} e + m + y &= 486{,}000 \\ 3y + (y + 14{,}000) + y &= 486{,}000 \\ 5y + 14{,}000 &= 486{,}000 \\ 5y &= 472{,}000 \end{aligned}$$

Thus
$$\begin{aligned} \boldsymbol{y} &= \mathbf{\$94{,}400} \\ \boldsymbol{m} &= \mathbf{\$108{,}400} \\ \boldsymbol{e} &= \mathbf{\$283{,}200} \end{aligned}$$

6. Let e be the length of the equal length pieces.

Then $2e + (e - 3) = 120$ inches
$$3e - 3 = 120$$
$$3e = 123 \Rightarrow e = 41$$

So the equal length pieces are **41 inches** and the short piece is **38 inches** (disregarding the width of the saw cuts).

7. Let d be the number of dimes; then $67 - d$ is the number of nickels. $0.10d$ is the amount of money in dimes and $0.05(67 - d)$ is the amount of money in nickels.

Thus $0.10d + 0.05(67 - d) = 4.20$, or
$$10d + 5(67 - d) = 420$$
$$10d + 335 - 5d = 420$$
$$5d = 85 \Rightarrow \quad d = \textbf{17 dimes}$$
$$67 - d = \textbf{50 nickels}.$$

8. Let m be Miriam's age now; then $m - 10$ is Ricardo's age now $m - 2$ was Miriam's age two years ago. Thus $m - 2 = 3(m - 10) \Rightarrow m - 2 = 3m - 30 \Rightarrow 28 = 2m$, or **Miriam is 14**
Ricardo is 4.

9. Let g be the number of graduate students; $20g$ is the number of undergraduates. Then
$$g + 20g = 21{,}000$$
$$21g = 21{,}000 \Rightarrow 1000 \text{ graduate students.}$$

10. Let the perpendicular sides to the river be of length a; then the parallel side is of length b = 2a. Thus $a + 2a + a = 1200, 4a = 1200 \Rightarrow a = 300.$
The perpendicular sides are then **300 yards**; the parallel side is **600 yards**.

11. In the given sequence $d = 3$, thus
$$n + (n+3) + (n+6) = 903$$
$$3n + 9 = 903$$
$$3n = 894$$
$$n = 298$$

The three terms are 298, 301, and 304.

12. Let a be the shorter side and b be the longer side. The shorter side is $\frac{1}{3}$ of the length of the longer side; $a = \frac{1}{3}b$. The perimeter is 100 ft or
$$2a + 2b = 100 \Rightarrow 2\left(\frac{1}{3}b\right) + 2b = 100$$
$$\Rightarrow \frac{2}{3}b + \frac{6}{3}b = 100 \Rightarrow \frac{8}{3}b = 100$$
$$b = \frac{300}{8} = \frac{75}{2} \text{ and } a = \frac{1}{3} \cdot \frac{75}{2} = \frac{25}{2}$$

Using Pythagorean Theorem, the length of the diagonal is
$$a^2 + b^2 = diagonal^2 \Rightarrow$$
$$\left(\frac{25}{2}\right)^2 + \left(\frac{75}{2}\right)^2 = diagonal^2$$
$$\frac{625}{4} + \frac{5625}{4} = diagonal^2 \Rightarrow \frac{6250}{4} = diagonal^2$$
$$\frac{3125}{2} = diagonal^2 \Rightarrow$$
$$diagonal = \sqrt{\frac{3125}{2}} = \sqrt{1562.5} = 39.53\ ft$$

Assessment 8-3B

1. If $\square = 2\Delta$, then $\Delta < \square$.
If $\Delta = 2O$, then $O < \Delta$.
So $O < \Delta < \square$.

(a) $\square$ weighs the most.
(b) **O** weighs the least.

2. **(a)** $x - \sqrt{2} = 3\sqrt{2} - x \Rightarrow 2x = 3\sqrt{2} + \sqrt{2}$

$$x = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

(b) $^{-}\frac{3}{2}x - 2 = x - \sqrt{2} \Rightarrow {}^{-}3x - 4$

$= 2x - 2\sqrt{2} \Rightarrow {}^{-}5x = 4 - 2\sqrt{2}$

$\Rightarrow x = \dfrac{2\sqrt{2} - 4}{5}$

(c) $5(2x - \sqrt{3}) - 7(2x - \sqrt{3}) = 2$

${}^{-}2(2x - \sqrt{3}) = 2 \Rightarrow 2x - \sqrt{3} = {}^{-}1$

$x = \dfrac{\sqrt{3} - 1}{2}$

(d) $2(x^2 - 3) = (x^2 - 3)$

$2(x^2 - 3) - (x^2 - 3) = 0$

$x^2 - 3 = 0$

$x = \pm\sqrt{3}$

(e) $(\sqrt{{}^{-}x} - 5)^2 = 9 \Rightarrow \sqrt{{}^{-}x} - 5 = \pm 3$

$\sqrt{{}^{-}x} = 5 \pm 3 \Rightarrow \sqrt{{}^{-}x} = 8$ or $\sqrt{{}^{-}x} = 2$

${}^{-}x = 64$ or ${}^{-}x = 4 \Rightarrow x = {}^{-}64$ or $x = {}^{-}4$

3. Let m be the number of matchsticks Ryan uses. Then $m_1 = 4, m_2 = 7, m_3 = 10, \ldots$. Assume an arithmetic sequence with $d = 3$. Let m_n be the number of matchsticks in the last figure; then $m_n + (m_n - 3) = 599 \Rightarrow m_n = 301$ matchsticks. Thus the last two figures used **298** and **301** matchsticks, respectively.

4. Step 1: Write a mathematical model. To do this, let x represent the number of tickets sold to students. Then, $723 - x$ represents the number of tickets sold to non-students. Thus,

$$\begin{aligned}
3x + 5(723 - x) &= 2815 \\
3x + 3615 - 5x &= 2815 \\
-2x + 3615 - 3615 &= 2815 - 3615 \\
-2x &= -800 \\
x &= 400
\end{aligned}$$

The number of non-student tickets sold is $723 - 400 =$ **323**.

5. Let e be the amount left to the eldest, m be the amount left to each of two middle siblings, and y be the amount left to the youngest.

It is given that $e = 3y$ and each $m = y + 16{,}000$.

Thus if $e + m + m + y = 2{,}000{,}000$ then

$3y + y + 16{,}000 + y + 16{,}000 + y = 2{,}000{,}000 \Rightarrow$

$6y + 32{,}000 = 2{,}000{,}000 \Rightarrow y = 328{,}000$.

The youngest will receive \$328,000, the middle children each will receive \$344,000, and the eldest will receive \$984,000.

6. Let a be the Alex's age now. Her age ten years from now will be $a + 10$. Then $a + 10 = 3a \Rightarrow a = 5$. Alex is **5 years old**.

7. Let d be the number of Dave's stickers; Matt then has $2d$ stickers.

So $d + 2d = 2(120) \Rightarrow d = 80$ and $2d = 160$.

If **Matt gives Dave 40 stickers**, each will then have 120.

8. **(a)** Let m be Miriam's age now and r be Ricardo's age now, so $r = m - 4$. Miriam's age 10 years ago was $m - 10$ and Ricardo's age 10 years ago was $(m - 4) - 10 = m - 14$.

Thus $m - 10 = 3(m - 14) \Rightarrow m - 10 = 3m - 42 \Rightarrow \boldsymbol{m = 16}$ and $\boldsymbol{r = 12}$.

(b) Ten years ago Miriam was 6 and Ricardo was 2; Miriam was then three times as old as Ricardo.

9. Let s be the number of students; then the number of professors is $\frac{1}{13}s$.

Thus $s + \frac{1}{13}s = 28{,}000 \Rightarrow 1\frac{1}{13} \cdot s = 28{,}000$

$\Rightarrow s =$ **26,000 students**.

10. Let a be the length of the parallel side, so the length of each of the perpendicular sides is $2a$.

Then $2a + a + 2a = 800 \Rightarrow a = 160$. The pasture's dimensions are thus **320 by 160 yards**.

11. In a geometric sequence, $a_n = a_1 r^{n-1}$ and the first two terms are thus a_1 and $a_1 r$. Then $a_1 + a_1 r = 100a_1 \Rightarrow a_1(1 + r) = 100a_1 \Rightarrow 1 + r = 100$

$\Rightarrow \boldsymbol{r = 99}$.

12. Let a be the shorter side and b be the longer side. Then $a = \frac{1}{3}b$. Using Pythagorean Theorem:

$$a^2 + b^2 = diagonal^2$$
$$\left(\frac{1}{3}b\right)^2 + b^2 = 9^2$$
$$\frac{1}{9}b^2 + b^2 = 81$$
$$\frac{10}{9}b^2 = 81 \Rightarrow b^2 = \frac{729}{10}$$
$$b = \sqrt{\frac{729}{10}} = \frac{27}{\sqrt{10}} = \frac{27\sqrt{10}}{10} \text{ and } a = \frac{9}{\sqrt{10}} = \frac{9\sqrt{10}}{10}$$

Mathematical Connections 8-3: Review Problems

13. $\boldsymbol{x = 3y}$.

14. If the middle even number is $2n$, then the next two are $2n + 2$ and $2n + 4$; the previous two are $2n - 4$ and $2n - 2$. Their sum is $2n - 4 + 2n - 2 + 2n + 2n + 2 + 2n + 4 = \mathbf{10n}$.

15. Let $\boldsymbol{x}$ be the number of Jack's CD's. Then the number of Julie's is $\mathbf{2x}$; the number of Tyto's is $3(2x) = \mathbf{6x}$.

16. **(a)** $P = 30 + (30 + d) + ((30 + d) + d) = 90 + 3d$

(b) $\$(d + 2d + 4d + 8d) = \mathbf{\$15d}$.

17. **(a)** If the middle term is x and the difference is d, then the terms are $x - d, x, x + d$. Their sum is $x - d + x + x + d = 3x$. Thus, the sum of any three consecutive terms in an arithmetic sequence is three times the middle term.

(b) If the middle term is x and the ratio is r then the terms are $\frac{x}{r}, x, xr$. Their product is $\frac{x}{r} \cdot x \cdot xr = x^3$. Thus, the product of any three consecutive terms in a geometric sequence is the middle term to the third power.

18. The sixth term is given by $\sqrt{5}(0.5)^5 = \sqrt{5} \cdot \left(\frac{1}{2}\right)^5 = \sqrt{5}\left(\frac{1}{32}\right) = \mathbf{\frac{\sqrt{5}}{32}}$.

19. We can write a mathematical model. Let r be the ratio. Then, the fourth term can be obtained by the expression $12 \cdot r^3$. Thus, $12 \cdot r^3 = \sqrt{5} \Rightarrow$ $\boldsymbol{r = \frac{\sqrt[6]{5}}{\sqrt[3]{12}}}$.

Assessment 8-4A: Functions

1. Where x is the first element in each ordered pair and $f(x)$ is the second:

(a) **Multiply** the input number by $^-2$; i.e., $f(x) = {}^-2x$.

(b) **Add 6** to the input number; i.e., $f(x) = x + 6$.

(c) **Square the input; i.e.** $f(x) = x^2$

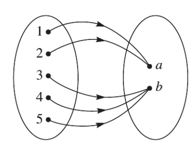
2. **(a)** **Not a function**. The element 1 is paired with both a and d.

(b) **A function**. Each element in the first set is paired with a unique element from the second.

(c) **A function**. Each distinct element in the first set is paired with an element from the second.

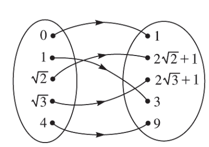
(d) **A function**. Each distinct element in the first set is paired with an element from the second.

3. **(a)** Answers may vary; e.g.:



(b) Each of the five elements in the domain have two choices for a pairing $\Rightarrow 2^5 = 32$ possible functions.

4. **(a)**

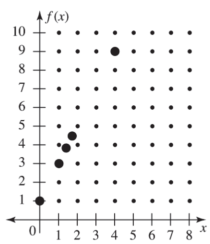


(b) $\{(0, 1), (1, 3), (\sqrt{2}, 2\sqrt{2} + 1), (\sqrt{3}, 2\sqrt{3} + 1), (4, 9)\}$.

(c)

x	$f(x)$
0	1
1	3
$\sqrt{2}$	$2\sqrt{2}+1$
$\sqrt{3}$	$2\sqrt{3}+1$
4	9

(d)



5. (a) **Function from $\mathbb{R}$ to {2}**, a subset of *R*.

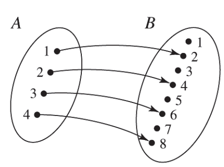
$f(x) = 2$ is called a constant function and its only output is 2.

(b) **Function from $\mathbb{R}$ to $\{x \mid x \geq 0\}$**, a subset of *R*. $f(x) = \sqrt{x}$ has output of nonnegative numbers only because $\sqrt{x}$ is the principal square root.

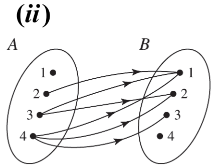
(c) **Not a function** because $(4, {}^-2)$ and $(4, 2)$ both satisfy the relation, but inputs can have only one associated output.

(d) **Function from $\mathbb{R}$ to $\mathbb{R}$**. $x = y^3 \Rightarrow y = \sqrt[3]{x}$ has output of all real numbers only because $\sqrt[3]{x}$ is the cubed root of all real numbers.

6. (a) **(*i*)**



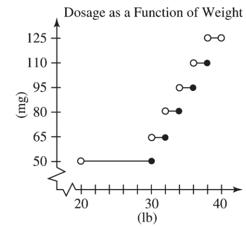
(*ii*)



(b) **(*i*)** **A function** from A to B because each element in A corresponds to one and only one element in B. The range is $\{2, 4, 6, 8\}$.

(*ii*) **Not a function**. Elements 3 and 4 in A correspond to more than one element in B; element 1 in A does not correspond to any element in B.

7. This is an example of **step function**; note in the following graph that a child weighing exactly 30, 32, 34,… pounds uses the lower dosage at the break point:

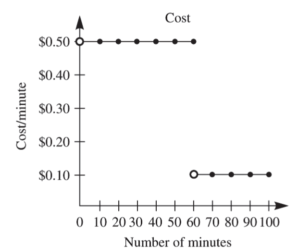


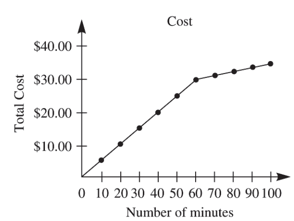
8. (a) $L(n)$ could be an arithmetic sequence with $a_3 = 8$ and $d = 3$. Then $8 = a_1 + (3-1)\cdot 3 \Rightarrow a_1 = 2$.

$L(n) = 2 + (n-1)\cdot 3 = \mathbf{3n - 1}$.

(b) $L(n)$ could be obtained by squaring n and adding 1, or $\boldsymbol{L(n) = n^2 + 1}$.

9. (a)





(b) It must be assumed that the company charges only for the **exact fraction of minutes used**.

(c) The two segments represent the **two different rates** per minute; the steeper one comes from the 50¢ per minute charge for the first 60 minutes.

(d) If $c(t)$ in dollars is cost as a function of time t in minutes, then

$$\boldsymbol{c(t) = \begin{cases} 0.50t & \text{if } t \leq 60 \\ 30 + 0.10(t - 60) & \text{if } t > 60 \end{cases}}$$

Where \$30 is the cost of the first 60 minutes.

10. Let n be the elements of the domain:

(a) Assume an arithmetic sequence with $a_1 = 3$ and $d = 5$. Thus $\boldsymbol{f(n) = 3 + (n-1)\cdot 5 = 5n - 2}$.

(b) The output is obtained if 3 is raised to the *nth* power, so $\boldsymbol{f(n) = 3^n}$.

(c) A constant function, so $f(n) = 3$.

(d) Assume a geometric sequence with $a_1 = 3$ and $r = 3^{\frac{1}{3}}$. Thus

$$f(n) = 3\left(3^{\frac{1}{3}}\right)^{n-1} = 3^1 3^{\frac{1}{3}n-\frac{1}{3}} = 3^{\frac{1}{3}n+\frac{2}{3}} = 3^{\frac{n+2}{3}}$$

11. **(a)** $(g \circ f)(6) = g[f(6)] = f(6) - 5 = 7(6) - 5 = 37$

(b) $(g \circ f)(10) = g[f(10)] = f(10) - 5 = 7(10) - 5 = 65.$

(c) $(g \circ f)(\sqrt{10}) = g[f(\sqrt{10})] = f(\sqrt{10}) - 5 = 7(\sqrt{10}) - 5.$

(d) $(g \circ f)(0) = g[f(0)] = f(0) - 5 = \boldsymbol{{}^-5}.$

(e) $(g \circ f)(\frac{5}{8}) = g[f(\frac{5}{8})] = f(\frac{5}{8}) - 5 = 7(\frac{5}{8}) - 5 = \frac{35}{8} - \frac{40}{8} = {}^-\frac{5}{8}.$

(f) $(g \circ f)(n) = g[f(n)] = f(7n) - 5 = 7n - 5.$

12. $7n - 5 = 2n \Rightarrow 5n = 5 \Rightarrow n = 1.$

13. **(a)** **(i)** $(f \circ g)(6) = f[g(6)] = f[6-5] = f(1) = 7(1) = 7.$

(ii) $(f \circ g)(10) = f[g(10)] = f(10-5) = f(5) = 7(5) = 35.$

(iii) $(f \circ g)(\sqrt{10}) = f[g(\sqrt{10})] = f(\sqrt{10} - 5) = 7(\sqrt{10} - 5) = 7\sqrt{10} - 35.$

(iv) $(f \circ g)(0) = f[g(0)] = f(0-5) = f({}^-5) = ({}^-5)(7) = {}^-35.$

(v) $(f \circ g)(\frac{5}{8}) = f[g(\frac{5}{8})] = f(\frac{5}{8} - 5) = f(\frac{5}{8} - \frac{40}{8}) = f(\frac{{}^-35}{8}) = \frac{{}^-35}{8}\cdot 7 = {}^-\frac{245}{8}.$

(vi) $(f \circ g)(n) = f[g(n)] = f(x-5) = 7(x-5) = 7x - 35.$

$(f \circ g)(x) = x$

(b) $7x - 35 = x \Rightarrow 6x = 35$

$x = \frac{35}{6}.$

14. **(a)** $f(f(0)) = f({}^-0 + b) = {}^-({}^-0 + b) + b = 0.$

(b) $f(f(2) = f({}^-2 + b) = {}^-({}^-2 + b) + b = 2.$

(c) $f(f(x)) = f({}^-x + b) = {}^-({}^-x + b) + b = x.$ This works for all real numbers.

15. **(a)** **(*i*)** **Yes**. $4n - 3 = 1 \Rightarrow t(1) = 1.$

(*ii*) **Yes**. $4n - 3 = 385 \Rightarrow t(97) = 385.$

(*iii*) **Yes**. $4n - 3 = 389 \Rightarrow t(98) = 389.$

(*iv*) **No**. If $4n - 3 = 392$, $n = \frac{395}{4}$ which is not a natural number.

(b) **(*i*)** **No**. 0 is not a natural number.

(*ii*) **Yes**. $n^2 = 25 \Rightarrow t(5) = 25.$

(*iii*) **Yes**. $n^2 = 625 \Rightarrow t(25) = 625.$

(*iv*) **No**. If $n^2 = 1000$, $n = \sqrt{1000}$ which is not a natural number.

(*v*) **No**. If $n^2 = 90$, $n = \sqrt{90}$, which is not a natural number.

(c) **(*i*)** **Yes**. $n(n-1) = 0 \Rightarrow t(1) = 0.$

(*ii*) **Yes**. $n(n-1) = 2 \Rightarrow t(2) = 2.$

(*iii*) **Yes**. $n(n-1) = 5 \Rightarrow t(5) = 20.$

(*iv*) **No**. If $n(n-1) = 999 \Rightarrow t(32.11) \approx 999$, but 32.11 is not a natural number.

16. **(a)** $f(x) = x^2 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}.$

(b) $f(x) = x^3 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2}.$

17. **(a)** **(*i*)** $(1,7) \Rightarrow 2\cdot 1 + 2\cdot 7 = \mathbf{16}.$

(*ii*) $(2,6) \Rightarrow 2\cdot 2 + 2\cdot 6 = \mathbf{16}.$

(***iii***) $(6,2) \Rightarrow 2 \cdot 6 + 2 \cdot 2 = \mathbf{16}$.

(***iv***) $(\sqrt{5},\sqrt{5}) \Rightarrow 2 \cdot \sqrt{5} + 2 \cdot \sqrt{5} = \mathbf{4\sqrt{5}}$.

(b) If output (or perimeter) $= 20$ then
$2\ell + 2w = 20 \Rightarrow 2(\ell + w) = 20$
$\Rightarrow \ell + w = 10.$

The possibilities are **any ordered pair** $(l, 10 - l)$ **with** $\mathbf{0 < l < 10}$.

(c) Domain:
$\{(\ell, w) \mid \ell$ and w are any positive real numbers$\}$.
The range is any positive real number.

18. **(a)** There were 600 cars on the road at 6:30; there were (as nearly as can be determined from the graph) 650 cars at 7:00 $\Rightarrow 650 - 600 =$ **50 cars** increase.

(b) The graph rises most steeply **between 6:00 and 6:30**; i.e., that is the period in which the increase in number of cars was greatest (by 200 cars).

(c) The graph is flat between 8:00 and 8:30; i.e., there was **no increase** in the number of cars.

(d) Only one period of the graph shows a drop: 700 cars at **8:30,** decreasing to 600 at **9:00**, or $700 - 600 =$ **100 cars** decrease.

(e) Segments are used because the data are continuous rather than discrete; e.g., there is a unique number of cars at 5:47 A.M. The assumption of a linear increase or decrease in traffic between hours may, however, be invalid and could lead to erroneous conclusions.

19. **(a)** (***i***) $H(2) = 128(2) - 16(2)^2 = \mathbf{192\ feet}$.

(***ii***) $H(6) = 128(6) - 16(6)^2 = \mathbf{192\ feet}$.

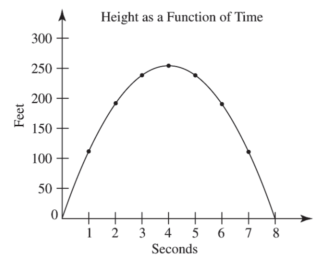
(***iii***) $H(3) = 128(3) - 16(3)^2 = \mathbf{240\ feet}$.

(***iv***) $H(5) = 128(5) - 16(5)^2 = \mathbf{240\ feet}$.

Some of the heights correspond to the height of the ball as it rises; some to its height as it falls.

(b) Plot:

t	$H(t) = 128t - 16t^2$
0	0
1	112
2	192
3	240
4	256
5	240
6	192
7	112
8	0



The ball will reach its highest point of **256 feet** at $t = \mathbf{4\ seconds}$.

(c) $H(t) = 0$ at $t = 8$; i.e., the ball will hit the ground in **8 seconds**. Note that $H(t) = 0$ also at $t = 0$.

(d) Domain: $\mathbf{\{t | 0 \le t \le 8\}}$ **seconds.**

(e) Range: $\mathbf{\{H(t) | 0 \le H(t) \le 256\}}$ **feet**.

20. 7, 17, 31,… The number of vertical matchsticks in each shape are square numbers: 4, 9, 16, … $= (n+1)^2$. The horizontal match sticks are $3 \cdot 1, 4 \cdot 2, 5 \cdot 3, \ldots : (n+2)n$ So $S(n) = (n+1)^2 + (n+2)n$.

21. 1, 4, 16. …It is a geometric sequence with $a_1 = 1$ and $r = 4$. Then $S(n) = 4^{n-1}$.

22. **(a)** **A function**. $x + y = 2 \Rightarrow y = 2 - x$; for any input x, y is unique.

(b) **Not a function**. $x - y < 2 \Rightarrow y > x - 2$; for any input x, y may be any value greater than $x - 2$.

(c) **A function**. y is unique for any input x.

(d) **A function**. $xy = 2 \Rightarrow y = \frac{2}{x}$; for any input x (except $x = 0$, for which y is undefined), y is unique.

(e) **A function**. y is unique for any input x.

(f) $|y| = x$ is not a function from x to y, $x = 1 \Rightarrow y = \pm 1$ which is not unique.

(g) $x^2 + y^2 = 1$ is not a function from x to y, $x = 0 \Rightarrow y = \pm 1$ which is not unique.

(h) **A function**. y is unique for any input x.

23. (a) **A function**. There is only one value of y for each value of x.

(b) **Not a function**. When $x = 1$ there are four values of y.

(c) **A function**. There is only one value of y for each value of x.

24. (a) B and H must be boys since there are no "is the sister of arrows" from either. The remainder; A, C, D, G, I, J, E, and F; must be girls.

(b) $\{(A, B), (A, C), (A, D), (C, A), (C, B), (C, D), (D, A), (D, B), (D, C), (E, H), (F, G), (G, F), (I, J), (J, I)\}$.

(c) **No**. A "is the sister of" three different people $\Rightarrow$ the relation is not a function.

25. (a) **Yes**. The set of all people has exactly one mother (assuming biological mothers only) each–more or less than one is not possible.

(b) **No**. Some elements of the set of all boys do not have a brother or may have more than one.

26. (a) **Not an equivalence relation:**

(*i*) Not reflexive. A person cannot be a parent to him/herself.

(*ii*) Not symmetric. John can be a parent to Jane, but Jane cannot be a parent to John.

(*iii*) Not transitive. If John is the parent of James and James is the parent of Joseph, John is not the parent of Joseph.

(b) **An equivalence relation** (reflexive, symmetric, and transitive):

(*i*) Reflexive. Juan is the same age as Juan.

(*ii*) Symmetric. If Juan is the same age as Juanita, then Juanita is the same age as Juan.

(*iii*) Transitive. If Juan is the same age as Jose and Jose is the same age as Victor, then Juan is the same age as Victor.

(c) **An equivalence relation:**

(*i*) Reflexive. Jo Ann has the same last name as herself.

(*ii*) Symmetric. If Jo Ann has the same last name as Cheryl, then Cheryl has the same last name as Jo Ann.

(*iii*) Transitive. If Jo Ann has the same last name as Cheryl and Cheryl has the same last name as Penelope, then Jo Ann has the same last name as Penelope.

(d) **An equivalence relation:**

(*i*) Reflexive. Vicky is the same height as herself.

(*ii*) Symmetric. If Barbara is the same height as Margarita, then Margarita is the same height as Barbara.

(*iii*) Transitive. If Willy is the same height as Billy and Billy is the same height as Don, then Willy is the same height as Don.

(e) **Not an equivalence relation:**

(*i*) Not reflexive. Cindy cannot be married to herself.

(*ii*) Symmetric. If Arnold is married to Pam, then Pam is married to Arnold.

(*iii*) Not transitive. If John is married to Clara and Clara is married to John, then John is not married to John.

(f) **Not an equivalence relation:**

(*i*) Reflexive. Peter lives within 10 miles of himself.

(*ii*) Symmetric. If Jon lives within 10 miles of Evangeline, then Evangeline lives within 10 miles of Jon.

(*iii*) Not transitive. If Fred lives within 10 miles of Jim and Jim lives within 10 miles of Herb, then Fred does not necessarily live within 10 miles of Herb.

(g) **Not an equivalence relation:**

(*i*) Not reflexive. Juan cannot be older than himself.

(*ii*) Not symmetric. If Jose is older than Mireya then Mireya cannot be older then Jose.

(*iii*) Transitive. If Jean is older than Mike and Mike is older than Cybil, then Jean is older than Cybil.

Assessment 8-4B

1. (a) **Multiply $^{-}$3** to the input number; i.e., $f(x) = (^{-}3)x$.

   (b) **Add 5** the input number ; i.e., $f(x) = x + 5$.

   (c) **Square** the input number; i.e., $f(x) = x^2$.

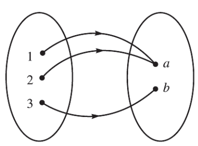
2. (a) **Not a function**. The element 2 from $\{1, 2, 3\}$ is not paired with any element from the set $\{a, b, c, d\}$.

   (b) **Not a function**. 1 is paired to more than one element from $\{a, b, c, d\}$, while 2 and 3 are not paired at all.

   (c) **Function.**

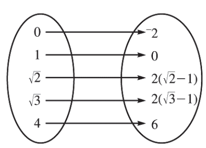
   (d) **Not a function.** $(2, b)$, and $(2, c)$ both satisfy the relation, but inputs can have only one associated output.

3. (a) Answers may vary; for example:



   (b) Each of the three elements in the domain have two choices for a pairing $\Rightarrow$ $\mathbf{2^3 = 8}$ possible functions.

4. (a)

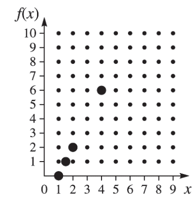


   (b) $\{(0, {}^{-}2), (1, 0), (\sqrt{2}, 2(\sqrt{2} - 1)), (\sqrt{3}, 2(\sqrt{3} - 1)), (4, 6)\}$.

   (c)

x	$f(x)$
0	$^{-}2$
1	0
$\sqrt{2}$	$2(\sqrt{2} - 1)$
$\sqrt{3}$	$2(\sqrt{3} - 1)$
4	6

   (d)



5. (a) **A function**. The function is $\{(0, 0), (1, 0), (2, 0), (3, 0), (x, 3)\}$ where x is any real number other than 0,1,2 or 3.

   (b) **Not a function**. Some elements of the domain correspond with more than one element of the range; e.g., (3, 0) and (3, 1), (4, 0) and (4, 1),….

   (c) **A function**. $\{(0, 0), (1, 1), (2, 2), \ldots, (9, 9)\}$.

   (d) **Not a function** because (2, 2) and (2, $^{-}$2) both satisfy the relation, but inputs can only have one associated output in a function.

6. (a) If each element in A is doubled and then increased by 1, it corresponds to a unique element in B. Let the elements in A be x and the elements in B be $f(x)$; the rule is $f(x) = 2x + 1$.

   (b) If each first element of the ordered pairs in A is squared and then added to the second element of the ordered pair, they correspond to a unique element in B. Let the elements in A be (x, y) and the elements in B be $f((x, y))$; the rule is $f((x, y)) = x^2 + y$.

7. (a) $T = 70°$ F. Then $C = 70 - 40 = 30$ chirps per 15 seconds $=$ **2 chirps** each second.

   (b) 40 chirps per minute $=$ 40 chirps per 60 seconds $=$ 10 chirps per 15 seconds. $10 = T - 40 \Rightarrow T = 10 + 40 = \mathbf{50°F}$.

8. These solutions are possible if the patterns continue:

   (a) $\boldsymbol{L(n) = n(n+1)}$; e.g., if $n = 6$ then $n + 1 = 7$ and $6 \cdot 7 = 42$. Note that n and $L(n)$ do not have to be in increasing or decreasing order.

   (b) $L(n)$ is obtained by taking the nth power of 2, or $\boldsymbol{L(n) = 2^n}$; e,g., if $n = 5$, $L(n) = 2^5 = 32$.

9. (a) The "Cost per Minute" graph shows 45¢ for the **cost of the sixth minute**; the "Total Cost for Calls" graph shows the **cost for a six-minute call** to be \$2.70.

   (b) It must be assumed that the cost per minute is always the same regardless of the length of the call, and that the company charges 45¢ per minute for each fraction of a minute.

(c) **(*i*)** $\boldsymbol{c = 45¢}$ per minute in the first graph.

(*ii*) At 45¢ per minute, $\boldsymbol{c = 45t}$ in the second graph, where c is cost in cents and t is time in minutes.

10. Let n be the elements of the domain; if the patterns continue:

(a) The output is obtained if n is doubled, or, alternatively, this is an arithmetic sequence with $a_1 = 2$ and $d = 2$. Thus $f(n) = 2 + (n-1)\cdot 2 = \boldsymbol{2n}$.

(b) This is a geometric sequence with first tern 3 and common ratio 3. Thus $f(n) = \mathbf{3^{n-1}}$.

(c) This is a geometric sequence with first tern $\sqrt{3}$ and common ratio 1. Thus $f(n) = \sqrt{3}(1)^{n-1} = \sqrt{3}$.

(d) This is a geometric sequence with first tern 3^{-1} and common ratio $3^{\frac{1}{2}}$. Thus

$$f(n) = 3^{-1}(3^{\frac{1}{2}})^{n-1} = 3^{-1}3^{\frac{1}{2}n-\frac{1}{2}}$$
$$= 3^{\frac{n-3}{2}}$$

11. **(a)** $(g \circ f)(5) = g[f(5)] = g(5-1) = 7\cdot 4 = \mathbf{28}$.

(b) $(g \circ f)(\sqrt{3}) = g[f(\sqrt{3})] = g(\sqrt{3}-1) = 7\cdot(\sqrt{3}-1) = 7\sqrt{3}-7$.

(c) $(g \circ f)(10) = g[f(10)] = g(10-1) = 7\cdot 9 = \mathbf{63}$.

(d) $(g \circ f)(a) = g[f(a)] = g(a-1) = \boldsymbol{7(a-1)}$.

(e) $(g \circ f)(\frac{7}{6}) = g[f(\frac{7}{6})] = g(\frac{7}{6}-1) = g(\frac{1}{6}) = 7\cdot\frac{1}{6} = \frac{7}{6}$.

12. $7x - 7 = 2x \Rightarrow 5x = 7 \Rightarrow x = \frac{7}{5}$

13. **(a)** **(i)** $(f \circ g)(5) = f[g(5)] = f(35) = 35 - 1 = 34$.

(ii) $(f \circ g)(\sqrt{3}) = f[g(\sqrt{3})] = f(7\sqrt{3}) = 7\sqrt{3} - 1$.

(iii) $(f \circ g)(10) = f[g(10)] = f(70) = 70 - 1 = 69$.

(iv) $(f \circ g)(a) = f[g(a)] = g(7a) = 7a - 1$.

(v) $(f \circ g)(\frac{7}{6}) = f[g(\frac{7}{6})] = g(\frac{49}{6}) = \frac{49}{6} - 1 = \frac{43}{6}$.

(b) $7x - 1 = x \Rightarrow 6x = 1 \Rightarrow x = \frac{1}{6}$

14. **(a)** $f(f(0)) = f(^-(0+b)) = {}^-(^-(0+b)+b) = 0$

(b) $f(f(\sqrt{2})) = f(^-(\sqrt{2}+b)) = {}^-(^-(\sqrt{2}+b)+b) = \sqrt{2}$

(c) $f(f(x)) = f(^-(x+b)) = {}^-(^-(x+b)+b) = x$

So for all real numbers.

15. **(a)** $\frac{2x-1}{2x+1} = 2 \Rightarrow 2x-1 = 2(2x+1)$

$2x - 1 = 4x + 2 \Rightarrow {}^-2x = 3 \Rightarrow x = {}^-\frac{3}{2}$

(b) $\frac{2x-1}{2x+1} = 0 \Rightarrow 2x-1 = 0(2x+1)$

$2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$

(c) $\frac{2x-1}{2x+1} = {}^-\frac{1}{2} \Rightarrow 2x-1 = {}^-\frac{1}{2}(2x+1)$

$2x - 1 = {}^-x - \frac{1}{2} \Rightarrow 3x = \frac{1}{2} \Rightarrow x = \frac{1}{6}$

(d) $\frac{2x-1}{2x+1} = 1 \Rightarrow 2x-1 = 1(2x+1)$

$2x - 1 = 2x + 1 \Rightarrow 0x = 2 \Rightarrow 0 = 2$

This is not possible.

16. **(a)** **(*i*)** **Yes**. $n^2 = 1 \Rightarrow t(1) = 1$.

(*ii*) **Yes**. $n^2 = 4 \Rightarrow t(2) = 4$.

(*iii*) **Yes**. $n^2 = 9 \Rightarrow t(3) = 9$.

(*iv*) **No**. If $n^2 = 10$, then $n = \sqrt{10}$ which is not a natural number.

(*v*) **Yes**. $n^2 = 900 \Rightarrow t(30) = 900$.

(b) **(*i*)** **Yes**. $n(n+1) = 2 \Rightarrow t(1) = 2$.

(*ii*) **Yes**. $n(n+1) = 12 \Rightarrow t(3) = 12$.

(*iii*) **Yes**. $n(n+1) = 2550 \Rightarrow$ $t(50) = 2550$.

(*iv*) **No**. $n(n+1) = 2600 \Rightarrow t(50.49) \approx$ 2600, but 50.49 is not a natural number.

17. **(a)** (*i*) $(1, 4) \Rightarrow 2 \cdot 1 + 2 \cdot 4 = \mathbf{10}$.

(*ii*) $(2, 1) \Rightarrow 2 \cdot 2 + 2 \cdot 1 = \mathbf{6}$

(*iii*) $(1, 2) \Rightarrow 2 \cdot 1 + 2 \cdot 2 = \mathbf{6}$.

(*iv*) $(\sqrt{3}, \sqrt{3}) \Rightarrow 2 \cdot \sqrt{3} + 2 \cdot \sqrt{3} = \mathbf{4\sqrt{3}}$.

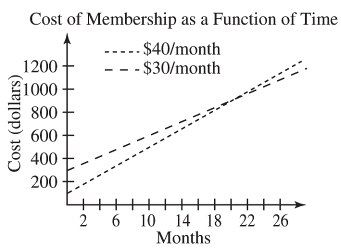
(*v*) $(x, y) \Rightarrow \mathbf{2 \cdot x + 2 \cdot y}$.

(b) If output (or perimeter) = 20 then $2\ell + 2w = 20 \Rightarrow 2(\ell + w) = 20 \Rightarrow \ell + w = 10$. The possibilities are **{(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9, 1)}**.

(c) **No**. Outputs are single numbers, not ordered pairs.

18. **(a)** $C(x) = \$100 + \$40$ per month $(x) \Rightarrow$ $\mathbf{C(x) = \$(100 + 40x)}$.

(b) and (c)



(d) The graphs cross at **20 months** where $300 + 30x = 100 + 40x$; after that the second plan is less expensive.

19. **(a)** (*i*) $H(2) = 128(2) - 16(2)^2 = \mathbf{192}$ **feet**.

(*ii*) $H(6) = 128(6) - 16(6)^2 = \mathbf{192}$ **feet**.

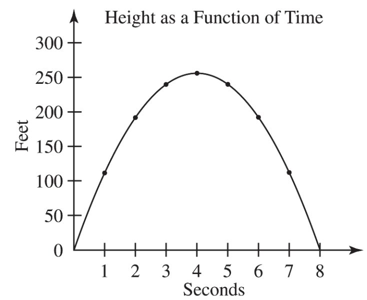
(*iii*) $H(3) = 128(3) - 16(3)^2 = \mathbf{240}$ **feet**.

(*iv*) $H(5) = 128(5) - 16(5)^2 = \mathbf{240}$ **feet**.

Some of the heights correspond to the height of the ball as it rises; some to its height as it falls. The graph is a parabola with vertex (4, 256), so **256** is the highest point.

(b) Plot:

t	$H(t) = 128t - 16t^2$
0	0
1	**112**
2	192
3	240
4	256
5	240
6	192
7	**112**
8	0



The ball will reach its highest point of **256 feet** at $\boldsymbol{t = 4}$ **seconds**.

(c) $H(t) = 0$ at $t = 8$; i.e., the ball will hit the ground in **8 seconds**. Note that $H(t) = 0$ also at $t = 0$.

(d) Domain: $\mathbf{\{t \mid 0 \leq t \leq 8\}}$ **seconds**.

(e) Range: $\mathbf{\{H(t) \mid 0 \leq H(t) \leq 256\}}$ **feet**.

20. **(a)** Assuming the matchstick figure patterns continue, expand each to the fourth figure and count: Each figure adds one row and one column to the preceding figure, so the fourth figure would have five rows and five columns. The two matchsticks at the lower left, though, are removed, leaving **58** matchsticks.

(b) If we include the missing 2 matchsticks from the bottom left then there are $2 \cdot 3 = (n+1)(n+2)$ vertical matchsticks and $2 \cdot 3 = (n+1)(n+2)$ horizontal matchsticks in the first shape. There are $3 \cdot 4 = (n+1)(n+2)$ vertical matchsticks and $3 \cdot 4 = (n+1)(n+2)$ horizontal matchsticks in the second shape. There are $4 \cdot 5 = (n+1)(n+2)$ vertical matchsticks and $4 \cdot 5 = (n+1)(n+2)$ horizontal matchsticks in the third shape

Each shape has $(n+1)(n+2) + (n+1)(n+2) = 2(n+1)(n+2)$

matchsticks. If the 2 are subtracted from the bottom left, there are

$S(n) = 2(n+1)(n+2) - 2$ matchsticks.

21. **(a)** 2,5,9,… $S(n) = \frac{n(n+1)}{2} + n$.

(b) $S(n) = n + 2 + 2n = \mathbf{3n + 2}$.

22. **(a)** **A function**. $x - y = 2 \Rightarrow y = x - 2$; for any input x, y is unique.

(b) **Not a function**. $x + y < 20 \Rightarrow y < 20 - x$; for any input x, y may be any value less than $20 - x$. Thus, it is easy to find two outputs y for each input x.

(c) **A function**. For any input x, y is unique.

(d) **A function**. For any input x, y is unique.

(e) **A function**. For any input x, y is unique.

(f) $|y| = |x|$ is not a function; if $x = \sqrt{2}$ then $y = \pm 1$ so there is no unique output.

(g) $x^2 - y^2 = 1$ is not a function; if $x = 1$ then $y = \pm 1$ so there is no unique output.

(h) **A function**. For any input x, y is unique.

23. **(a)** **A function**. For any value x, there is a unique value of y.

(b) **A function**. For any value x, there is a unique value of y.

(c) **A function**. For any value x, there is a unique value of y.

(d) **A function**. For any value x, there is a unique value of y.

(e) **Not A function**. For example, for $x=1$, there is two distinct y values.

24. **(a)** **(*i*)** **Symmetric**. If $x = a$ and $y = b$ satisfies the relation, then $x = b$ and $y = a$ also satisfies it (by the commutative property).

(*ii*) **Not symmetric**. $a - b \neq b - a$.

(*iii*) **Symmetric**. If $x = a$ and $y = b$ satisfies the relation, then $x = b$ and $y = a$ also satisfies it.

(*iv*) **Symmetric**. If $y = x$, then $x = y$.

(*v*) **Not symmetric**. $(2, 4) \neq (4, 2)$.

(b) **All** are functions of y in terms of x. In each, for any input x there is a unique output y.

25. **(a)** B and F must be boys, since others are "the sister of" them but they are not "the sister of" in return. E and H are not paired, so their gender is indeterminate. The remainder; A, C, D, I, J, and G, must be girls.

(b) $\{(A, B), (A, C), (A, D), (C, A), (C, B), (C, D), (D, A), (D, B), (D, C), (G, F), (I, J), (J, I)\}$.

(c) **No**. A "is the sister of" three different people $\Rightarrow$ the relation is not a function.

26. **(a)** **A function**. Each element of the domain is paired with one and only one element of the range.

(b) A relation but **not a function**. New York from the domain is paired with two elements of the range.

(c) A relation but **not a function**. "mother" from the domain could be paired with more than one element of the range.

(d) **A function**. Each element of the domain is paired with one and only one element of the range.

(e) A relation but **not a function**. The element 1 from the domain could be paired with any odd number from the range to produce an even number.

27. In each of the following, use the nonempty set $\{1, 2, 3\}$ as an example

(a) **An equivalence relation**:

(*i*) Reflexive. Assume that the nonempty set is called A. The relation "is equal to" is reflexive, because for any subset of A, call it B, $B = B$ by the definition of set equality.

(*ii*) Symmetric. Assume that B and C are subsets of A, and that $B = C$. This means that every element in B is in C and every element in C is in B. Thus, $C = B$ is also true, making the relation "is equal to" symmetric on the set of subsets of A.

(*iii*) Transitive. Assume that B, C, and D are subsets of A and that $B = C$ and $C = D$. Then because of set equality, every element of B is an element of C and every element of C is an element of D, making B a subset of D. Further, every element of D is an element of C and every element of B, so D is a subset of B. Thus $B = D$, proving that "is equal to" is transitive.

(b) Not an equivalence relation:

(*i*) Not reflexive. $\{1\} \not\subset \{1\}$.

(*ii*) Not symmetric. If $\{1\} \subset \{1, 2\}$ then $\{1, 2\} \not\subset \{1\}$.

(*iii*) Transitive. If $\{1\} \subset \{1, 2\}$ and $\{1, 2\} \subset \{1, 2, 3\}$ then $\{1\} \subset \{1, 2, 3\}$.

(c) Not an equivalence relation:

(*i*) Not reflexive. $\{1\}$ is not $\neq$ $\{1\}$.

(*ii*) Symmetric. If $\{1\} \neq \{2\}$ then $\{2\} \neq \{1\}$.

(*iii*) Not transitive. If $\{1\} \neq \{2\}$ and $\{2\} \neq \{1\}$ then $\{1\}$ is not $\neq$ $\{1\}$.

(d) An equivalence relation:

(*i*) Reflexive. $n(\{1, 2\}) = n(\{2, 1\})$.

(*ii*) Symmetric. If $n(\{1, 2\}) = n(\{1, 3\})$ then $n(\{1, 3\}) = n(\{1, 2\})$.

(*iii*) Transitive. If $n(\{1, 2\}) = n(\{1, 3\})$ and $n(\{1, 3\}) = n(\{2, 3\})$ then $n(\{1, 2\}) = n(\{2, 3\})$.

Mathematical Connections 8-4: Review Problems

22. **(a)**
$$\begin{aligned}2\sqrt{2} - \sqrt{2}x &= x\\ 2\sqrt{2} &= x + \sqrt{2}x\\ 2\sqrt{2} &= x(1+\sqrt{2})\\ x &= \frac{2\sqrt{2}}{1+\sqrt{2}}\end{aligned}$$

(b)
$$\frac{3}{4}x = \frac{\sqrt{2}}{2}x + 3 \Rightarrow \frac{3}{4}x - \frac{\sqrt{2}}{2}x = 3$$
$$x\left(\frac{3}{4} - \frac{\sqrt{2}}{2}\right) = 3 \Rightarrow x\left(\frac{3}{4} - \frac{2\sqrt{2}}{4}\right) = 3$$
$$x\left(\frac{3-2\sqrt{2}}{4}\right) = 3 \Rightarrow x = \frac{12}{3-2\sqrt{2}}$$

(c)
$$(x-1)^2 = 2 \Rightarrow x - 1 = \pm\sqrt{2}$$
$$x = 1 \pm \sqrt{2}$$

(d)
$$\begin{aligned}&\sqrt{2}(2x-3) = \sqrt{3}(2x-3)\\ &\sqrt{2}(2x-3) - \sqrt{3}(2x-3) = 0\\ &(2x-3)(\sqrt{2}-\sqrt{3}) = 0\\ &2x - 3 = 0 \Rightarrow x = \frac{3}{2}\end{aligned}$$

(e) $\sqrt{(x-1)^2} = x - 1 \Rightarrow$ True for all $x \geq 1$.

(f) $\sqrt{(x-1)^2} = 1 - x \Rightarrow$ True for all $x \leq 1$.

23. Since distance $=$ rate $\times$ time, the fast car will travel $70t$ miles in the same time as the slow car travels $60t$ miles. Then $70t - 40 = 60t \Rightarrow 10t = 40 \Rightarrow \boldsymbol{t = 4}$ **hours**.

24. **(a)** Answers may vary. $\sqrt{3} \approx 1.7320508$, so **1.8** and **1.85** are two rational numbers between $\sqrt{3}$ and 2.

(b) Answers may vary. $\frac{11}{13} \approx 0.84615$ and $\frac{12}{13} \approx 0.92307$ Irrational numbers between $\frac{11}{13}$ and $\frac{12}{13}$ include **0.901001000…** and **0.9020020002…** .

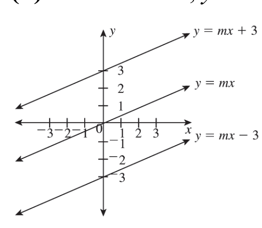
25. Answers may vary. For example, $\sqrt{2} + {}^{-}\sqrt{2} = 0$, a rational number.

Assessment 8-5A: Equations in a Cartesian Coordinate System

1. The graph of $y = mx + 3$ contains the point $(0, 3)$ and is parallel to the line $y = mx$. Similarly, the graph of $y = mx - 3$ contains the point $(0, {}^{-}3)$ and is parallel to $y = mx$ (see below).

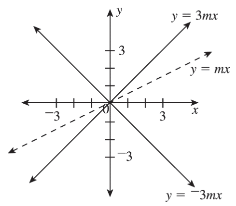
(a) Parallel line; y-intercept $= 3$.

(b) Parallel line; y-intercept $= {}^{-}3$.



(c) The graph of $y = mx$ with a slope three times m and the same y-intercept.

(d) The graph of $y = mx$ with a slope negative three times m and the same y-intercept.



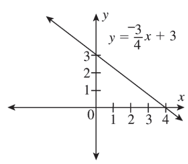
2. Each of the following is in the form $y = mx + b$, where m represents the slope of the line and b represents the y-coordinate of the y-intercept. The x-intercept in each case can be found by setting y equal to 0 and solving for x.

(a) Given $y = \frac{^{-}3}{4}x + 3$:

(***i***) $m = \frac{^{-}3}{4}$ and $b = 3$.

(***ii***) If $y = 0 = \frac{^{-}3}{4}x + 3 \Rightarrow x = 4$; the x-intercept is at $(4, 0)$.

(***iii***) Draw a line through $(0, 3)$ and $(4, 0)$.

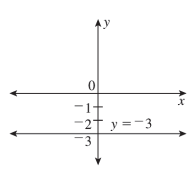


(b) Given $y = {^{-}3}$:

(***i***) $m = 0$ and $b = {^{-}3}$.

(***ii***) There is no x-intercept; i.e., $y = {^{-}3}$ is a horizontal line.

(***iii***) Draw a line through $(0, {^{-}3})$ parallel to the x-axis.

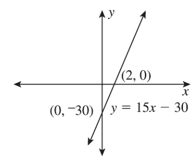


(c) Given $y = 15x - 30$:

(***i***) $m = 15$ and $b = {^{-}30}$.

(***ii***) If $y = 0 = 15x - 30 \Rightarrow x = 2$.

(***iii***) Draw a line through $(0, {^{-}30})$ and $(2, 0)$.



3. When the equations in Problem 2 are in the form $y = mx + b$, b is the y-intercept and is located at $(0, b)$. The x-intercept may be found by setting y equal to 0 and solving for x. If x is some value a, the x-intercept is located at $(a, 0)$.

(a) $y = \frac{^{-}3}{4}x + 3$:

(***i***) $b = 3$. y-intercept is at **(0, 3)**.

(***ii***) If $y = 0 = \frac{^{-}3}{4}x + 3 \Rightarrow x = 4$. x-intercept is at **(4, 0)**.

(b) $y = {^{-}3}$:

(***i***) $b = {^{-}30}$. y-intercept is at **(0, ¯3)**.

(***ii***) There is **no x-intercept**; i.e., $y = {^{-}3}$ is a horizontal line parallel to the x-axis.

(c) $y = 15x - 30$.

(***i***) $b = {^{-}30}$. y-intercept is at **(0, ¯30).**

(***ii***) If $y = 0 = 15x - 30 \Rightarrow x = 2$. x-intercept is at **(2, 0)**.

4. **(a)** For ease in calculating, use the points $(0, 32)$ and $(100, 212)$. Slope $(m) = \frac{212-32}{100-0} = \frac{9}{5} \Rightarrow$ $F = \frac{9}{5}C + b$.

The F-intercept is at $(0,32)$. $b = 32$; thus $\boldsymbol{F = \frac{9}{5}C + 32}$.

(b) $F = \frac{9}{5}C + 32 \Rightarrow \frac{9}{5}C = F - 32 \Rightarrow$ $\boldsymbol{C = \frac{5}{9}(F - 32)}$.

(c) Consider the relationship $F = \frac{9}{5}C + 32$. Substitute C for F:

$$
\begin{aligned}
C &= \tfrac{9}{5}C + 32 \\
5c &= 5\left(\tfrac{9}{5}C + 32\right) \\
5c &= 9c + 160 \\
-4c &= 160 \\
c &= -40.
\end{aligned}
$$

The ordered-pair $\mathbf{(-40, -40)}$ represents where degrees Celsius and degrees Fahrenheit are the same.

5. When the equations are in slope-intercept form, $y = mx + b$, m is the slope of the line and b is the y-intercept, located at $(0, b)$.

(a) (***i***) $3y - x = 0 \Rightarrow 3y = x \Rightarrow \boldsymbol{y = \frac{1}{3}x}$.

(***ii***) $m = \frac{1}{3}$; y-intercept is at $(0, 0)$.

(b) (***i***) $x + y = 3 \Rightarrow \boldsymbol{y = {}^{-}x + 3}$.

(***ii***) $m = {}^{-}1$; y-intercept is at $(0, 3)$.

(c) (***i***) $x = 3y \Rightarrow \boldsymbol{y = \frac{1}{3}x}$.

(***ii***) $m = \frac{1}{3}$; y-intercept is at $(0, 0)$.

6. **(a)** $m = \frac{{}^{-}2-3}{1-{}^{-}4} = \frac{{}^{-}5}{5} = {}^{-}1$, thus $y = {}^{-}x + b$.

Substituting x and y from $({}^{-}4, 3)$: $(3) = {}^{-}({}^{-}4) + b$, thus $b = {}^{-}1$.

So $\boldsymbol{y = {}^{-}x - 1}$.

(b) $m = \frac{1-0}{2-0} = \frac{1}{2}$, thus $y = \frac{1}{2}x + b$. The line goes through the origin, thus $b = 0$.

So $\boldsymbol{y = \frac{1}{2}x}$.

(c) $m = \dfrac{0 - \frac{{}^{-}1}{2}}{\frac{1}{2} - 0} = \dfrac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = 1$, thus $y = x + b$.

Substituting x and y from $\left(\frac{1}{2}, 0\right)$: $(0) = \left(\frac{1}{2}\right) + b$, thus $b = \frac{{}^{-}1}{2}$.

So $\boldsymbol{y = x - \frac{1}{2}}$.

(d) $m = \frac{{}^{-}3 - {}^{-}3}{{}^{-}1 - 0} = 0$, thus $y = 0x + b$.

${}^{-}3 = 0x + b \Rightarrow b = {}^{-}3$

So $y = {}^{-}3$.

7. Answers may vary.

(a) Both points include the coordinate $y = 2$, thus the equation of the line is $y = 2$. Other points could be $({}^{-}3, 2), (5, 2), \ldots$.

(b) Both points include the coordinate $x = 0$, thus the line is the y-axis. Other points could be $(0, 1), (0, {}^{-}6), \ldots$.

8. **(a)** $({}^{-}2, 0), ({}^{-}2, 1)$, and (x, y) are collinear, thus the value of x at each point is ${}^{-}2$.

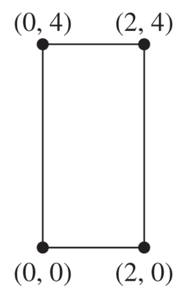
(***i***) $\boldsymbol{x = {}^{-}2}$ is the equation of the line.

(***ii***) y may assume **any real value**.

(b) (***i***) The fourth quadrant implies that $\boldsymbol{x > 0}$ and $\boldsymbol{y < 0}$.

(***ii***) x and y may assume **any real value** greater than zero or less than zero, respectively.

9. The rectangle is shown below:



It has dimensions of $2 - 0 = 2$ and $4 - 0 = 4$.

(***i***) Area $= 2 \cdot 4 =$ **8 square units**.

(***ii***) Perimeter $= 2 \cdot 2 + 2 \cdot 4 =$ **12 units**.

10. **(a)** $\boldsymbol{x = 3}$, a vertical line through $(3, 0)$.

(b) $\boldsymbol{y = 5}$, a horizontal line through $({}^{-}4, 5)$.

11. In each case, slope $(m) = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$.

(a) $m = \frac{0-3}{{}^{-}5-4} = \frac{{}^{-}3}{-9} = \boldsymbol{\frac{1}{3}}$.

(b) $m = \frac{2-2}{1-\sqrt{5}} = \frac{0}{1-\sqrt{5}} = \mathbf{0}$.

(c) $m = \frac{b-a}{b-a} = \mathbf{1}$ (if $b \neq a$).

(d) $m = \frac{{}^{-}2-2}{\sqrt{5}-\sqrt{5}} = \frac{{}^{-}4}{0}$ does not exist.

12. In each case, since m is known, substitute x and y values into the equation $y = mx + b$ and solve for b.

(a) Use the point $(4,3)$. $(3) = \frac{1}{3}(4) + b \Rightarrow$ $b = \frac{5}{3}$, thus $\boldsymbol{y = \frac{1}{3}x + \frac{5}{3}}$.

(b) $\boldsymbol{y = 2}$ (a horizontal line).

(c) Use the point $(a, a) \Rightarrow (a) = 1(a) + b \Rightarrow$ $b = 0$, thus $\boldsymbol{y = x}$ (if $b \neq a$).

13. Answers may vary, depending on estimates from the fitted line; e.g.:

(a) From the fitted line, estimate point coordinates of $(50, 10)$ and $(80, 40)$. $m = \frac{40-10}{80-50} = \frac{30}{30} = 1$.

Use the point $(50, 10)$ and substitute $T = 50$ and $C = 10$ into $C = 1T + b$ (i.e., an equation of the form $y = mx + b$). $(10) = 1(50) + b \Rightarrow b = {}^-40$. The equation is then $\boldsymbol{C = T - 40}$.

(b) **If** $T = 90°$, then $C = (90) - 40 =$ **50 chirps** per 15-second interval.

(c) $N = 4C$; i.e., there are four 15-second intervals in one minute. $N = 4(T - 40)$, or $\boldsymbol{N = 4T - 160}$.

14. **(a)** $y = 2x - 20$ is line $\overleftrightarrow{BC}$ because the x-intercept is (10,0) and the y-intercept is (0,$^-$20). Then $y = 4 - 2x$ is line $\overleftrightarrow{AB}$.

(b) Point D has the same x-coordinate as point C which is 10. Point D lies on $\overleftrightarrow{AB}$, so $y = 4 - 2(10) = {}^-16$. Point D has coordinates (10, $^-$16).

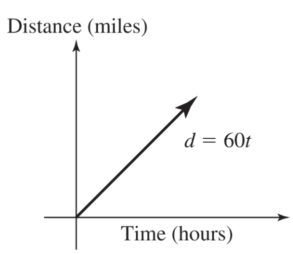
15. **(a)** The lines have the same x-intercept, $^-$3.

(b) The lines have the same x-intercept, 1.

(c) The lines have the same slope, $^-$2.

(d) The lines have the same slope, $^-$1.

16. **(a)** Graphs may vary.



(b) Use the points $(0, 0)$ and $(1, 60)$. Then $m = \frac{60-0}{1-0} =$ **60**.

17. **(a)** If $y = 3x - 1$ and $y = x + 3$, then $3x - 1 = x + 3 \Rightarrow 2x = 4 \Rightarrow x = 2$.

Substituting $x = 2$ into $y = x + 3 \Rightarrow y = 5$; or a unique solution of **(2, 5)**.

(b) Multiply $3x + 4y = {}^-17$ by $2 \Rightarrow$ $6x + 8y = {}^-34$. Multiply $2x + 3y = {}^-13$ by ${}^-3 \Rightarrow {}^-6x + {}^-9y = 39$.

Add the equations:
$$\begin{array}{rcrcr} 6x & + & 8y & = & {}^-34 \\ {}^-6x & + & {}^-9y & = & 39 \\ \hline & & {}^-y & = & 5 \end{array}$$

or $y = {}^-5$.

Substitute $y = {}^-5$ into $3x + 4y = {}^-17$.

$3x + 4({}^-5) = {}^-17 \Rightarrow x = 1$; or a unique solution of $\boldsymbol{(1, {}^-5)}$.

(c) Multiply ${}^-\frac{2}{3}x + y = \frac{1}{3}$ by $3 \Rightarrow {}^-2x + 3y = 1$

and add the equations:
$$\begin{array}{r} {}^-2x + 3y = 1 \\ 2x - 3y = {}^-1 \\ \hline 0 = 0 \end{array}$$
There are infinitely many solutions of the form $\left(x, \frac{2}{3}x + \frac{1}{3}\right)$.

(d) Substitute $y = 1 - x$ in $y = x - 1$: $1 - x = x - 1 \Rightarrow 2 = 2x \Rightarrow x = 1$. Substitute 1 for x: $y = 1 - 1 \Rightarrow y = 0$. There is a unique solution (1,0).

18. Let g be the number of gallons of gasoline and let k be the number of gallons of kerosene on the truck.

Then: $g + k = 5000 \Rightarrow g = 5000 - k$.
$\$0.13g + \$0.12k = \$640 \Rightarrow 13g + 12k = 64000$.

Substitute $g = 5000 - k$ into $13g + 12k = 64000 \Rightarrow 13(5000 - k) + 12k = 64000 \Rightarrow$ $k =$ **1000 gallons of kerosene**.

Substitute $k = 1000$ into $g = 5000 - k \Rightarrow$ $g =$ **4000 gallons of gasoline**.

19. Let d be the number of dimes and let q be the number of quarters. Then:

$d + q = 27 \Rightarrow d = 27 - q.$

$\$0.10d + \$0.25q = \$5.25 \Rightarrow 10d + 25q = 525.$

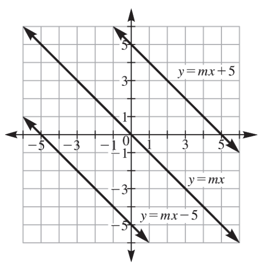
Substitute $d = 27 - q$ into $10d + 25q = 525 \Rightarrow$ $10(27 - q) + 25q = 525.$ $q =$ **17 quarters**.

Substitute $q = 17$ into $d = 27 - q.\, d =$ **10 dimes**.

Assessment 8-5B

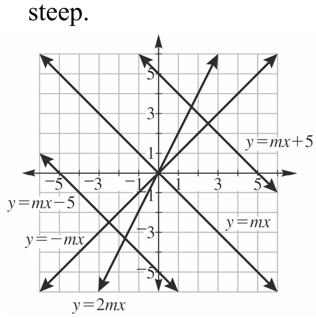
1. (a) The graph of $y = mx + 5$ contains the point $(0, 5)$ and is parallel to the line $y = mx$.

(b) Similarly, the graph of $y = mx - 5$ contains the point $(0, -5)$ and is parallel to the line $y = mx$.



(c) The graph of $y = {}^-mx$ has the same intercept as $y = mx$, but a negative slope.

(d) The graph of $y = 2mx$ has the same intercept as the graph of $y = mx$, but a slope twice as steep.



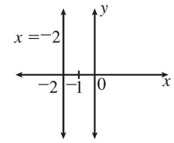
2. Each of the following [except part (a)] is in the form $y = mx + b$, where m represents the slope of the line and b represents the y-coordinate of the y-intercept. The x-intercept in each case can be found by setting y equal to 0 and solving for x.

(a) Given $x = {}^-2$:

(*i*) m is undefined, thus there is no y-intercept.

(*ii*) The x-intercept is at $x = {}^-2$.

(*iii*) Draw a line through $({}^-2, 0)$ parallel to the y-axis.

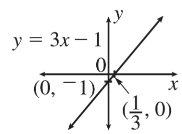


(b) Given $y = 3x - 1$:

(*i*) $m = 3$ and $b = {}^-1$.

(*ii*) If $y = 0 = 3x - 1 \Rightarrow x = \frac{1}{3}$.

(*iii*) Draw a line through $(0, {}^-1)$ and $\left(\frac{1}{3}, 0\right)$.

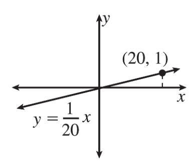


(c) Given $y = \frac{1}{20}x$:

(*i*) $m = \frac{1}{20}$ and $b = 0$.

(*ii*) To find another point, arbitrarily assign a value to x and solve for y; e.g., if $x = 20 \Rightarrow y = 1$.

(*iii*) Draw a line through $(0, 0)$ and $(20, 1)$.



3. When the equations in problem lare in the form $y = mx + b, b$ is the y-intercept and is located at $(0, b)$. The x-intercept may be found by setting y equal to 0 and solving for x. If x is some value a, the x-intercept is located at $(a, 0)$.

(a) $x = {}^-2$:

(*i*) There is **no *y*-intercept**; i.e., $x = {}^-2$ is a vertical line.

(*ii*) The x-intercept is at $x = {}^-2$, or $\mathbf{({}^-2, 0)}$.

(b) $y = 3x - 1$:

(*i*) $b = {}^-1$. y-intercept is at $\mathbf{(0, {}^-1)}$.

(*ii*) if $y = 0 = 3x - 1 \Rightarrow x = \frac{1}{3}$. x-intercept is at $\left(\mathbf{\frac{1}{3}, 0}\right)$.

(c) $y = \frac{1}{20}x$:

(*i*) $b = 0$. y-intercept is at **(0, 0)**.

(*ii*) If $y = 0 = \frac{1}{20}x \Rightarrow x = 0$. x-intercept is at **(0, 0)**.

4. Think of slope as a ratio: $\frac{39.37\text{ in}}{1\text{ m}}$. If x is a measure in meters, then $39.37\text{ in/m} \cdot x = 39.37\text{ in}$. If y is the measure in inches, then the equation is $\boldsymbol{y = 39.37x}$.

5. When the equations are in slope-intercept form, $y = mx + b$, m is the slope of the line and b is the y-intercept, located at $(0, b)$.

(a) (*i*) $\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow 12\left(\frac{x}{3} + \frac{y}{4}\right) = 12(1) \Rightarrow$
$4x + 3y = 12 \Rightarrow 3y = {}^-4x + 12$.
$\boldsymbol{y = \frac{^-4}{3}x + 4}$.

(*ii*) $m = \frac{^-4}{3}$; y-intercept is at $(0, 4)$.

(b) (*i*) $3x - 4y + 7 = 0 \Rightarrow {}^-4y = {}^-3x - 7 \Rightarrow 4y = 3x + 7$. $\boldsymbol{y = \frac{3}{4}x + \frac{7}{4}}$.

(*ii*) $m = \frac{3}{4}$; y-intercept is at $\left(0, \frac{7}{4}\right)$.

(c) (*i*) $x - y = 4(x - y) \Rightarrow x - y = 4x - 4y \Rightarrow 0 = 3x - 3y$. $\boldsymbol{y = x}$.

(*ii*) $m = 1$; y-intercept is at $(0, 0)$.

6. **(a)** Both points include the coordinate $y = 1$. The equation of the line is $\boldsymbol{y = 1}$.

(b) Both points include the coordinate $x = 2$. The equation of the line is $\boldsymbol{x = 2}$.

(c) Both points include the coordinate $y = 0$. The equation of the line is $\boldsymbol{y = 0}$.

(d) Both points include the coordinate $y = {}^-\sqrt{3}$. The equation of the line is $y = {}^-\sqrt{3}$.

7. Answers may vary.

(a) Both points include the coordinate $x = {}^-1$, thus the equation of the line is $x = {}^-1$. Other points could be $({}^-1, 7), ({}^-1, {}^-5), \ldots$.

(b) Both points have the same x and y value; i.e., the equation of the line is $y = x$. All other points are of the form (a, a); other points could be $(3, 3), ({}^-1, {}^-1), \ldots$.

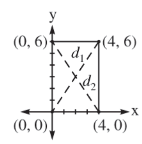
8. **(a)** $({}^-2, 1), (0, 1)$, and (x, y) are collinear, thus the value of y at each point is 1.

(*i*) $\boldsymbol{y = 1}$ is the equation of the line.

(*ii*) x may assume **any real value**.

(b) $(x, y), (0, 0)$, and $({}^-1, {}^-1)$ are collinear, and in $(0, 0)$ and $({}^-1, {}^-1)$ the values of x and y are the same. Since the points are collinear, $x = y$.

9. **(a)** Sketches may vary.



(b) Coordinates of the other vertices are (0, 0), (0, 6), and (4, 0).

(c) There are two diagonals, one with positive slope and one with negative slope.

(*i*) Positive slope: $m = \frac{6-0}{4-0} = \frac{3}{2}$. The y-intercept is at (0, 0). Thus $\boldsymbol{y = \frac{3}{2}x}$.

(*ii*) Negative slope: $m = \frac{0-6}{4-0} = \frac{^-3}{2}$. The y-intercept is at (0, 6). Thus $\boldsymbol{y = \frac{^-3}{2}x + 6}$.

(d) The point of intersection is at the intersection of the two diagonals, or where $\frac{3}{2}x = \frac{^-3}{2}x + 6 \Rightarrow 3x = 6 \Rightarrow x = 2$. If $x = 2$, $y = \frac{3}{2}\ (2) = 3$. The coordinates of the intersection are then **(2, 3)**.

10. **(a)** $\boldsymbol{y = {}^-2}$, a horizontal line through $(0, {}^-2)$.

(b) $\boldsymbol{x = {}^-4}$, a vertical line through $({}^-4, 5)$.

11. **(a)** $m = \frac{2-1}{5-{}^-4} = \boldsymbol{\frac{1}{9}}$.

(b) $m = \frac{198-81}{^-3 - {}^-3} = \frac{117}{0} \Rightarrow$ **undefined** (or no slope).

(c) $m = \frac{10-12}{1-1.0001} = \frac{{}^{-}2}{{}^{-}0.0001} = \mathbf{20,000}$.

(d) $m = \frac{\sqrt{5}-\sqrt{5}}{{}^{-}2-2} = 0$.

12. (a) Use the point $(5,2)$. $(2) = \frac{1}{9}(5) + b \Rightarrow$ $b = \frac{13}{9}$, thus $\boldsymbol{y = \frac{1}{9}x + \frac{13}{9}}$.

(b) $\boldsymbol{x = {}^{-}3}$ (a vertical line).

(c) Use the point $(1,10) \Rightarrow (10) = 20,000(1) + b \Rightarrow$ $b = {}^{-}19,990$, thus $\boldsymbol{y = 20,000x - 19,990}$.

(d) $y = \sqrt{5}$.

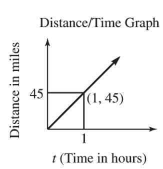
13. (a) In eight months, $\$2180 - \$2100 = \$80$ simple interest was earned, or $\$10$ per month. (10 months)$\cdot$(\$10 per month) $= \$100$ interest earned in the first ten months. The original balance was $\$2100 - \$100 = \mathbf{\$2000}$.

(b) Simple interest (I) $=$ principal (p). annual rate $(r)\cdot$ time (t) in years.
$r = \frac{1}{pt} = \frac{\$100}{\$2000\cdot\frac{10}{12}} = 0.06$, or **6%**.

14. $y = x-4$ is line $\overleftrightarrow{BC}$; $y = 2-2x$ is line $\overleftrightarrow{AB}$; and $y = 5$ is line $\overleftrightarrow{CD}$. Point A is the x-intercept of line $\overleftrightarrow{AB}$, $0 = 2-2x \Rightarrow x = 1$ so (1,0). Point B is the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$, $2-2x = x-4 \Rightarrow 6 = 3x \Rightarrow x = 2$ so $(2,{}^{-}2)$. Point C is the intersection of $\overleftrightarrow{BC}$ and $\overleftrightarrow{CD}$, $x-4 = 5 \Rightarrow x = 9$ so (9,5).

15. (a) The lines have the same x-intercept, 3.
(b) The lines have the same x-intercept, 3.
(c) The lines have the same slope, 1.
(d) All non-vertical lines through the point (1,1).

16. (a)



(b) Think of slope as a ratio, 45 miles per 1 hour. So slope or $\frac{45}{1} = \mathbf{45}$.

17. (a) Rewrite each equation into the form $y = mx + b$. $2x - 6y = 7 \Rightarrow y = \frac{1}{3}x - \frac{7}{6}$ and $3x - 9y = 10 \Rightarrow y = \frac{1}{3}x - \frac{10}{9}$.

Since both equations have slope $\frac{1}{3}$ and different y-intercepts, they are parallel. Since they never intersect at a single point there is **no solution**. This can be illustrated by solving the system. Set $\frac{1}{3}x - \frac{7}{6} = \frac{1}{3}x - \frac{10}{9} \Rightarrow 18\left(\frac{1}{3}x - \frac{7}{6}\right) = 18\left(\frac{1}{3}x - \frac{10}{9}\right) \Rightarrow 6x - 21 = 6x - 20 \Rightarrow 0 = 1$. Since $0 \neq 1$, it can be seen that there is no solution.

(b) Rewrite each equation into the form $y = mx + b$. $4x - 6y = 1 \Rightarrow y = \frac{2}{3}x - \frac{1}{6}$ and $6x - 9y = 1.5 \Rightarrow y = \frac{2}{3}x - \frac{1}{6}$.

Since both equations represent the same line, $y = \frac{2}{3}x - \frac{1}{6}$, there are an infinite number of solutions of the form $\left(x, \frac{2}{3}x - \frac{1}{6}\right)$.

In attempting to solve these types of systems, a solution such as $0 = 0$ will always be obtained. An answer of this type means an infinite number of solutions is possible.

(c) Substitute $y = 2-x$ in $y = x-2$:

$2-x = x-2 \Rightarrow 4 = 2x \Rightarrow x = 2$. Substitute 2 for x: $y = 2-2 \Rightarrow y = 0$. There is a unique solution (2,0).

(d) Multiply ${}^{-}\frac{3}{2}x + y = \frac{1}{2}$ by $2 \Rightarrow {}^{-}3x + 2y = 1$

$$3x - 2y = {}^{-}1$$

and add the equations: $\begin{array}{r} {}^{-}3x + 2y = \phantom{{}^{-}}1 \\ \hline 0 = 0 \end{array}$. There are infinitely many solutions of the form $\left(x, \frac{3}{2}x + \frac{1}{2}\right)$.

18. The equation of the segment connecting $(5,0)$ and $(6,8)$ is $\boldsymbol{y = 8x - 40}$ (determined by finding m; then substituting x and y from one of the points into the equation $y = mx + b$ to find b).

The equation of the segment connecting $(10,0)$ and $(3,4)$ is $\boldsymbol{y = \frac{-4}{7}x + \frac{40}{7}}$.

The equation of the segment connecting $(0,0)$ and $(8,4)$ is $\boldsymbol{y = \frac{1}{2}x}$.

Equating these three values of y (i.e., $8x - 40 = \frac{^-4}{7}x + \frac{40}{7} = \frac{1}{2}x$) yields $x = \frac{16}{3}$.

Substituting $x = \frac{16}{3}$ into any of the three equations yields $y = \frac{8}{3}$. The coordinates of the common intersection are $\left(\boldsymbol{\frac{16}{3}, \frac{8}{3}}\right)$.

Mathematical Connections 8-5: Review Problems

15. **(a)** $\boldsymbol{\sqrt[3]{x} = x^2 - 3}$.

(b) $\boldsymbol{x^2 + y^2 = 36}$.

(c) $\boldsymbol{\sqrt{0.\overline{9}} = 1}$.

16. $\sqrt{6} \approx \mathbf{2.45}$.

17. **(a)** $x\sqrt{2} - 3y = 0.\overline{4} \Rightarrow x\sqrt{2} = 0.\overline{4} + 3y \Rightarrow x = \frac{0.\overline{4}+3y}{\sqrt{2}}$.

(b) $x^2 - 81 = 0 \Rightarrow (x-9)(x+9) = 0$
$\Rightarrow x - 9 = 0$ or $x + 9 = 0$
$\Rightarrow \boldsymbol{x = 9}$ or $\boldsymbol{x = {}^-9}$.

(c) $3x < {}^-\sqrt{7} \Rightarrow x < \frac{^-\sqrt{7}}{3}$.

18. **(a)** $f(3) = 3\sqrt{7} - \sqrt{7} = \mathbf{2\sqrt{7}}$.

(b) $f(\sqrt{7}) = \sqrt{7} \cdot \sqrt{7} - \sqrt{7} = \mathbf{7 - \sqrt{7}}$.

(c) $f({}^-4) = {}^-4\sqrt{7} - \sqrt{7} = {}^-5\sqrt{7}$.

19. $12 = 3x - \sqrt{2} \Rightarrow 12 + \sqrt{2} = 3x$
$\Rightarrow x = \boldsymbol{\frac{12+\sqrt{2}}{3}}$.

20. **(a)** For $f(x) = \sqrt{x+1}$ to have real-valued range values, $x + 1$ must not be negative. So $x + 1 \geq 0 \Rightarrow \boldsymbol{x \geq {}^-1}$. Write as $\{x \mid x \in \mathbb{R}$ and $x \geq {}^-1\}$.

(b) For $f(x) = \sqrt{^-x}$ to have real-valued range values, $-x$ must not be negative. So $-x \geq 0 \Rightarrow \boldsymbol{x \leq 0}$. Write as $\boldsymbol{\{x \mid x \in R \text{ and } x \leq 0\}}$.

Chapter 8 Review

1. **(a)** **Irrational**. The pattern never repeats in blocks of the same length.

(b) **Irrational**. Any non-zero rational number divided by any irrational number is irrational.

(c) **Rational**. The ratio of two integers.

(d) **Rational**. The pattern repeats in blocks of 0011.

(e) **Irrational**. The pattern does not repeat.

2. **(a)** $\sqrt{484} = 22$

(b) $\sqrt{288} = \sqrt{144 \cdot 2} = \mathbf{12\sqrt{2}}$.

(c) $\sqrt{180} = \sqrt{36 \cdot 5} = \mathbf{6}\sqrt{5}$.

(d) $\sqrt[3]{162} = \sqrt[3]{27 \cdot 6} = \mathbf{3\sqrt[3]{6}}$.

3. **(a)** **No**. ${}^-\sqrt{2} + \sqrt{2} = 0$ is rational.

(b) **No**. $\sqrt{2} - \sqrt{2} = 0$ is rational.

(c) **No**. $\sqrt{2} \cdot \sqrt{2} = 2$ is rational.

(d) **No**. $\frac{\sqrt{2}}{\sqrt{2}} = 1$ is rational.

4. $4 < \sqrt{23} < 5$.

$(4.7)^2 = 22.09$ and $(4.8)^2 = 23.04 \Rightarrow$
$4.7 < \sqrt{23} < 4.8$;
$(4.79)^2 = 22.9441$ and $(4.80)^2 = 23.04 \Rightarrow$
$4.79 < \sqrt{23} < 4.80$;
$(4.795)^2 = 22.992025$ and $(4.796)^2 = 23.001616 \Rightarrow 4.795 < \sqrt{23} < 4.796$;
23 is closer to 23.001616 than to 22.992025, thus $\sqrt{23} \approx \mathbf{4.796}$.

5. $1 < \sqrt[3]{2} < 2$

$1.2^3 = 1.728$ and $1.3^3 = 2.197$.

$1 \cdot 2 < \sqrt[3]{2} < 1.3$.

$1.25^3 = 1.953125$ and $1.26^3 = 2.000376$.

Thus, $1.25 < \sqrt[3]{2} < 1.26$.

2 is closer to 2.000376 that it is to 1.953125, so $\sqrt[3]{2} \approx \mathbf{1.26}$.

6. **(a)** Since $a_1 = 5$ and $a_3 = 10, 10 = 5(r)^2$. This implies that $r^2 = 2$ and $r = \sqrt{2}$ or $r = {}^{-}\sqrt{2}$. So, $a_2 = \mathbf{5\sqrt{2}}$ **or** $\mathbf{{}^{-}5\sqrt{2}}$.

(b) Since $a_1 = 1$ and $a_5 = \frac{1}{4}, \frac{1}{4} = 1(r)^4$. This implies that $r^4 = \frac{1}{4}$ and $r = \frac{1}{\sqrt[4]{4}}$ or $\frac{-1}{\sqrt[4]{4}}$.

Thus, two possible solutions are: $\mathbf{1, \frac{1}{\sqrt[4]{4}}, \frac{1}{\sqrt[4]{16}} = \frac{1}{2}, \frac{1}{2\sqrt[4]{4}}}, \frac{1}{4}$ and $\mathbf{1, \frac{-1}{\sqrt[4]{4}}, \frac{1}{2}, \frac{-1}{2\sqrt[4]{4}}},$

7. $\sqrt{7}({}^{-}1)^{10} = \sqrt{7}$. The ratio $r = {}^{-}1$.

8. The difference is $a_2 - a_1 = \sqrt{2} - 1$. Thus, $\boldsymbol{a_n = 1 + (n - 1)(\sqrt{2} - 1)}$.

9. $\boldsymbol{S = 13P}$.

10. There are 103 times as many girls as boys.

11. $\boldsymbol{f = 3y}$.

12. Let $S = n_1 + n_2 + n_3 + \cdots$. Then $S_{new} = (10n_1 - 10) + (10n_2 - 10) + (10n_3 - 10) + \cdots = 10[(n_1 - 10) + (n_2 - 10) + (n_3 - 10) + \cdots] = \mathbf{10S - 10n}$.

13. Let n be the whole number. Then $12\left(\frac{n}{13}\right) - 20 + 89 = 93 \Rightarrow 12\left(\frac{n}{13}\right) = 24 \Rightarrow \frac{n}{13} = 2 \Rightarrow \boldsymbol{n = 26}$.

14. **(a)** Let n be the number. Then:

(i) $n + 17$

(ii) $2(n + 17) = 2n + 34$

(iii) $(2n + 34) - 4 = 2n + 30$

(iv) $2(2n + 30) = 4n + 60$

(v) $(4n + 60) + 20 = 4n + 80$

(vi) $\frac{4n+80}{4} = n + 20$

(vii) $n + 20 - 20 = \boldsymbol{n}$.

(b) Answers may vary; for example, the next two lines could be to add 3, divide by 4, and subtract 17. Then $\frac{[4(n+18)-7]+3}{4} - 17 = n$.

15. **(a)** $4(7x - 21) = 14(7x - 21) \Rightarrow 28x - 84 = 98x - 294 \Rightarrow 210 = 70x \Rightarrow \boldsymbol{x = 3}$.

(b) $3(\sqrt{x} - 1) = \sqrt{9x} + 5 \Rightarrow 3\sqrt{x} - 3 = 3\sqrt{x} + 5 \Rightarrow 3\sqrt{x} - 3\sqrt{x} = 8 \Rightarrow 0 \neq 8$ so there is no solution.

(c) $4x - 2 = 3x + \sqrt{10} \Rightarrow \boldsymbol{x = 2 + \sqrt{10}}$.

(d) $2(x + \sqrt{3}) = 3(x - \sqrt{3}) \Rightarrow 2x + 2\sqrt{3} = 3x - 3\sqrt{3} \Rightarrow 5\sqrt{3} = x$.

16. Let m be the number of Mikes's cards, j be the number of Jordan's cards, and p be the number of Paige's cards.

It is given that $j = 2p$ and $m = 3j = 3(2p) = 6p$.

Then If $m + j + p = 999 \Rightarrow p + 2p + 6p = 999 \Rightarrow 9p = 999 \Rightarrow p = 111$.

So Paige has **111 cards**

Jordan has **222 cards**

Mike has **666 cards.**

17. Let s be the number of science book overdue days and c be the number of children's book overdue days. It is given that $c = s - 14$.

Then $8(0.20)(s - 14) + 2(0.20)s = 11.60 \Rightarrow 1.6s - 22.4 + 0.4s = 11.60 \Rightarrow 2s = 34 \Rightarrow s = 17$.

Each science book was overdue by **17 days**

Each children's book was overdue by $17 - 14 = \mathbf{3}$ **days**.

18. Let j be the number of papers delivered by Jacobo, d be the number of papers delivered by Dahlia, and r be the number of papers delivered by Rashid.

It is given that $d = r + 100$ and $j = 2d = 2(r + 100)$.

Then if $j + d + r = 500 \Rightarrow r + (r + 100) + 2(r + 100) = 500 \Rightarrow 4r + 300 = 500 \Rightarrow r = 50$.

So Rashid delivered **50 papers**

Dahlia delivered $50 + 100 = \mathbf{150}$ **papers**

Jacobo delivered $2(50 + 100) = \mathbf{300}$ **papers**.

19. **(a)** **A function**. Each component of the domain corresponds to a unique component of the range.

(b) **Not a function**. a and b both correspond to two components of the range.

(c) **A function**. a and b correspond to unique components of the range.

(d) **A function**. Each component of the domain corresponds to a value of 2 in the range.

20. **(a)** $\{0, 1, 2, 3\} + 3 \Rightarrow$ Range is $\{x + 3 | x \in \{0, 1, 2, 3\}\} = \mathbf{\{3, 4, 5, 6\}}$.

(b) Range is $\{3x - 1 | x \in \mathbb{R}\} = \mathbb{R}$

(c) The domain is $\{x^2 | x \in \mathbb{R}\}$ so the range is $\{x | x \geq 0\}$.

(d) $\{0, 1, 2\}^2 + 3\{0, 1, 2\} + 5 \Rightarrow$ Range is $\{x^2 + 3x + 5 \mid x \in \{0, 1, 2\}\} = \mathbf{\{5, 9, 15\}}$.

21. **(a)** **Not a function**. A student may have more than one major.

(b) **A function**. The range is the subset of the natural numbers that includes the number of pages in each book in the library.

(c) **A function**. The range is $\{x | x \geq 6$ and x is even$\}$.

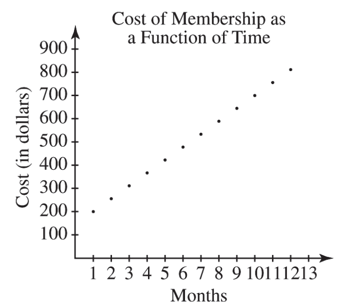
(d) **A function**. The range is $\{0, 1\}$.

(e) **A function**. The range is the set of all natural numbers N.

22. **(a)** $\mathbf{C(x) = \$[200 + 55(x - 1)]}$ where x is the number of months.

(b) Plot:

x	$C(x)$
1	200
2	255
3	310
4	365
5	420
6	475
7	530
8	585
9	640
10	695
11	750
12	805



(c) The cost will exceed \$600 beginning with the **ninth month**.

(d) If $C(x) = 200 + 55(x - 1) = 6000$ then $55x - 55 = 5800 \Rightarrow x = 106.5$.

Or the cost will exceed \$6000 in the **107th month**.

23. **(a)** $4x - 5 = 15 \Rightarrow 4x = 20 \Rightarrow x = 5$.

(b) $x^2 - 1 = 2 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$.

(c) $\sqrt{^-x} = 3 \Rightarrow {^-x} = 9 \Rightarrow x = {^-9}$.

(d) $\sqrt{x} = {^-3}$ has no solutions. All square roots have positive answers.

(e) $\dfrac{x+1}{x+2} = {^-1} \Rightarrow x + 1 = ({^-1})(x + 2)$

$x + 1 = {^-x} - 2 \Rightarrow 2x = {^-3} \Rightarrow x = \dfrac{^-3}{2}$

(f) $\dfrac{x+1}{x+2}=1 \Rightarrow x+1=x+2 \Rightarrow 0x=1 \Rightarrow 0 \neq 1$ no solution.

24. **(a)** **A function**. Each value on the x-axis corresponds to a unique value on the y-axis.

(b) **Not a function**. 4 and 5 on the x-axis each correspond to two values on the y-axis.

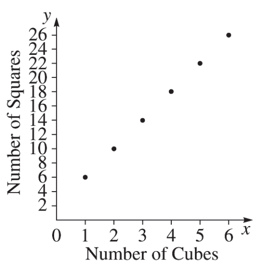
(c) **Not a function**. The x-value 5 corresponds to two values on the y-axis.

25. Assume Jilly starts with unpainted blocks each time:

(a)

Number of cubes	Number of squares to paint
1	6
2	10
3	14
4	18
5	22
6	26

(b)



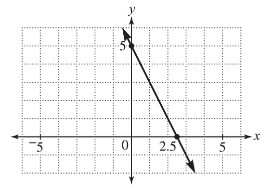
(c) This is a function which is an arithmetic sequence with $a_1 = 6$, $d = 4$, and $a_6 = 26 \Rightarrow a_n = 6 + (n-1)4 \Rightarrow 4n + 2$. The function is $y = \mathbf{4x + 2}$ for $x = 1, 2, ..., 6$.

(d) **No**. The graph does represent a function, but not a straight line because values of x cannot assume anything but natural numbers.

26. Use a spreadsheet approach, where values of x are in column A and corresponding values of the functions are in column B.

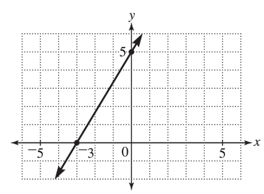
(a) $y = {}^{-}2x + 5$:

A	B
$^{-}1$	7
0	5
1	3
2	1
⋮	⋮



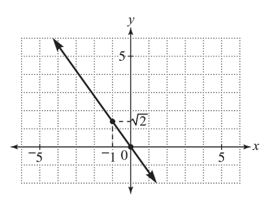
(b) $\dfrac{^{-}x}{3}+\dfrac{y}{5}=1 \Rightarrow {}^{-}5x+3y=15 \Rightarrow y=\dfrac{5}{3}x+5$ (values of x are multiples of 3):

A	B
$^{-}6$	$^{-}5$
$^{-}3$	0
0	5
3	10
6	15
⋮	⋮



(c) $y = {}^{-}\sqrt{2}x$ (values of x in column A are selected to yield whole-number values of y in column B):

A	B
$^{-}\sqrt{2} \approx {}^{-}1.4$	2
0	0
$\sqrt{2} \approx 1.4$	$^{-}2$
$\sqrt{8} \approx 2.8$	$^{-}4$
⋮	⋮



27. **(a)** $4x+3y-12=0$ is the graph of $\overleftrightarrow{CD}$.

(b) $x-y-1=0 \Rightarrow y=x-1$

$4x+3y-12=0 \Rightarrow y=4-\frac{4}{3}x$

$x-1=4-\frac{4}{3}x \Rightarrow 3x-3=12-4x$

$7x=15 \Rightarrow x=\frac{15}{7}$

$y=\frac{15}{7}-1 \Rightarrow y=\frac{8}{7}$

The coordinates of point E are $\left(0,\frac{8}{7}\right)$.

(c) $x-y-1=0 \Rightarrow 0-y-1=0 \Rightarrow y={}^{-}1$, the coordinates of point B are $(0,{}^{-}1)$. Point F has the same y-coordinate. Using the equation for $\overleftrightarrow{CD}$ substitute ${}^{-}1$ for y and solve for x.

$4x+3({}^{-}1)-12=0 \Rightarrow 4x-15=0 \Rightarrow x=\frac{15}{4}$. The distance of $\overleftrightarrow{BF}$ is $\frac{15}{4}-0=\frac{15}{4}$.

28. **(a)** ${}^{-}3x+12=23 \Rightarrow {}^{-}3x=11 \Rightarrow x={}^{-}\frac{11}{3}$

(b)

$\sqrt{2}x-5=\sqrt{8}x+1$

$\sqrt{2}x-5=2\sqrt{2}x+1$

$\sqrt{2}x-2\sqrt{2}x=6$

${}^{-}\sqrt{2}x=6 \Rightarrow x={}^{-}\frac{6}{\sqrt{2}}={}^{-}3\sqrt{2}$

(c) $x^3={}^{-}2 \Rightarrow x=\sqrt[3]{{}^{-}2}$

(d) $4x^2-33=3 \Rightarrow 4x^2=36 \Rightarrow x^2=9 \Rightarrow \sqrt{x^2}=\sqrt{9} \Rightarrow |x|=3 \Rightarrow \boldsymbol{x=\pm 3}$.

(e) $(x-1)^3=2 \Rightarrow x-1=\sqrt[3]{2} \Rightarrow x=1+\sqrt[3]{2}$.

(f) $(2x+1)^4=5 \Rightarrow 2x+1=\pm\sqrt[4]{5} \Rightarrow x=\frac{{}^{-}1\pm\sqrt[4]{5}}{2}$.

(g) Set each factor equal to zero and solve.

So $x-1=0 \Rightarrow x=1$
or $2x+1=0 \Rightarrow x=\frac{1}{2}$.

29. Yes. The new ratio is the cube root of the original ratio. If x and y are successive terms in the original sequence with ratio r, so that $r=\frac{y}{x}$, then $\frac{\sqrt[3]{y}}{\sqrt[3]{x}}\sqrt[3]{\frac{y}{x}}=\sqrt[3]{r}$ which is constant.

CHAPTER 9

PROBABILITY

Assessment 9-1A: Determining Probabilities

1. **(a)** **No.** There are fewer face cards than there are not face cards.

   (b) **Yes.** Each suit has the same number of cards.

   (c) **Yes.** There are equal numbers of black and red cards.

   (d) **No.** There are 4 each kings, queens, jacks, and aces; 20 even-numbered cards; and 16 odd-numbered cards in a standard deck.

2. The probability is not likely because there is only one whole number 0 in the sample space that is not a natural number. $P(0) = \frac{1}{1000} = 0.001$ which is not likely.

3. **(a)** $P(vowel) = \frac{n(vowel)}{n(S)} = \mathbf{\frac{5}{26}}$.

   (b) $P(consonant) = 1 - P(vowel)$
   $= 1 - \frac{5}{26} = \mathbf{\frac{21}{26}}$.

   (c) P(*vowel or letter from* "*probability*") $=$ $P(vowel) + P(letter) - P(vowel \cap letter) =$ $\frac{5}{26} + \frac{9}{26} - \frac{3}{26} = \mathbf{\frac{11}{26}}$. There are 11 letters in the word "probability," but *b* and *i* are repeated. There are only nine unique choice of letters; the probabilities of a vowel and a letter from "probability" are not mutually exclusive.

4. **(a)** $P(1, 5 \text{ or } 7) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \mathbf{\frac{3}{8}}$. Factors of 35 are 1, 5, or 7 and their probabilities are mutually exclusive.

   (b) $P(3 \text{ or } 6) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \mathbf{\frac{1}{4}}$. Multiples of 3 are 3 or 6 and their probabilities are mutually exclusive.

   (c) $P(2, 4, 6 \text{ or } 8) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \mathbf{\frac{1}{2}}$.

   (d) $P(6 \text{ or } 2) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \mathbf{\frac{1}{4}}$.

   (e) $P(11) = \mathbf{0}$. The probability of an impossible event is 0.

   (f) $P(4, 6 \text{ or } 8) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \mathbf{\frac{3}{8}}$. 4, 6, and 8 are the only composite numbers on the spinner.

   (g) $P(1) = \mathbf{\frac{1}{8}}$. 1 is the only number on the spinner which is neither prime nor composite.

5. **(a)** $P(red\ card) = \frac{n(red)}{n(S)} = \frac{26}{52} = \mathbf{\frac{1}{2}}$.

   (b) $P(face\ card) = \frac{n(face\ card)}{n(S)} = \frac{12}{52} = \mathbf{\frac{3}{13}}$.

   (c) $P(red\ card\ or\ 10) = \frac{n(red)}{n(S)} + \frac{n(10)}{n(S)} - \frac{n(red\ and\ 10)}{n(S)} = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \mathbf{\frac{7}{13}}$. Red and 10 are not mutually exclusive.

   (d) $P(queen) = \frac{n(queen)}{n(S)} = \frac{4}{52} = \mathbf{\frac{1}{13}}$.

   (e) $P(not\ a\ queen) = 1 - P(queen) =$ $1 - \frac{1}{13} = \mathbf{\frac{12}{13}}$.

   (f) $P(face\ card\ or\ club) = \frac{n(face\ card)}{n(S)} + \frac{n(club)}{n(S)} - \frac{n(face\ card\ and\ club)}{n(S)} = \frac{12}{52} + \frac{13}{52} - \frac{3}{52} = \frac{22}{52} =$ $\mathbf{\frac{11}{26}}$. Face card and club are not mutually exclusive.

   (g) $P(face\ card\ and\ club) = \frac{n(face\ card\ and\ club)}{n(S)} =$ $\mathbf{\frac{3}{52}}$.

   (h) $P(not\ a\ face\ card\ and\ not\ a\ club) =$ $1 - P(face\ card\ or\ club) = 1 - \frac{11}{26} = \mathbf{\frac{15}{26}}$. Use a Venn diagram to verify this use of the complementary property.

6. Assume the socks are randomly mixed in the drawer.

   (a) $P(brown) = \frac{n(brown)}{n(S)} = \frac{4}{12} = \mathbf{\frac{1}{3}}$. There are four brown socks out of the twelve in the drawer.

   (b) $P(black\ or\ green) = \frac{n(black)}{n(S)} + \frac{n(green)}{n(S)} =$ $\frac{6}{12} + \frac{2}{12} = \frac{8}{12} = \mathbf{\frac{2}{3}}$. The two events are mutually exclusive.

(c) $P(red) = \mathbf{0}$. There are no red socks in the drawer, so this is an impossible event.

(d) $P(not\ black) = 1 - P(black) = 1 - \frac{n(black)}{n(S)} = 1 - \frac{6}{12} = \frac{6}{12} = \mathbf{\frac{1}{2}}$.

or

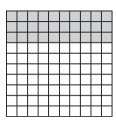
$P(not\ black) = \frac{n(broun)}{n(S)} + \frac{n(green)}{n(S)} = \frac{4}{12} + \frac{2}{12} = \frac{6}{12} = \mathbf{\frac{1}{2}}$.

(e) $P(pair\ of\ same\ color) = \mathbf{1}$. Since there are only three colors, if four socks are pulled out at least two of them must be of the same color.

7. **(a)** $P(English\ files) = \frac{n(English\ files)}{n(S)} = \mathbf{\frac{1}{6}}$.

(b) $P(neither\ math\ nor\ chemistry) = \frac{n(neither\ math\ nor\ chemistry)}{n(S)} = \frac{4}{6} = \mathbf{\frac{2}{3}}$.

8. **(a)** Answers will vary. For example, choose one cell of the grid at random.



(b) $P(no\ rain)$
$= 1 - P(rain) = 1 - 0.30 = 0.70$,
or **70%**.

9. There are $17+13+12=42$ dolls. The probability of drawing a doll with no red hair is

$P(\text{no red}) = 1 - P(\text{red}) = 1 - \frac{12}{42} = \frac{30}{42} = \frac{5}{7}$.

10. There are 10 plays that are considered tragedies.

(a) $P(\text{"King"}) = \frac{1}{10}$. *King Lear* is the only title that contains the word "King".

(b) $P(\text{Starts with "T"}) = \frac{2}{10} = \frac{1}{5}$. *Timon of Athens* and *Titus Andronicus* are the two titles that start with the letter "T".

(c) $P(\text{Contains "Q"}) = \frac{0}{10} = 0$ **.** No titles contain the letter "Q".

(d) $P(\text{two names}) = \frac{2}{10} = \frac{1}{5}$. *Antony and Cleopatra* and *Romeo and Juliet* are the two titles that contain the names of two people.

11. $P(o) = \frac{1}{5}$. There is 1 letter "o" in "*e-i-e-i-o*".

12. There are $350 + 320 + 310 + 400 = 1380$ students in all. $P(freshman) = \frac{n(freshman)}{n(S)} = \frac{350}{1380} = \mathbf{\frac{35}{138}}$.

13. Because A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = 0.3 + 0.4 = \mathbf{0.7}$.

14. $n(S) = n(men) + N(women) = 35 + 45 = 80$.

(a) $P(female) = \frac{n(female)}{n(S)} = \frac{45}{80} = \mathbf{\frac{9}{16}}$.

(b) $P(computer\ science\ major) = \frac{n(computer\ science)}{n(S)} = \frac{10}{80} = \mathbf{\frac{1}{8}}$.

(c) $P(not\ math\ major) = 1 - P(math\ major) = 1 - \frac{n(math)}{n(S)} = 1 - \frac{20}{80} = \frac{60}{80} = \mathbf{\frac{3}{4}}$. It is easier to use the complementary property in this case than to calculate the sum of the non-math majors.

(d) $P(computer\ science\ or\ math\ major) = \frac{n(computer\ science)}{n(s)} + \frac{n(math)}{n(S)} = \frac{10}{80} + \frac{20}{80} = \frac{30}{80} = \mathbf{\frac{3}{8}}$. Since there are no double majors, the events are mutually exclusive.

15. **(a)** If r red balls are added then $P(red\ ball) = \frac{2+r}{10+r} = \frac{3}{4} \Rightarrow 4(2 + r) = 3(10 + r)$.
$r = $ **22 red balls**.

(b) There are originally $\frac{5 \text{ white}}{10 \text{ total}}$ balls. b black balls are added so that $P(white) = \frac{5}{10+b} = \frac{1}{4} \Rightarrow 20 = 10 + b$. $b = $ **10 black balls** added.

16. **No**.

(***i***) $P(white\ ball\ from\ \#1) = \frac{n(white\ balls)}{n(S)} = \mathbf{\frac{3}{4}}$.

(***ii***) $P(white\ ball\ from\ \#2) = \frac{n(white\ balls)}{n(S)} = \mathbf{\frac{5}{8}}$.

Because $\frac{3}{4} = \frac{6}{8} > \frac{5}{8}$, the probability of drawing a white ball from #1 is greater.

17. A fair coin will always exhibit equal probabilities of a head or a tail when flipped, regardless of past history. Thus the probability of a head on toss 16 is $\mathbf{\frac{1}{2}}$.

18. The word "numbers" has seven letters, two of which are vowels and five of which are consonants.

(a) $P(vowel) = \frac{n(vowels)}{n(S)} = \frac{2}{7}$.

(b) $P(consonant) = \frac{n(consonants)}{n(S)} = \frac{5}{7}$.

19. $P(\text{correct digit}) = \dfrac{1}{10}$. There are ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and only one can be the correct digit.

20. $P(\text{first letter is a vowel}) = \dfrac{1}{9}$. There are nine letters and one vowel.

21. $P(4) = \dfrac{24}{100}$, the theoretical probability is $P(4) = \dfrac{1}{6}$

The difference is $\dfrac{24}{100} - \dfrac{1}{6} = \dfrac{36}{150} - \dfrac{25}{150} = \dfrac{11}{150}$.

22. $P(4) = \dfrac{1}{6} \approx \dfrac{167}{1000}$. So about 167 times.

Assessment 9-1B

1. **(a)** $P(point\ up) = \frac{n(up)}{n(S)} = \frac{56}{56+24} = \frac{56}{80} = \frac{7}{10}$.

(b) $P(point\ down) = \frac{n(down)}{n(S)} = \frac{24}{80} = \frac{3}{10}$.

Another method of solving this problem is to recognize that *point up* and *point down* are mutually exclusive events; thus $P(point\ down) = 1 - P(point\ up) = 1 - \frac{7}{10} = \frac{3}{10}$.

(c) **Probably not**. When running the experiment again with only a relatively small number of tries the outcome will probably not be identical. Experimental probability is based only on the number of times the experiment is repeated, not what will happen in the long run.

(d) **Yes**. The thumbtack would be unchanged, so the results would be determined by the same dynamics. Experimental probability, though, does not guarantee exact results.

2. Let S be the sample space and E be the elements of the listed event:

(a) $S = \mathbf{\{0,1,2,3,4,5,6,7,8,9\}}$.

(b) $E = \mathbf{\{0,1,2,3,4\}}$.

(c) $E = \mathbf{\{1,3,5,7,9\}}$.

(d) $E = \mathbf{\{0,1,3,4,5,6,7,8,9\}}$.

(e) (***i***) $P(digit < 5) = \frac{n(<5)}{n(S)} = \frac{5}{10} = \mathbf{\frac{1}{2}}$.

(***ii***) $P(digit\ odd) = \frac{n(odd)}{n(S)} = \frac{5}{10} = \mathbf{\frac{1}{2}}$.

(***iii***) $P(digit\ not\ 2) = \frac{n(not\ 2)}{n(S)} = \mathbf{\frac{9}{10}}$.

3. There are 435 members of the House of Representatives and 100 senators. The probability that the congressperson drawn is a member of the House of Representatives is $\dfrac{435}{535}$ which is greater than the probability this congressperson is a senator, $\dfrac{100}{535}$.

4. $P(\text{corrected vision}) = \dfrac{3}{4} = \dfrac{225}{300}$. 225 million people have corrected vision.

5. $n(S) = 28$:

(a) $P(diet\ soda) = \frac{n(diet\ sodas)}{n(S)} = \frac{16}{28} = \mathbf{\frac{4}{7}}$.

(b) $P(regular\ soda) = \frac{n(regular\ sodas)}{n(S)} = \frac{8}{28} = \mathbf{\frac{2}{7}}$.

(c) $P(water) = \frac{n(waters)}{n(S)} = \frac{4}{28} = \mathbf{\frac{1}{7}}$.

Part (*iii*) could also be solved by realizing that the probability of drawing a bottle $=$ $1 \Rightarrow P(water) = 1 - \left(\frac{4}{7} + \frac{2}{7}\right) = \mathbf{\frac{1}{7}}$.

6. If $P(\leq 10) = \frac{4}{10}$, then $\frac{n(\leq 10)}{n(S)} = \frac{4}{10}$. Thus $4 \cdot n(S) = 10 \cdot n(\leq 10) \Rightarrow n(S) = \frac{10}{4} \cdot n(\leq 10) \Rightarrow n(S) = \frac{5}{2} \cdot 10.$ $n(S) =$ **25 cards**.

7. **(a)** **No**. $P(I\ win\ or\ you\ lose) = P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1.$ Each player's probability of winning is not equal.

(b) **Yes**. $P(heads\ I\ win) = P(H) = \frac{1}{2}$. $P(otherwise) = P(T) = \frac{1}{2}$.

(c) **Yes**. $P(1; I\ win) = \frac{1}{6}$. $P(6; you\ win) = \frac{1}{6}$.

(d) **Yes**. $P(even; I\ win) = \frac{n(even)}{n(S)} = \frac{3}{6}$. $P(odd; you\ win) = \frac{n(odd)}{n(S)} = \frac{3}{6}$.

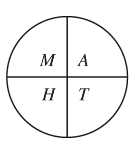
(e) **No**. $P(\geq 3; I\ win) = \frac{n(\geq 3)}{n(S)} = \frac{4}{6}$. $P(< 3; you\ win) = \frac{n(<3)}{n(S)} = \frac{2}{6}$.

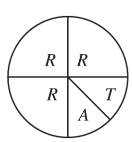
(f) **Yes**. $P(1\ on\ each; I\ win) = \frac{1}{36}$; $P(6\ on\ each; you\ win) = \frac{1}{36}$.

(g) **No**. $P(3\ ; I\ win) = \frac{n(3)}{n(S)} = \frac{2}{36}$. $P(2; you\ win) = \frac{n(2)}{n(S)} = \frac{1}{36}$.

8. In these "spinner" problems, the probabilities are represented by angles through which the spinner rotates.

(a) Equally likely probability means equal angles through which the spinner rotates. See the first picture below for an example.





(b) Answers may vary. $P(R) = \frac{3}{4}$ means R must be represented by $\frac{3}{4}$ of the $360°$ through which the spinner rotates, or $270°$. $P(A) = P(T) = \frac{1}{8}$ means A and T must each be represented by $\frac{1}{8}$ of $360°$, or $45°$. See the second picture in (a) for an example.

9. Assume drawing without replacement; i.e., there will be 50 cards remaining in the deck after the first two have been dealt.

(a) There are five cards between 5 and jack in each of the four suits $\Rightarrow P(between\ 5\ and\ jack) = \frac{5 \cdot 4}{50} = \frac{20}{50} = \mathbf{\frac{2}{5}}$.

(b) There are ten cards between 2 and king in each of the four suits. $P(between\ 2\ and\ king) = \frac{10 \cdot 4}{50} = \frac{40}{50} = \mathbf{\frac{4}{5}}$.

(c) $P(between\ 5\ and\ 6) = \mathbf{0}$. This is an impossible event.

10. **(a)** $P(A \cup C) =$ the probability of students taking Algebra **or** Chemistry, or both.

(b) $P(A \cap C) =$ the probability of students taking both Algebra **and** Chemistry.

(c) $1 - P(C) =$ the complement of the probability of a student taking Chemistry, or the probability of a student **not** taking chemistry.

11. Let n be the number of balls in the box. There are $0.25n$ black balls originally, and then another $0.25n$ black balls are added. Thus $P\ (drawing\ black\ ball) = \frac{0.25n+0.25n}{n+0.25n} = \frac{n(0.25+0.25)}{n(1+0.25)} = \frac{0.5}{1.25} = \mathbf{0.4}$.

12. $P(not\ study) = \frac{n(not\ studied)}{n(S)} = \mathbf{\frac{2}{3}}$.

13. **(a)** $P(B) = \frac{n(B)}{n(S)} = \mathbf{\frac{1}{15}}$.

(b) $P(not\ B) = 1 - P(B) = 1 - \frac{1}{15} = \mathbf{\frac{14}{15}}$.

14. **(a)** **No**. Suppose x white, x black, and x red balls are added to the box. Then $P(black\ ball) = \frac{3+x}{(5+x)+(3+x)+(2+x)} = \frac{3+x}{10+3x}$.

If $P(black\ ball) = \frac{1}{3}$ then $\frac{3+x}{10+3x} = \frac{1}{3} \Rightarrow 3(3 + x) = 10 + 3x \Rightarrow 9 = 10.$ There is no solution to the problem.

(b) **Yes**. If x white, x black, and x red balls are added to the box then $P(black\ ball) = \frac{3+x}{10+3x}$.

If P (*black ball*) $= 0.32$ then $\frac{3+x}{10+3x} = 0.32 \Rightarrow 3 + x = 0.32(10 + 3x) \Rightarrow 3 + x = 3.2 + 0.96x \Rightarrow 0.04x = 0.2.$
$x = 5;$ i.e., add five balls of each color.

15. **No**. If the spinner has seven sections, then $P(odd) = \frac{4}{7}$ and $P(even) = \frac{3}{7}$. These are not equal probabilities.

16. **(a)** Since there are currently fewer women than men on the U.S. Supreme Court this event is **likely**.

(b) **Unlikely**, there are fewer African–Americans than non-African–Americans in the current senate.

(c) **Likely**, there are more people with Asian roots than people with non-Asian roots.

17. $P(H) = \frac{1}{2}, P(\text{even number}) = \frac{3}{6} = \frac{1}{2}$. The probability of flipping a coin and getting heads and the probability of rolling a die and getting an even number are the same so it does not matter what you pick.

18. $P(\text{vowel}) = \frac{12}{50} = \frac{6}{25}, P(\text{consonant}) = \frac{38}{50} = \frac{19}{25}$ so I am more likely to choose a state whose name starts with a consonant.

19. $P(\text{vowel}) = \frac{5}{7}, P(\text{consonant}) = \frac{2}{7}$ so the probability of choosing a continent that does not start with a vowel is $\frac{2}{7}$.

20. The probability of seeing a car built before 1940 is unlikely. There would be more cars built after 1940 on the road.

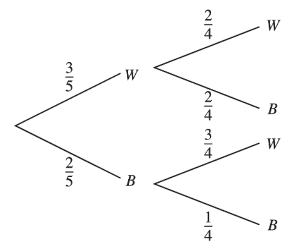
21. $P(6) = \frac{16}{100}$, the theoretical probability is $P(6) = \frac{1}{6}$

The difference is $\frac{1}{6} - \frac{16}{100} = \frac{25}{150} - \frac{24}{150} = \frac{1}{150}$.

22. $P(6) = \frac{1}{6} \approx \frac{167}{1000}$. So, about 167 times.

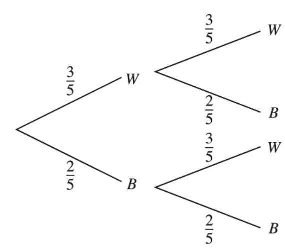
Assessment 9-2A: Multistage Experiments and Modeling Games

1. **(a)** **(*i*)** A tree diagram showing paths for each possible outcome is shown below. Paths are defined by finding the probability of each event in draw one followed by the probability of each event in draw two; for example, the probability of drawing a white ball first is $\frac{3}{5}$, and the probability of drawing another is then $\frac{2}{4}$, since there will be two white balls of a total of four:



(*ii*) The possible outcomes to obtain different colors are white and black or black and white. $P(\textit{different colors})$
$= \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{5} \cdot \frac{3}{4} = \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \mathbf{\frac{3}{5}}$.

(b) **(*i*)** A tree diagram is shown below. Probabilities in each path differ from (a) above in that balls are replaced; thus the probability of drawing a white or black ball is the same in each path:



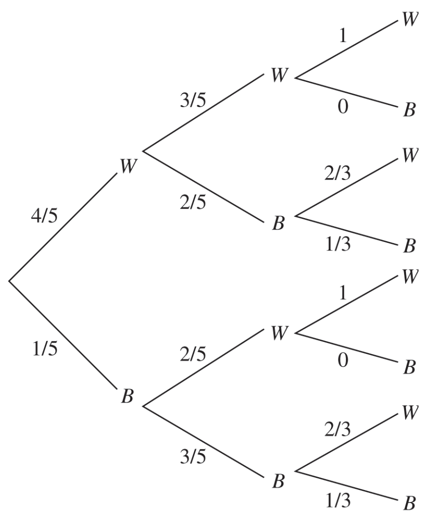
(*ii*) $P(\textit{different colors}) = \frac{3}{5} \cdot \frac{2}{5} + \frac{2}{5} \cdot \frac{3}{5} = \mathbf{\frac{12}{25}}$.

2. **(a)** This is a multistage event with replacement. $P(DAN) = P(D) \cdot P(A) \cdot P(N) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \mathbf{\frac{1}{216}}$.

(b) Without replacement $P(DAN) = P(D) \cdot P(A) \cdot P(N) = \frac{1}{6} \cdot \frac{1}{5} \cdot \frac{1}{4} = \mathbf{\frac{1}{120}}$.

3. This is a multistage event without replacement. $P(\textit{three women}) = \frac{4}{10} \cdot \frac{3}{9} \cdot \frac{2}{8} = \frac{24}{720} = \mathbf{\frac{1}{30}}$.

4. **(a)** The tree diagram below illustrates the event paths leading to a white ball:



$P(\textit{white})$
$= \frac{4}{5} \cdot \frac{3}{5} \cdot 1 + \frac{4}{5} \cdot \frac{2}{5} \cdot \frac{2}{3} + \frac{1}{5} \cdot \frac{2}{5} \cdot 1 + \frac{1}{5} \cdot \frac{3}{5} \cdot \frac{2}{3} = \frac{12}{25} + \frac{16}{75} + \frac{2}{25} + \frac{6}{75} = \frac{36+16+6+6}{75} = \mathbf{\frac{64}{75}}$.

(b) $P(\textit{black}) = 1 - P(\textit{white}) = 1 - \frac{64}{75} = \mathbf{\frac{11}{75}}$.

5. $P(\textit{all boys}) = P\ (\textit{first child boy}) \cdot P(\textit{second child boy}) \cdot P(\textit{third child boy}) \cdot P(\textit{fourth child boy}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{\frac{1}{16}}$.

6. Draw a tree diagram with two stages of branches:

(*i*) One stage with each of the numbers 4, 6, 7, 8, or 9 as branches, each with probability $\frac{1}{5}$.

(*ii*) A second stage: for each branch of (*i*) extend four branches with the remaining numbers as choices, each with probability $\frac{1}{4}$.

(*iii*) There are $5 \cdot 4 = 20$ branches, each with probability $\frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$.

(a) Add the two numbers in each branch of the tree diagram; there are eight even sums. $P(\textit{even sum}; \textit{win}) = \frac{8}{20} = \mathbf{\frac{2}{5}}$.

(b) **Yes**. Only two of the 20 products are odd ($7 \cdot 9$ and $9 \cdot 7$). $P(\textit{even product}; \textit{win}) = \frac{18}{20} = \mathbf{\frac{9}{10}}$.

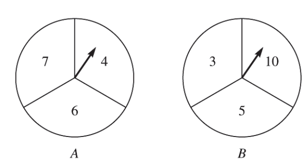
7. Arthur wins if the two marbles match. The probability of both marbles matching is $P(BB) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ or $P(WW) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ so the probability of Arthur winning is $\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.
Gwen wins if the marbles don't match. The probability of both marbles not matching is $P(BW) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ or $P(WB) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ so the probability of Gwen winning is $\frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$.
This game is fair because the probabilities of winning are equal.

8. **(a)** Spinner A wins with the following combinations:

Outcome on spinner A	Outcome on spinner B
4	3
6	3 or 5
8	3 or 5

Assuming each spinner is divided evenly into thirds, $P(A > B; \textit{win}) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}$; $P(B > A; \textit{win}) = 1 - \frac{5}{9} = \mathbf{\frac{4}{9}}$. Therefore, choose **spinner A**.

(b) Answers may vary; one example is shown below:



$P(A > B) = \frac{5}{9}; P(B > A) = \frac{4}{9}$.

9. Each question has a probability of $\frac{1}{2}$ of being right, and the result of each question has no effect on subsequent questions. $P(100\%) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{\frac{1}{32}}$.

10. **(a)** $P(Paxson\ loses) = P(Rattlesnake\ wins)$. $P(4\ Paxson\ losses) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \mathbf{\frac{16}{81}}$.

(b) There are six ways in which each school wins two games: $S = \{(PPRR), (PRPR), (PRRP), (RPPR), (RPRP), (RRPP)\}$; each occurs with probability $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{4}{81}$.

$P(draw) = 6 \cdot \frac{4}{81} = \frac{24}{81} = \mathbf{\frac{8}{27}}$.

11. The total area of the dart board $= 5x$ by $5x = 25x^2$.

(a) The area of $A = x$ by $x = x^2$. $P(section\ A) = \frac{x^2}{25x^2} = \mathbf{\frac{1}{25}}$.

(b) The area of $B = (3x)^2 - x^2 = 8x^2$.

$P(section\ B) = \frac{8x^2}{25x^2} = \mathbf{\frac{8}{25}}$.

(c) The area of $C = (5x)^2 - (3x)^2 = 16x^2$.

$P(section\ C) = \frac{16x^2}{25x^2} = \mathbf{\frac{16}{25}}$.

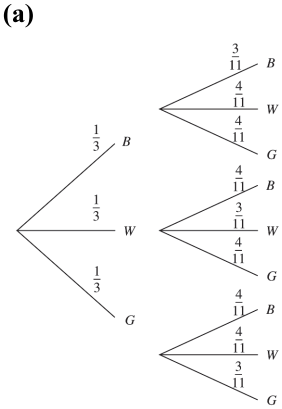
or

$P(section\ C) = 1 - [P(section\ A + P(section\ B)] = 1 - \left(\frac{1}{25} + \frac{8}{25}\right) = 1 - \frac{9}{25} = \mathbf{\frac{16}{25}}$.

12. Assuming an analog and not a digital display, the second hand will cover the distance between 3 and 4 in 5 seconds. $P(between\ 3\ and\ 4) = \frac{5\text{ sec}}{60\text{ sec}} = \mathbf{\frac{1}{12}}$.

13. Of the ten secretaries, two are male. $P(secretary\ given\ male) = \frac{n(male\ and\ secretary)}{n(male)} = \frac{2}{28} = \mathbf{\frac{1}{14}}$.

14. **(a)**



(b) The probability of two the same color is composed of three mutually exclusive events: two blues, two whites, or two grays. $P(same\ color) = P(two\ blues) + P(two\ whites) + P(two\ grays) = \frac{1}{3} \cdot \frac{3}{11} + \frac{1}{3} \cdot \frac{3}{11} + \frac{1}{3} \cdot \frac{3}{11} = \frac{9}{33} = \mathbf{\frac{3}{11}}$.

(c) $P(two\ gray) = \frac{1}{3} \cdot \frac{3}{11} = \frac{3}{33} = \mathbf{\frac{1}{11}}$.

(d) $P(two\ same\ color) = \mathbf{1}$. There are only three colors among the four socks.

15. **(a)** "At least as many heads as tails" means:

0 tails and 4 heads, which can occur in 1 way;
1 tail and 3 heads, which can occur in 4 ways;
2 tails and 2 heads, which can occur in 6 ways.

The total number of possible outcomes is $2^4 = 16$.

$P(as\ many\ heads\ as\ tails) = \frac{1+4+6}{16} = \mathbf{\frac{11}{16}}$.

(b) If the quarter is fair (i.e., heads and tails occur the same number of times) then this probability is the same as in (a), $\mathbf{\frac{11}{16}}$.

16. The probability that any smoke detector taken at random will work is $\frac{450}{500} = 0.9$. The probability in any given sample size that at least 1 will work is the complement of the probability that none will work.

Sample size 1: $P(work) = 0.9$

Sample size 2: $P(at\ least\ one\ work) =$

$1 - P(0\ work) = 1 - (0.1)^2 = 0.99$.

Sample size 3: $P(at\ least\ one\ work) =$

$1 - P(0\ work) = 1 - (0.1)^3 = 0.999$.

Sample size 4: $P(at\ least\ one\ work) =$

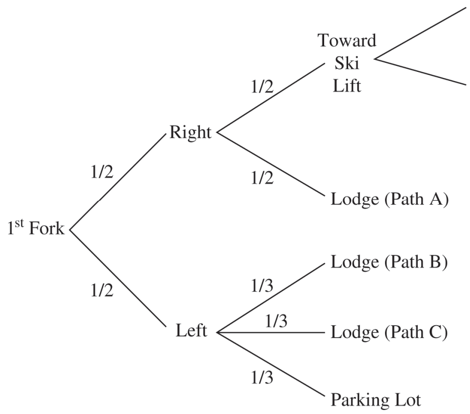
$1 - P(0\ work) = 1 - (0.1)^4 = 0.9999$.

Sample size 5: $P(at\ least\ one\ work) =$

$1 - P(0\ work) = 1 - (0.1)^5 = 0.99999$.

At least four should be installed to assure a probability of at least 99.9% that one will work.

17.



There are three paths to the lodge. These paths represent mutually exclusive events since skiers cannot be on more than one path at a time. $P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{4} + \frac{1}{6} + \frac{1}{6} = \mathbf{\frac{7}{12}} \approx \mathbf{0.58}$.

18. Carolyn can win only if she wins the first two games or the last two games:

(*i*) Billie-Bobby-Billie – $P(\textit{beats Billie and beats Bobby}) = 0.5 \cdot 0.8 = 0.4$; $P(\textit{loses to Billie; beats Bobby and Billie}) = 0.5 \cdot 0.8 \cdot 0.5 = 0.2$. $P(\textit{wins prize}) = 0.4 + 0.2 = 0.6$.

(*ii*) Bobby-Billie-Bobby – $P(\textit{beats Bobby and beats Billie}) = 0.8 \cdot 0.5 = 0.4$; $P(\textit{loses to Bobby; beats Billie and Bobby}) = 0.2 \cdot 0.5 \cdot 0.8 = 0.08$. $P(\textit{wins prize}) = 0.4 + 0.08 = 0.48$.

She should play **Billie-Bobby-Billie**.

19. Answers may vary**.** There are $3+4+3=10$ marbles in the bag. Three out of ten are blue so the probability of drawing a blue marble is $\frac{3}{10} = \frac{30}{100} = 30\%$. To change that probability to $75\% = \frac{75}{100} = \frac{3}{4}$ we could add 18 blue marbles. There would now be $21+4+3=28$ marbles in the bag. Twenty-one out of twenty eight would be blue, changing the probability of drawing a blue marble to $P(\text{blue}) = \frac{21}{28} = \frac{3}{4} = 75\%$.

20. A sum of two can only be obtained by rolling two ones. If one die is a one then the probability of rolling a one on the second die is $P(1) = \frac{1}{6}$.

21. A sum greater than 9 is a sum of 10 or 11 or 12. Given that the first die shows a six, the probability of rolling a sum of ten is equal to the probability of rolling a 4 (6 + 4 = 10): $P(4) = \frac{1}{6}$; the probability of rolling a sum of eleven is equal to the probability of rolling a 5 (6 + 5 = 11): $P(5) = \frac{1}{6}$; and the probability of rolling a sum of twelve is equal to the probability of rolling a 6 (6 + 6 = 11): $P(6) = \frac{1}{6}$. The probability of rolling a sum greater than nine is

$$P(4) + \text{P}(5) + \text{P}(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}.$$

22. **(a)** $P(\text{snowboarding}) = \frac{69}{126} = \frac{23}{42}$.

(b) $P(\text{skiing, given 6th grader}) = \frac{18}{45} = \frac{2}{5}$**.**

(c) $P(\text{7th grade \& snowboarding}) = \frac{22}{38}$

$P(\text{8th grade \& snowboarding}) = \frac{20}{43}$

$$P(\text{both snowboarding}) = \frac{22}{38} \cdot \frac{20}{43} = \frac{440}{1634} = \frac{220}{817}$$

(d) $P(\text{8th grade \& snowboarding, given that 7th grader snowboards}) = \frac{20}{43}$

Assessment 9-2B

1. **(a)** $S = \{\mathbf{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)}\}$.

(b) $A = \{\mathbf{(2, 2)}\}$.

(c) $B = \{\mathbf{(1, 2), (2, 1), (2, 2), (2, 3)}\}$.

(d) $C = \{\mathbf{(1, 2), (2, 1), (2, 3)}\}$.

2. **(a)** This is a multistage event without replacement. $P(HAT) = P(H) \cdot P(A) \cdot P(T) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} = \mathbf{\frac{1}{24}}$.

(b) With replacement $P(HAT) = P(H) \cdot P(A) \cdot P(T) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \mathbf{\frac{1}{64}}$.

(c) $P(HAT) = P(H) \cdot P(A) \cdot P(T) = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{7} = \mathbf{\frac{1}{84}}$.

(d) $P(box\ 1) = P(box\ 2) = P(box\ 3) = \frac{1}{3}$.

$P(A\ from\ box\ 1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$;

$P(A\ from\ box\ 2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$;

$P(A\ from\ box\ 3) = \frac{1}{3} \cdot 0 = 0$.

These are mutually exclusive events. Thus $P(A) = \frac{1}{12} + \frac{1}{9} + 0 = \frac{3}{36} + \frac{4}{36} = \mathbf{\frac{7}{36}}$.

3. **(a)** The denomination of the coins is irrelevant; "at least three heads" means either three heads or four heads. There are $2^4 = 16$ possible outcomes if four coins are tossed. There is only one way four heads can be obtained, with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$. There are four ways three heads can be obtained (*HHHT*, *HHTH*, *HTHH*, *THHH*), each with probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$.

$P(at\ least\ three\ heads) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \mathbf{\frac{5}{16}}$.

(b) If the first toss is heads and there are three heads total, the possibilities of the next three tosses are: HHT or HTH or THH. The corresponding probabilities are $P(HHT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$,

$P(HTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$, or

$P(THH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. The probability of obtaining three heads given that the first toss is heads is

$P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$.

4. **(a)** **Box 1** because:

(***i***) $P(SOS\ from\ box\ 1) = \frac{2}{3} \cdot \frac{1}{2} \cdot 1 = \frac{2}{6} = \frac{1}{3}$.

(***ii***) $P(SOS\ from\ box\ 2) = \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{24}{120} = \frac{1}{5}$.

(b) **Either box** because:

(***i***) $P(SOS\ from\ box\ 1) = \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{27}$.

(***ii***) $P(SOS\ from\ box\ 2) = \frac{4}{6} \cdot \frac{2}{6} \cdot \frac{4}{6} = \frac{32}{216} = \frac{4}{27}$.

5. A tree diagram would show three event paths:

(***i***) First inspector will catch the defect, with a probability of 0.95; or

(***ii***) First inspector will miss the defect and the second will catch it, with a probability of $(0.05)(0.99) = 0.0495$; or

(***iii***) First inspector will miss the defect and the second will miss it, with a probability of $(0.05)(0.01) = 0.0005$.

The probability of a defect passing both inspectors (i.e., both inspectors missing) $=$ **0.0005**.

6. **(a)** $P(white\ from\ box\ 1) = \frac{3}{5}$;

$P(white\ from\ box\ 2) = \frac{2}{6}$;

$P(two\ whites) = \frac{3}{5} \cdot \frac{2}{6} = \frac{6}{30} = \mathbf{\frac{1}{5}}$.

(b) "At least one" black means either one or two; i.e., black from box 1 and black from box 2 *or* black from box l and white from box 2 *or* white from box 1 and black from box 2 (all mutually exclusive). $P(at\ least\ one\ black) = \frac{2}{5} \cdot \frac{4}{6} + \frac{2}{5} \cdot \frac{2}{6} + \frac{3}{5} \cdot \frac{4}{6} = \frac{8}{30} + \frac{4}{30} + \frac{12}{30} = \frac{24}{30} = \mathbf{\frac{4}{5}}$.

(c) "At most one" black means either zero or one. At a maximum there can be two black balls, so it is simpler to find the probability of two black balls and then use the complementary events property. $P(two\ black) = \frac{2}{5} \cdot \frac{4}{6} = \frac{8}{30} = \frac{4}{15}$.

$P(at\ most\ one\ black) = 1 - P(two\ black) = 1 - \frac{4}{15} = \mathbf{\frac{11}{15}}$.

(d) $P(black\text{-}white\ or\ white\text{-}black) = \frac{2}{5} \cdot \frac{2}{6} + \frac{3}{5} \cdot \frac{4}{6} = \frac{4}{30} + \frac{12}{30} = \frac{16}{30} = \mathbf{\frac{8}{15}}$.

7. **(a)** If we think of S, the sample space, as ordered pairs (1st die, 2nd die), then there are $6 \times 6 = 36$ possible outcomes. The number of ways to roll a 7 or 11 can be seen in the table below.

1st die	2nd die	Total
1	6	7
2	5	7
3	4	7
4	3	7
5	2,6	7,11
6	1,5	7,11

Thus, $P(\textit{rolling a } 7 \text{ or } 11) = \frac{8}{36} = \mathbf{\frac{2}{9}} \approx \mathbf{.22}$

(b) There is one way to roll a $2:(1,1)$. There are two ways to roll a $3:(2,1)$ and $(1,2)$. There is one way to roll a $12:(6,6)$. Thus, P(rolling a 2, 3, or 12) $= \frac{4}{36} = \mathbf{\frac{1}{9}} \approx \mathbf{.11}$.

(c)

Total	Ways of rolling the total	# of ways
4	$(1,3),(2,2),(3,1)$	3
5	$(1,4),(2,3),(3,2),(4,1)$	4
6	$(1,5),(2,4),(3,3),(4,2),(5,1)$	5
8	$(2,6),(3,5),(4,4),(5,3),(6,2)$	5
9	$(3,6),(4,5),(5,4),(6,3)$	4
10	$(4,6),(5,5),(6,4)$	3

$P(\textit{rolling} \text{ a } 4, 5, 6, 8, 9, \text{ or } 10) = \frac{3+4+5+5+4+3}{36} = \frac{24}{36} = \mathbf{\frac{2}{3}} \approx \mathbf{.66}$.

(d) Either **6 or 8**, since there are more ways to roll this sum.

(e) **0**, the smallest possible sum is 2.

(f) **1**, all sums are less than 13.

(g) There are six ways to roll a 7. So the probability of the event of rolling a 7 when two dice are rolled is $\frac{6}{36} = \frac{1}{6}$. In 60 rolls of two dice, we expect to see **10** sevens.

8. There are $4 \cdot 4 = 16$ possible numerator-denominator combinations, each with probability $\frac{1}{16}$. Of these, there are two combinations in which the numerator is more than $1\frac{1}{2}$ times the denominator (6 and 3; 5 and 3). $P(\textit{fraction} > 1\frac{1}{2}) = \frac{2}{16} = \mathbf{\frac{1}{8}}$.

9. There are five different ways of ascending four steps: $S = \{(1,1,1,1), (1,2,1), (1,1,2), (2,1,1), (2,2)\}$. Their probabilities are, respectively:

(i) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1$ (i.e., once she ascends the first three steps the probability is 1 that she will take one step to ascend the fourth) $= \frac{1}{8}$;

(ii) $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$;

(iii) $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$;

(iv) $\frac{1}{2} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4}$;

(v) $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

(a) $P(\textit{two strides}) = \mathbf{\frac{1}{4}}$.

(b) $P(\textit{three strides}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{4} = \mathbf{\frac{5}{8}}$.

(c) $P(\textit{four strides}) = \mathbf{\frac{1}{8}}$.

10. **(a)** There are $4 \cdot 2 = 8$ possible outcomes, each with probability $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$. There are five outcomes in which the number on the spinner will be greater than the number on the coin: $\{(2,1), (3,1), (3,2), (4,1), (4,2)\}$. $P(\textit{spinner} > \textit{coin}) = 5 \cdot \frac{1}{8} = \mathbf{\frac{5}{8}}$.

(b) There are three outcomes in which there are two consecutive integers: $\{(1,2), (2,1), (3,2)\}$. $P(\textit{consecutive integers}) = \mathbf{\frac{3}{8}}$.

11. **(a)** 10 units by 10 units $=$ **100 square units**.

(b) ***(i)*** $P(\textit{region } A) = \frac{4}{100} = \mathbf{\frac{1}{25}}$.

(ii) $P(\textit{region } B) = \frac{12}{100} = \mathbf{\frac{3}{25}}$.

(iii) $P(\textit{region } C) = \frac{20}{100} = \mathbf{\frac{1}{5}}$.

(iv) $P(\textit{region } D) = \frac{28}{100} = \mathbf{\frac{7}{25}}$.

(v) $P(\textit{region } E) = \frac{36}{100} = \mathbf{\frac{9}{25}}$.

(c) 20 points can be scored if and only it both darts land in region A. $P(20\ \textit{pts}) = \frac{1}{25} \cdot \frac{1}{25} = \mathbf{\frac{1}{625}}$.

(d) $P(A \text{ or } B \text{ or } C) = \frac{4}{100} + \frac{12}{100} + \frac{20}{100} = \frac{36}{100} = \mathbf{\frac{9}{25}}$.

12. The set of multiples of $9 = \{0, 9, 18, 27, 36\}$; the set of multiples of $4 = \{0, 4, 8, 12, 16, 20, 24, 28, 32, 36\}$. There are five multiples of 9 and ten multiples of 4 in the 40 inclusive numbers between 0 and 39, so $P(9 \times 9 \times 4) = \frac{5}{40} \cdot \frac{5}{40} \cdot \frac{10}{40} = \frac{250}{64{,}000} = \mathbf{\frac{1}{256}}$.

13. $P(M\,I\,S\,S\,I\,S\,S\,I\,P\,P\,I) = P(M) \cdot P(I) \cdot P(S) \cdot P(S) \cdot P(I) \cdot P(S) \cdot P(S) \cdot P(I) \cdot P(P) \cdot P(P) \cdot P(I) = \frac{1}{11} \cdot \frac{4}{10} \cdot \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot 1$ (i.e., without replacement) $= \frac{1152}{39{,}916.800} = \mathbf{\frac{1}{34{,}650}}$.

14. The total area of the earth is area of land + area of water $\approx 197{,}100{,}000 \text{ mi}^2$. $P(\textit{hitting water}) = \frac{139{,}600{,}000 \text{ mi}^2}{197{,}100{,}000 \text{ mi}^2} \approx \mathbf{0.7}$.

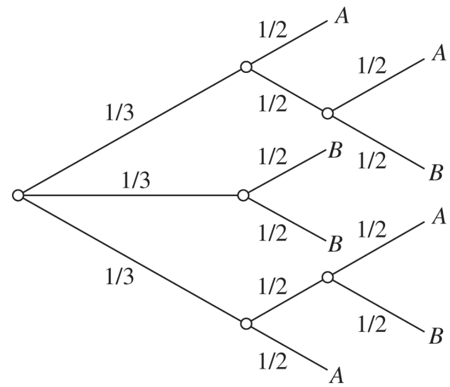
15. "At least one" means one, two, or three children. There are seven ways of having at least one child inherit the disease (where *y* indicates yes and *n* indicates no): $S = \{(ynn), \{nyn), (nny), (yyn), (yny), (nyy), (yyy)\}$. Three of these have probabilities of $0.1 \cdot 0.9 \cdot 0.9 = 0.081$; three have probabilities of $0.1 \cdot 0.1 \cdot 0.9 = 0.009$; and one has probability of $(0.1)^3 = 0.001$. $P(\textit{at least one}) = 3 \cdot 0.081 + 3 \cdot 0.009 + 0.001 = \mathbf{0.271}$.

or

$P(\textit{at least one}) = 1 - P(\textit{none}) = 1 - (0.9)^3 = \mathbf{0.271}$.

16. It is known that of the 30 smokers, 25 have lung cancer. $P(\textit{cancer given smoker}) = \frac{n(\textit{cancer and smoker})}{n(\textit{smoker})} = \frac{25}{30} = \mathbf{\frac{5}{6}}$.

17. Each room has the same probability of being chosen. An equivalent tree diagram is shown below:



$P(A) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{2}$.

$P(B) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$.

Thus it makes **no difference** where the car is placed.

18. (***i***) Hard-soft: $P(\textit{win}) = P(\textit{first in}) \cdot P(\textit{winning the point}) + P(\textit{first out}) \cdot P(\textit{second in}) \cdot P(\textit{winning the point}) = 0.5 \cdot 0.75 + 0.5 \cdot 0.75 \cdot 0.5 = 0.5625$.

(***ii***) Hard-hard: $P(\textit{win}) = 0.5 \cdot 0.75 + 0.5 \cdot 0.5 \cdot 0.75 = 0.5625$.

(***iii***) Soft-hard: $P(\textit{win}) = 0.75 \cdot 0.5 + 0.25 \cdot 0.5 \cdot 0.75 = 0.46875$.

(***iv***) Soft-soft: $P(\textit{win}) = 0.75 \cdot 0.5 + 0.25 \cdot 0.75 \cdot 0.5 = 0.46875$.

She should always **serve hard the first time**; it does not matter what her second serve is.

19. It is known that $P(A \textit{ given } B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(B) \cdot P(A \textit{ given } B)$. (See the discussion of conditional probabilities in Section 9-2 of the text.) Thus:

$P(\textit{eaten} \cap \textit{sickly}) = P(\textit{sickly}) \cdot P(\textit{eaten given sickly}) = \frac{1}{20} \cdot \frac{1}{3} = \frac{1}{60}$.

$P(\textit{eaten} \cap \textit{not sickly}) = P(\textit{not sickly}) \cdot P(\textit{eaten given not sickly}) = \frac{19}{20} \cdot \frac{1}{150} = \frac{19}{3000}$.

$P(\textit{eaten}) = \frac{1}{60} + \frac{19}{3000} = \frac{69}{3000} = \mathbf{\frac{23}{1000}}$.

20. If the sum of the faces is 11 and one die is 6 then the other die has to be a 5 (6 + 5 =11). The probability of the other die being a 4 is $P(4) = 0$.

21. A sum less than 7 is a sum of 2 or 3 or 4 or 5 or 6. If the first die shows a 3 then a sum of 2 or 3 is not possible. Given that the first die shows a three, the probability of rolling a sum of four is equal to the probability of rolling a 1 (3 + 1 = 4): $P(1)=\frac{1}{6}$; the probability of rolling a sum of five is equal to the probability of rolling a 2 (3 + 2 = 5): $P(2)=\frac{1}{6}$; and the probability of rolling a sum of six is equal to the probability of rolling a 3 (3 + 3 = 6): $P(3)=\frac{1}{6}$. The probability of rolling a sum less than seven is

$$P(1)+\text{P}(2)+\text{P}(3)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}=\frac{1}{2}.$$

22. **(a)** $P(\text{skiing})=\frac{57}{126}=\frac{19}{42}$.

(b) $P(\text{snowboarder, given 7th grader})=\frac{22}{38}=\frac{11}{19}$.

(c) $P(\text{6th grade \& skier})=\frac{18}{45}$

$P(\text{8th grade \& skier})=\frac{23}{43}$

$P(\text{both skiers})=\frac{18}{45}\cdot\frac{23}{43}=\frac{414}{1935}=\frac{46}{215}$

(d) $P(\text{both skiers, given that 6th grader skis})=\frac{23}{43}$

Mathematical Connections 9-2: Review Problems

21. **(a)** A certain event $\leftrightarrow$ ***(v)***. A certain event has a probability of 1.

(b) An impossible event $\leftrightarrow$ ***(iii)***. An impossible event has a probability of 0.

(c) A very likely event $\leftrightarrow$ ***(ii)***. A very likely event has a probability close to 1.

(d) An unlikely event $\leftrightarrow$ ***(i)***. An unlikely event has a probability close to 0.

(e) A 50% chance $\leftrightarrow$ ***(iv)***. A 50% chance has a probability of $\frac{1}{2}$.

22. **(a)** $P(\textit{April}\,7)=\mathbf{\frac{1}{30}}$.

(b) $P(\textit{April}\,31)=\mathbf{0}$.

(c) $P(\textit{before April}\,20)=\mathbf{\frac{19}{30}}$.

23. If two of the men always lie and the third always tells the truth, then the probability of a truthful answer is the probability of choosing the truth-sayer from the liars, or $\mathbf{\frac{1}{3}}$.

24. **(a)** $P(\text{black})=\frac{18}{38}=\frac{9}{19}$.

(b) $P(0\text{ or }00)=P(0)+P(00)$
$$=\frac{1}{38}+\frac{1}{38}=\frac{2}{38}=\frac{1}{19}$$

(c) $P(\text{not in 1-12})=1-P(1\text{-}12)$
$$=1-\frac{12}{38}=\frac{26}{38}=\frac{13}{19}$$

(d) $P(\text{odd or green})=P(\text{odd})+P(\text{green})$
$$=\frac{18}{38}+\frac{2}{38}=\frac{20}{38}=\frac{10}{19}$$

25. $P(0\text{ or }00)=\frac{1}{19}=\frac{10}{190}$. The ball should land on 0 or 00 is ten times out of 190.

Assessment 9-3A: Simulations and Applications in Probability

In discussing methods of using simulations, all the following answers could vary.

1. If the use of the thumbtack was to involve dropping it and then observing whether it landed point up or point down, it would not be possible. The probabilities for birth of boys versus girls are approximately equal, but the thumbtack probabilities are not.

2. One method might be to let the digits 1 and 2 represent Diamonds, the digits 3 and 4 represent Clubs, the digits 5 and 6 represent Hearts, and the digits 7 and 8 represent Spades. Disregard digits 9 and 0, and read one digit at a time.

3. **(a)** Let the numbers 1, 2, 3, 4, 5, and 6 represent numbers on the die; ignore the numbers 0, 7, 8, and 9

(b) **(*i*)** Number the persons $01, 02, 03, \ldots, 20$

(*ii*) Mark off groups of two in a random digit table

(*iii*) The three persons chosen are the first three whose numbers appear (ignore two-digit numbers > 20)

(c) Represent Red by the numbers 0 through 4; Green by the numbers 5, 6, and 7; Yellow by the number 8; and White by the number 9; then pick a number from a random digit table

4. Number the students from 001 to 500; then, in the random digit table, mark off blocks large enough so that 30 three digits from 001 to 500 are in each block (disregard 000 or any numbers > 500).

These are the numbers of the 30 students who will be chosen.

5. To simulate Monday, let the digits 1 through 8 represent rain and 0 and 9 represent no rain. If rain occurred on Monday, repeat the same process for Tuesday. If it did not rain on Monday, let the digits 1 through 7 represent rain and 0, 8, and 9 represent dry. Repeat a similar process for the rest of the week.

For example, 60304 13976 was chosen from the random digits table

Day of the week	digit	weather
Mon	6	rain
Tues	0	no rain
Wed	3	rain
Thurs	0	no rain
Fri	4	rain
Sat	1	rain

6. Mark off blocks of two digits and let the digits 00, 01, 02,..., 13, 14 represent contracting the disease; let the digits 15, 16,..., 98, 99 represent not contracting the disease. Mark off three blocks of two digits (for a total of six digits) to represent the three children. If at least one of the two-digit numbers is in the range 00 through 14 it represents a child in the three-child family having contracted strep.

7. $P(<30) = \frac{n(\textit{numbers} < 30)}{n(\textit{possible two-digit numbers})} = \frac{30}{100} = \mathbf{\frac{3}{10}}.$

8. Assuming an unbiased random sample of fish from the pond is caught, then $\frac{50}{300} = \frac{1}{6}$ of the total population is marked. Let n be the fish population; $\frac{1}{6}n = 200 \Rightarrow n = 6 \cdot 200 = \mathbf{1200 \text{ fish}}$.

9. **(a)** **7 games**. It is possible for the losing team to win three games while the winning team wins four.

(b) Answers may vary. E.g., because the teams are evenly matched use a table of random digits and let a number between 0 and 4 represent a win by Team A; let a number between 5 and 9 represent a win by Team B.

Pick a starting spot and count the number of digits it takes before a Team A or Team B series win is recorded. Repeat the experiment many times and then base your answers on:

(*i*)

$$P(\textit{four-game series}) = \frac{n(\textit{four-game series})}{n(\textit{total series})}.$$

(*ii*)

$$P(\textit{seven-game series}) = \frac{n(\textit{seven-game series})}{n(\textit{total series})}.$$

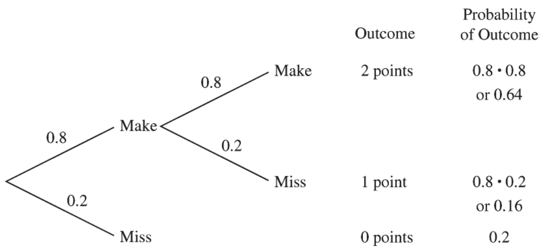
Given evenly matched teams, the probability of a four-game series would be expected to be low.

10. Answers may vary. E.g.,

(*i*) From a random-digit table use numbers 0 through 7 to simulate making the basket and 8 or 9 to simulate missing the basket; or

(*ii*) Use a spinner constructed with 80% of its swept angle representing the making of a basket and 20% representing the missing of a basket.

By using the tree diagram shown below, the theoretical estimates for the number of points scored in 25 attempts may be computed and compared with the experimental probability obtained by simulation. Those are shown in the table following the tree diagram:



Number of Points	Expected Number of Times Points are Scored in 25 Attempts
0	**5**
1	**4**
2	**16**

11. **(a)** $P(\textit{drawing a face card}) = \frac{12}{52} = \frac{3}{13}$. Odds in favor $= \frac{P(\textit{face card})}{1-P(\textit{face card})} = \frac{\left(\frac{3}{13}\right)}{\left(\frac{10}{13}\right)} = \frac{3}{10}$ or **3:10**.

(b) If odds in favor are 3:10, then odds against are **10:3**.

12. $P(7) = \frac{1}{6}$. Odds against $= \frac{1-P(7)}{P(7)} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{1}{6}\right)} = \frac{5}{1}$ or **5:1**.

13. If $P(boy) = \frac{1}{2}$ then $P(\textit{four boys}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

Odds against $= \frac{1-P(\textit{four boys})}{P(\textit{four boys})} = \frac{\left(\frac{15}{16}\right)}{\left(\frac{1}{16}\right)} = \frac{15}{1}$ or **15:1**.

14. $\frac{1-P(win)}{P(win)} = \frac{3}{5}$ (given that odds against $=$ 3:5). $5[1 - P(win)] = 3[P(win)] \Rightarrow 5 - 5[P(win)] = 3[P(win)] \Rightarrow 5 = 8[P(win)]$. $P(win) = \mathbf{\frac{5}{8}}$

15. Given $P(cat) = 0.27$, then the odds against $= \frac{1-P(cat)}{P(cat)} = \frac{0.73}{0.27} =$ **73:27**.

16. $P(red) = \frac{5}{8} \Rightarrow$ odds in favor $= \frac{P(red)}{1-P(red)} = \frac{\left(\frac{5}{8}\right)}{\left(\frac{3}{8}\right)} = \frac{5}{3}$ or **5:3**.

17. Odds against $= \frac{1-P(red)}{P(red)} = \frac{\left(\frac{3}{8}\right)}{\left(\frac{5}{8}\right)} = \frac{3}{5}$ or **3:5**.

18. By Theorem 9-8, if the odds in favor of an event are $m{:}n$, then $P(E) = \frac{m}{m+n}$. Thus if the odds in favor of a grand slam are 2:9, then $P(\textit{grand slam}) = \frac{2}{2+9} = \mathbf{\frac{2}{11}}$.

19. The prime numbers of the six on a die are 2, 3, and 5, so $P(prime) = \frac{3}{6} = \frac{1}{2}$. Then odds of a prime are $\frac{P(prime)}{1-P(prime)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{1}$, or **1:1**.

20. Odds against $= \frac{1-P(event)}{P(event)} = \frac{\left(\frac{5}{93}\right)}{\left(\frac{88}{93}\right)} = \frac{5}{88}$ or **5:88**.

21. If the probability of rain is 90% or 0.9, then the probability of no rain is 10% or 0.1. The odds are 0.9 : 0.1, or equivalently, **9:1**.

22. If we think of the values of the dice as ordered pairs, e.g. (1, 3) represents a roll of one on one of the dice and 3 on the other, then there are 36 possible outcomes. Double sixes represents one of these outcomes. Thus, the odds of a double six is **1:35**.

23. $E = \$0.25 \cdot \frac{5}{25} + \$0.10 \cdot \frac{5}{25} + \$0.05 \cdot \frac{5}{25} + \$0.01 \cdot \frac{10}{25} = \$0.084$, or **about 8¢**.

24. Odds in favor of winning $= 5{:}2 \Rightarrow \frac{P(win)}{1-P(win)} = \frac{5}{2} \Rightarrow P(win) = \frac{5}{7}$. $E = \$14{,}000 \cdot \frac{5}{7} =$ **\$10,000**.

25. The probabilities of rolling all possible values with two dice are tabularized below:

Value	Number of Ways	Probability
2	1	$\frac{1}{36}$
3	2	$\frac{2}{36} = \frac{1}{18}$
4	3	$\frac{3}{36} = \frac{1}{12}$
5	4	$\frac{4}{36} = \frac{1}{9}$
6	5	$\frac{5}{36}$
7	6	$\frac{6}{36} = \frac{1}{6}$
8	5	$\frac{5}{36}$
9	4	$\frac{1}{9}$
10	3	$\frac{1}{12}$
11	2	$\frac{1}{18}$
12	1	$\frac{1}{36}$

$$E = \$2 \cdot \frac{1}{36} + \$3 \cdot \frac{1}{18} + \$4 \cdot \frac{1}{12} + \$5 \cdot \frac{1}{9} + \$6 \cdot \frac{5}{36} + \$7 \cdot \frac{1}{6} + \$8 \cdot \frac{5}{36} + \$9 \cdot \frac{1}{9} + \$10 \cdot \frac{1}{12} + \$11 \cdot \frac{1}{18} + \$12 \cdot \frac{1}{36} = \$7.$$

No. Expected gain is less then cost. In the long run you would expect to lose \$1 each time you roll the dice.

26. **(a)** The first ball was white, so two white and two red remain. $P(red) = \frac{2}{4} = \mathbf{\frac{1}{2}}$.

(b) The first ball was red, so three white and one red remain. $P(red) = \mathbf{\frac{1}{4}}$.

27. If the ratio of men to women is 5:4 then the odds of a man being chosen is 5:4.

28. If the probability of spilling soup on your tie is $\frac{1}{3}$ then the probability of not spilling soup on your tie is $\frac{2}{3}$ and the odds of not spilling on your tie is 2:1.

29. If 98% of the seeds will germinate 2% will not germinate. The odds of having a seed germinate is 98:2 or 49:1.

30. Answers may vary. Let the two-digit numbers 00, 01, 02, 03, …, 57 represent the 58 consonants. Choose a random place to start the random digit table. Read the numbers in pairs. If one of the pairs listed above appears in reading the table, a consonant is found. Here y is counted as a consonant.

Assessment 9-3B

In discussing methods of using simulations, all the following answers could vary.

1. The red half of the spinner might represent the birth of a boy and the blue half of the spinner might represent the birth of a girl. Spin the arrow and record the results.

2. One possibility would be assign digits 0 through 5 to represent rain; digits 6 through 9 to represent no rain.

3. One way would be to consider the people to be numbered as 00, 01, 02,..., 09, 10, 11,..., 24. Then from the random-digit generator or random-number table select a starting point; look at sets of numbers from the starting point in pairs. If, for example, the numbers 405809664176912311... are seen, the first pair that represents a person is 09. The next pair would be 23. Continue until the desired four people are selected.

4. Number people with triples, starting with 000, 001, 002,... , 199. Then from the random-digit generator or random-number table select a starting point; look at sets of numbers from the starting point in groups of three. If, for example, the numbers 405809664176912311... are seen, the first group of three that represents a person is 176. Continue until the desired four people are selected.

5. **(*i*)** Use a random-number table or random-digit generator and look at pairs of digits, where every odd digit could represent a tail and every even digit could represent a head. For example, 83 could represent a head and a tail, while 71 could represent two tails. Then look at 25 such pairs to determine the experimental probability.

(*ii*) Theoretically, the probability of obtaining two tails from two coin losses is $\frac{1}{4}$. The expected number of such results from 25 trials would be $25 \cdot \frac{1}{4}$, or about six times.

6. Assuming an unbiased random sample of frogs from the pond is caught, then $\frac{14}{50} = \frac{7}{25}$ of the total population is marked. Let n be the size of the frog population; $\frac{7}{25}n = 30 \Rightarrow n = \frac{25}{7} \cdot 30 \approx$ **107 frogs**. (107 probably represents an accuracy that does not exist; a better estimate might be to round to 100.)

7. The probability of a single student choosing the square with the dot is $\frac{1}{100}$. In 100 trials (i.e., 100 students making a choice) the expected number of successes would be $100 \cdot \frac{1}{100} = \mathbf{1}$.

8. Number the students from 1 to 7, corresponding to the sectors in the spinner. Spin it 100 times to simulate the 100 questions, and record the results. With 100 questions, each student could expect to be called on about 14 times.

9. For the simulation you choose two-digit numbers to represent the 12 zodiac signs. For example, we could chose 00, 01, 02,...,11 to represent these signs. One possible way to do this: On identically sized slips of paper, write the numbers 00 through 11. Put them into a bag. Draw one out, record your result, put it back into the bag, shake the bag (to randomize the numbers) and draw again. Continue until you draw 5 numbers, and check to see if you have at least a pair of similar numbers or not. The example below illustrates doing this five times:

$$\overbrace{04, 04, 04, 03, 02}^{\text{Pair}}, \overbrace{08, 09, 07, 05, 01}^{\text{No pair}}$$

$$\overbrace{06, 02, 07, 02, 06}^{\text{Pair}}, \overbrace{11, 00, 09, 07, 03}^{\text{No pair}}$$

$$\overbrace{03, 02, 01, 04, 04}^{\text{Pair}}$$

this simulation of at least one pair in zodiac signs is $\mathbf{\frac{3}{5}} = \mathbf{.6}$. If you repeat this experiment enough times, your answer should approach **.62**, which is very close to the theoretical probability.

10. Use a random number table and read the numbers in groups of two. A number less than 45 represents a type O donor. Record the number of random numbers needed at each trial to answer the questions. (a) **around 0.73**. (b) **around 9.**

11. **(a)** **No**.

(b) $P(\textit{black card}) = \frac{26}{52} = \frac{1}{2}$. Odds in favor are

$$\frac{P(\textit{black card})}{1-P(\textit{black card})} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{1} = \mathbf{1{:}1}.$$

12. **(a)** $P(\textit{tail on 10th toss}) = \mathbf{\frac{1}{2}}$. Because the probability of a head or a tail on any toss is not influenced by the results of the previous tosses, coin tosses are independent events.

(b) $P(\textit{ten more tails}) = \left(\frac{1}{2}\right)^{10} = \mathbf{\frac{1}{1024}}$.

(c) Odds against $= \frac{1-P(\textit{ten tails})}{P(\textit{ten tails})} = \frac{\left(\frac{1023}{1024}\right)}{\left(\frac{1}{1024}\right)} = \frac{1023}{1}$ or **1023:1.**

13. The probabilities of a boy or a girl are essentially $\frac{1}{2}$.

(a) Absent other genetic factors, $P(\textit{boy}) = \mathbf{\frac{1}{2}}$.

(b) There are four elements in the sample space for a two-child family: $\{b, b\}$, $\{b, g\}$, $\{g, b\}$, and $\{g, g\}$. Thus $P(\textit{girl} \cap \textit{boy}) = \frac{1}{4}$ and

$$P(\textit{boy}|\textit{girl}) = \frac{P(\textit{girl} \cap \textit{boy})}{P(\textit{girl})} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \frac{2}{4} = \mathbf{\frac{1}{2}}.$$ **Thus, the odds are 1:1.**

14. Use "exactly" 55 million to 1 to approximate the solution. Then from Theorem 9-8, if the odds against event E are m:n, then $P(E) = \frac{n}{m+n}$. Thus if the odds are 55 million to 1 then $P(\textit{winning lottery}) = \frac{1}{55{,}000{,}001}$.

15. $S = \{(hhh), (hht), (hth), (htt), (thh), (tht), (tth), (ttt)\}$. At least two heads appear in four of the eight outcomes, thus $P(\textit{at least two heads}) = \frac{1}{2}$. Odds in favor $= \frac{P(\textit{at least two})}{1-P(\textit{at least two})} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{1}$ **or 1:1**.

16. $P(\textit{drawing } S) = \frac{4}{11}$, so the odds in favor $=$

$$\frac{P(\textit{drawing } S)}{1-P(\textit{drawing } S)} = \frac{\left(\frac{4}{11}\right)}{\left(\frac{7}{11}\right)} = \mathbf{4{:}7}.$$

17. There are 7 unfavorable outcomes, i.e., letters not S, and 4 S_s. The odds are **7:4**.

18. $P(5 < \textit{whole number} < 10) = \frac{4}{10}$, so the odds in favor $= \frac{P(5<\textit{number}<10)}{1-P(5<\textit{number}<10)} = \frac{(\frac{4}{10})}{(\frac{6}{10})} = 4:6 = \mathbf{2:3}$.

19. Given $P(\textit{event}) = 0.\overline{3}$ then the odds against $= \frac{1-P(\textit{event})}{P(\textit{event})} = \frac{0.\overline{6}}{0.\overline{3}} = \mathbf{2:1}$.

20. **(a)** $P(\textit{landing on an odd}) = \frac{\#\text{ of odd numbers}}{\#\text{ of numbers}} = \frac{4}{8} = \mathbf{\frac{1}{2}}$.

(b) $P(\textit{landing on a number divisible by three}) = \frac{\#\text{ of numbers divisible by three}}{\#\text{ of numbers}} = \frac{2}{8} = \mathbf{\frac{1}{4}}$.

(c) $P(5, 6, \text{or } 7) = \frac{3}{8}$ $P(\text{not a } 5, 6, \text{ or } 7) = 1 - \frac{3}{8} = \mathbf{\frac{5}{8}}$.

(d) $P(\textit{number} < 4) = \mathbf{\frac{3}{8}}$.

21. Odds against $= \frac{1-P(\textit{rain})}{P(\textit{rain})} = \frac{0.4}{0.6} = \frac{2}{3}$ or $\mathbf{2:3}$.

22. $P(\textit{red slot}) = \frac{18}{38} = \frac{9}{19}$.

Odds against $= \frac{1-P(\textit{red})}{P(\textit{red})} = \frac{\left(\frac{10}{19}\right)}{\left(\frac{9}{19}\right)} = \frac{10}{9}$ or $\mathbf{10:9}$.

23. **(a)** $P(17) = \mathbf{\frac{1}{38}}$.

(b) Odds against $17 = \frac{1-P(17)}{P(17)} = \frac{\left(\frac{37}{38}\right)}{\left(\frac{1}{38}\right)} = \frac{37}{1}$ or **37:1**.

(c) $E = \$35 \cdot \frac{1}{38} - \$1 \cdot \frac{37}{38} = {}^{-}\$\frac{2}{38} = \mathbf{{}^{-}\$\frac{1}{19}}$, or about a $5¢$ loss. Another viewpoint is that only $\$\frac{18}{19}$, or about $95¢$, will be gained for each \$1 bet.

24. There are six ways in which two dice can show 7, so $P(7) = \frac{6}{36} = \frac{1}{6}$. $E = \$10 \cdot \frac{1}{6} + {}^{-}\$2 \cdot \frac{5}{6} = \frac{10}{6} - \frac{10}{6} = 0$. Since on average there will be 0 expected gain from each throw (assuming that if you win you keep your \$2), in the long run you will come out **about even**.

25. In order to make a fair game, the cost should be equal to the expected value. Each of the numbers 1 through 6 will show on the top of an unbiased die with equal probability $\frac{1}{6}$. Thus:

$$E = \$(6 + 5 + 4 + 3 + 2 + 1) \cdot \frac{1}{6}$$
$$= \$\frac{21}{6} = \mathbf{\$3.50}.$$

26. If a family has three children there are eight possible outcomes. (Think of these as all possible three letter combinations of B and G. For example, BGB represents one girl. There is 1 outcome with no girls, 3 with one girl, 3 with 2 girls, and 1 with three girls. Thus,
$E = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \mathbf{1.5}$.

27. If the ratio of men to women is 5:4 then the odds of a woman being chosen is 4:5.

28. If the probability of spilling soup on your tie is $\frac{1}{3}$ then the probability of not spilling soup on your tie is $\frac{2}{3}$ and the odds of spilling on your tie is 1:2.

29. If 98% of the seeds will germinate 2% will not germinate. The odds of having a seed not germinate are 2:98 or 1:49.

30. Answers may vary. Let the two-digit numbers 00, 01, 02, 03, …, 41 represent the 42 vowels. Choose a random place to start the random digit table. Read the numbers in pairs. If one of the pairs listed above appears in reading the table, a vowel is found.

Mathematical Connections 9-3: Review Problems

20. **No.** There are $2^4 = 16$ possible outcomes when tossing four coins; six of these have exactly two heads: $S = \{(hhtt), (htht), (htth), (thht), (thth), (tthh)\} \Rightarrow P(\textit{exactly two heads}) = \frac{6}{16} = \frac{3}{8}$.

For each player to have an equal chance of winning, the probability of a win for each must be $\frac{1}{2}$.

21. **(a)** $P(\textit{club}) = \frac{13}{52} = \mathbf{\frac{1}{4}}$. There are 13 cards of each suit in a deck of 52 cards.

(b) $P(\textit{queen and spade}) = \mathbf{\frac{1}{52}}$. There is only 1 Queen of Spades in a deck.

(c) $P(\textit{not a queen}) = 1 - P(\textit{queen}) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \mathbf{\frac{12}{13}}$. There are four queens in a deck of 52 cards.

(d) $P(\textit{not a heart}) = 1 - P(\textit{heart}) = 1 - \frac{13}{52} = 1 - \frac{1}{4} = \mathbf{\frac{3}{4}}$.

(e) $P(\textit{spade or heart}) = P(\textit{spade}) + P(\textit{heart}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \mathbf{\frac{1}{2}}$. These are mutually exclusive events.

(f) $P(6 \textit{ of diamonds}) = \mathbf{\frac{1}{52}}$. There is only one 6 of Diamonds.

(g) $P(\textit{queen or spade}) = P(\textit{queen}) + P(\textit{spade}) - P(\textit{queen and spade}) = \frac{4}{52} + \frac{13}{52} - \frac{4}{52} \cdot \frac{13}{52} = \frac{1}{13} + \frac{1}{4} - \frac{1}{13} \cdot \frac{1}{4} = \mathbf{\frac{4}{13}}$. These are non-mutually exclusive events.

(h) $P(\textit{either red or black}) = \frac{1}{2} + \frac{1}{2} = \mathbf{1}$. This is a certain event.

22. **(a)** $P(\textit{either red or blue}) = \frac{7}{19} + \frac{8}{19} = \mathbf{\frac{15}{19}}$.

(b) $P(\textit{first red, second blue}) = \frac{7}{19} \cdot \frac{8}{19} = \mathbf{\frac{56}{361}}$.

(c) $P(\textit{first red, second blue}) = \frac{7}{19} \cdot \frac{8}{18} = \frac{56}{342} = \mathbf{\frac{28}{171}}$.

23. If the probability of making a free throw is $\frac{1}{3}$, the probability of missing is $\frac{2}{3}$. The probability of missing three in a row, assuming the player's skill level is unchanged from shot to shot, would be $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \mathbf{\frac{8}{27}}$.

24. **(a)** $S = \{1, 2, 3, 4\}$.

(b) $S = \{\textit{red}, \textit{blue}\}$.

(c) $S = \{(1, \textit{red}), (1, \textit{blue}), (2, \textit{red}), (2, \textit{blue}), (3, \textit{red}), (3, \textit{blue}), (4, \textit{red}), (4, \textit{blue})\}$.

(d) $S = \{(\textit{red}, 1), (\textit{red}, 2), (\textit{red}, 3), (\textit{red}, 4), (\textit{red}, 5), (\textit{red}, 6), (\textit{blue}, 1), (\textit{blue}, 2), (\textit{blue}, 3), (\textit{blue}, 4), (\textit{blue}, 5), (\textit{blue}, 6)\}$.

(e) $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$.

(f) $S = \{(\textit{red}, \textit{red}), (\textit{red}, \textit{blue}), (\textit{blue}, \textit{red}), (\textit{blue}, \textit{blue})\}$.

25. $P(\textit{blue}, \textit{blue}) = P(\textit{blue}) \cdot P(\textit{blue.}) \Rightarrow P(\textit{blue}) = \sqrt{P(\textit{blue}, \textit{blue})} = \sqrt{\frac{25}{36}} = \frac{5}{6}$. Thus the **blue** section must have $\frac{5}{6} \cdot 360° = \mathbf{300}°$ and the **red** section must have $360° - 300° = \mathbf{60}°$.

26. $P(\textit{two vowels}) = \frac{5}{26} \cdot \frac{5}{26} = \mathbf{\frac{25}{676}}$.

Assessment 9-4A: Permutations and Combinations in Probability

1. The event "girls" can occur in 16 ways; the event "boys" can occur in 14 ways. Thus the event "girls *and* boys" can occur as $16 \cdot 14 = $ **224 unique pairings**.

2. The number of ways the four digits after the prefix may be arranged is $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = $ 10,000. Thus **10,000 numbers** can be associated with each prefix (assuming all can be used).

3. There are $3 \cdot 15 \cdot 4 = \mathbf{180}$ possible different three-course meals.

4. **(a)** **True**. $6! = 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 6 \cdot 5!$.

(b) **False**. $3! + 3! = 3 \cdot 2 \cdot 1 + 3 \cdot 2 \cdot 1 = 12 \neq 6!$.

(c) **False**. $\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 6 \cdot 5 \cdot 4 = 120 \neq 2!$.

5. **(a)** There are 8 unlike letters in SCRAMBLE. $\mathbf{8! = 40,320}$.

(b) There are 9 unlike letters in PERMUTATION and 2 like letters. $\mathbf{\frac{11!}{2!} = 19,958,400}$.

6. Since order is not distinct the number of two-person committees is a combination of six persons taken two at a time. ${}_6C_2 = \frac{6!}{2!(6-2)!} = \frac{6 \cdot 5}{2 \cdot 1} = $ **15 committees**.

7. **(a)** Order is distinct, implying a permutation of 30 persons taken three at a time. ${}_{30}P_3 = \frac{30!}{(30-3)!} = 30 \cdot 29 \cdot 28 = \mathbf{24{,}360}$ **ways** in which to select the three officers.

(b) Order is not distinct, implying a combination of 30 persons taken three at a time. ${}_{30}C_3 = \frac{30!}{(30-3)!3!} = 29 \cdot 14 \cdot 10 = \mathbf{4060}$ **ways** in which to choose three-person committees.

8. Five volumes may be placed in $5! = 120$ different ways; only one will be in correct order. $P(\textit{correct order}) = \mathbf{\frac{1}{120}}$.

9. The problem is to choose ten points two at a time. Since a line may be drawn either way order is not distinct. ${}_{10}C_2 = \frac{10!}{(10-2)!2!} = \mathbf{45}$ **lines**.

10. There are nine flags with four (red), three (green), and two (white) repeated. $\frac{9!}{4!3!2!} = \frac{362{,}880}{24 \cdot 6 \cdot 2} =$ **1260 possible** signals.

11. If there were n people at the party there were n combinations of people two at a time shaking hands. ${}_nC_2 = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2!} = 28 \Rightarrow n(n-1) = 56$. This is the product of two consecutive whole numbers; look for factors of 56 which yield $8 \cdot 7$. Thus $n(n-1) = 8 \cdot 7$, and $n = \mathbf{8}$ **people** at the party.

12. Order within hands is not distinct, thus a combination of 52 cards taken five at a time. ${}_{52}C_5 = \frac{52!}{(52-5)!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \mathbf{2{,}598{,}960}$ **possible** five-card hands.

13. Order is not important, implying a combination of 54 numbers taken six at a time. ${}_{54}C_6 = \frac{54!}{(54-6)!6!} = \frac{54 \cdot 53 \cdot 52 \cdot 51 \cdot 50 \cdot 49}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 25{,}827{,}165$ possible combinations of numbers, only one of which will win. Then $P(\textit{win}) = \mathbf{\frac{1}{25{,}827{,}165}}$.

14. **(a)** $P(\textit{all girls}) = \frac{{}_{12}C_4}{{}_{22}C_4} = \frac{495}{7315} = \mathbf{\frac{9}{133}}$.

(b) $P(\textit{all boys}) = \frac{{}_{10}C_4}{{}_{22}C_4} = \mathbf{\frac{6}{209}}$.

(c) $P(\textit{at least one girl}) = 1 - P(\textit{all boys}) = 1 - \frac{{}_{10}C_4}{{}_{22}C_4} = 1 - \frac{210}{7315} = 1 - \frac{6}{209} = \mathbf{\frac{203}{209}}$.

15. **(a)** $P(\textit{two Britons, four Italians, two Danes}) = \frac{{}_{20}C_2 \cdot {}_{21}C_4 \cdot {}_4C_2}{{}_{45}C_8} = \frac{190 \cdot 5985 \cdot 6}{215{,}553{,}195}$, or about **0.032**.

(b) $P(\textit{no Britons}) = \frac{{}_{20}C_0 \cdot {}_{25}C_8}{{}_{45}C_8}$, or about **0.005**.

(c) $P(\textit{at least one Briton}) = 1 - P(\textit{no Britons}) = 1 - \frac{{}_{20}C_0 \cdot {}_{25}C_8}{{}_{45}C_8} \approx \mathbf{0.995}$.

(d) $P(\text{all } \textit{Britons})\ \frac{{}_{20}C_8 \cdot {}_{25}C_0}{{}_{45}C_8}$, or about $\mathbf{5.84 \cdot 10^{-4}}$.

16. There are $5! = 120$ ways of arranging the five letters, only one of which is correct. $P(\textit{all correct}) = \mathbf{\frac{1}{120}}$.

17. **(a)** Since numbers can be repeated, there would be $10^4 = \mathbf{10{,}000}$ possibilities.

(b) If numbers cannot be repeated, there would be $10 \cdot 9 \cdot 8 \cdot 7 = \mathbf{5040}$ possibilities.

(c) There are five choices for each digit. So there are $5^4 = 625$ ways to create four digit numbers where the digits are all even. $P(\textit{all even}) = \frac{625}{10{,}000} = \mathbf{\frac{1}{16}}$.

18. Order is not important, so the number of ways is ${}_6C_4 = \frac{6!}{4! \cdot 2!} = \frac{720}{24 \cdot 2} =$ **15 ways** of presenting the awards.

19. Order is not important, so the number of choices is ${}_8C_3 = \frac{8!}{3! \cdot 5!} = \frac{40{,}320}{6 \cdot 120} =$ **56 choices** for the exercise.

20. If the vowels all must stay in the same place there are 6 arrangements that can be made. If there is only 1 choice for each vowel then there are 3 consonants that can move:

$\underline{e}\,q\,\underline{u}\,\underline{a}\,t\,\underline{i}\,\underline{o}\,n = \underline{1} \cdot 3 \cdot \underline{1} \cdot \underline{1} \cdot 2 \cdot \underline{1} \cdot \underline{1} \cdot 1 = 6$

21. **(a)** The word "quick" has 5 letters. If all letters are used $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ arrangements can be made.

(b) If three letters are used, $5 \cdot 4 \cdot 3 = 60$ arrangements can be made.

(c) If four letters are used, $5 \cdot 4 \cdot 3 \cdot 2 = 120$ arrangements can be made.

22. Seven people can stand in a line $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways.

Assessment 9-4B

1. Each coin toss will result in two possible outcomes (head or tail). Five tosses will result in $2^5 = \mathbf{32}$ **different combinations** of heads and tails.

2. Assume all letters after the first can be repeated; then:

(*i*) There are $2 \cdot 26 \cdot 26 = \mathbf{1352}$ possible **three-letter** call signs.

(*ii*) There are $2 \cdot 26 \cdot 26 \cdot 26 = \mathbf{35{,}152}$ possible **four-letter** call signs.

3. If we only consider gender, then we are arranging 7 objects, 3 of which are alike and 4 of which are alike. The number of ways the people can be arranged is $\frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \mathbf{35}$.

4. **(a)** **False**. $\frac{6!}{3} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3} = 6 \cdot 5 \cdot 4 \cdot 2 \cdot 1 \neq 2!$.

(b) **False**. $\frac{6!}{5!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 6$.

(c) **True**. $\frac{6!}{4!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{6 \cdot 5}{2 \cdot 1} = 15$.

(d) **True**. $(n+1) \cdot n! = (n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = (n+1)!$.

5. **(a)** There are two O's repeated in OH1O, so there are $\frac{4 \cdot 3 \cdot 2 \cdot 1}{2!} = \frac{4!}{2!} = \mathbf{12}$ **possible arrangements.**

(b) There are four A's repeated in ALABAMA, so there are $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 =$ **210 possible arrangements**.

(c) There are three I's and two L's repeated in ILLINOIS, so there are $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 1} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2 =$ **3360 possible arrangements**.

(d) There are four I's, four S's, and two P's in MISSISSIPPI, so there are $\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = \frac{11!}{4!4!2!} = 11 \cdot 10 \cdot 9 \cdot 7 \cdot 5 = \mathbf{34{,}650}$ **possible arrangements**.

(e) There are four E's, two N's, and two S's in TENNESSEE, so there are $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)(2 \cdot 1)} = \frac{9!}{4!2!2!} = 9 \cdot 7 \cdot 5 \cdot 4 \cdot 3 =$ **3780 possible arrangements**.

6. There would be $\frac{12!}{6!4!2!} = \mathbf{13{,}860}$ possible ways in which the twelve cars could finish.

7. Player's positions are not distinct, implying a combination of 12 persons taken five at a time ${}_{12}C_5 = \frac{12!}{(12-5)!5!} =$ **792 possible** team combinations.

8. **(a)** $P(C\ performs\ 1^{st}) = \mathbf{\frac{1}{7}}$.

(b) $P(F\ or\ G\ perform\ 1^{st}) = \frac{1}{7} + \frac{1}{7} = \mathbf{\frac{2}{7}}$.

(c) Since the performances represent unlike events, there are 7! ways to arrange the performances. C, D, E, A, E, F, G represents one possible outcome, so the probability is $\frac{1}{7!} = \mathbf{\frac{1}{5040}}$.

9. **(a)** From the starting point there are 3 possible paths, each of which can get to the next point in the shortest distance. From the ends of each of these paths there are 2 shortest, and from the ends of each of these there is only 1 path which will end at the other point. Thus there are $3 \cdot 2 \cdot 1 = 3! = \mathbf{6}$ **shortest paths** on the cubes.

(b) Using the fundamental principal of counting, there are $6 \cdot 6 =$ **36 shortest paths**.

10. If no letter or number repetitions are allowed, there are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = \mathbf{11{,}232{,}000}$ **possible** license plate numbers.

11. Order is not important, thus a combination of 24 people taken 12 at a time. ${}_{24}C_{12} = \mathbf{2{,}704{,}156}$ different juries.

12. There are $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = \mathbf{10^9}$, or 1,000,000,000, possible Social Security numbers (repetition allowed).

13. **(a)** $P(1 \textit{ on each roll}) = \left(\frac{\mathbf{1}}{\mathbf{6}}\right)^{\mathbf{8}}$.

(b) If $P(6) = \frac{1}{6}$ then $P(\textit{not } 6) = 1 - \frac{1}{6} = \frac{5}{6}$.

Thus $P(\textit{two } 6\textit{'s and six others}) = \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^6$.

There are ${}_8C_2 = \frac{8!}{6! \cdot 2!} = 28$ ways of rolling two sixes out of eight tries (order is not distinct).

$P(6 \textit{ exactly twice}) = {}_8C_2 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^6 = \frac{28 \cdot 5^6}{6^8} = \frac{437{,}500}{1{,}679{,}616} \approx \mathbf{0.260}$.

(c) $P(\textit{at least one } 6) = 1 - P(\textit{zero } 6\textit{'s}) = 1 - \left(\frac{5}{6}\right)^8 \approx \mathbf{0.767}$.

14. **(a)** The number of ways of selecting a committee of three from the group of seven Americans is ${}_7C_3 = 35$. There are ${}_{15}C_3 = 455$ ways of selecting three members from the set of 15. $P(\textit{all Americans}) = \frac{35}{455} = \frac{\mathbf{1}}{\mathbf{13}}$.

(b) If no Americans are selected there are ${}_8C_3$ ways of selecting the three members from among the French and English. Then $P(\textit{no Americans}) = \frac{{}_8C_3}{{}_{15}C_3} = \frac{56}{455} = \frac{\mathbf{8}}{\mathbf{65}}$.

15. Order is not important in the choosing of the executive committee, implying the combination ${}_{32}C_5 = \frac{32!}{5!(32-5)!} = 201{,}376$ choices. Once the committee members are selected there are then five possibilities for president, so the total number of choices is $5({}_{32}C_5) = \mathbf{1{,}006{,}880}$.

16. If six free throws are made, then four are missed. If $P(\textit{making free throw}) = \frac{2}{3}$, then $P(\textit{missing free throw}) = 1 - \frac{2}{3} = \frac{1}{3}$.

Thus $P(\textit{six made and four missed}) = \left(\frac{2}{3}\right)^6 \cdot \left(\frac{1}{3}\right)^4$.

There are ${}_{10}C_6 = \frac{10!}{(10-6)!6!} = 210$ ways of making six free throws out of ten (order is not distinct).

$P(\textit{exactly six made}) = 210 \cdot \left(\frac{2}{3}\right)^6 \cdot \left(\frac{1}{3}\right)^4 = \frac{210 \cdot 2^6}{3^6 \cdot 3^4} = \frac{13{,}440}{59{,}049}$, or **about 0.228**.

17. $P(\textit{royal flush}) = \frac{4}{{}_{52}C_5} = \frac{4}{2{,}598{,}960} = \frac{1}{649{,}740}$, where the 4 in the numerator is from choosing the suit.
$P(\textit{no royal flush}) = 1 - P(\textit{royal flush}) = 1 - \frac{1}{649{,}740} = \frac{649{,}739}{649{,}740}$.

Odds against royal flush $= \frac{P(\textit{no royal flush})}{P(\textit{royal flush})} = \frac{\left(\frac{649{,}739}{649{,}740}\right)}{\left(\frac{1}{649{,}740}\right)} = \frac{649{,}739}{1} = \mathbf{649{,}739{:}1}$.

18. Since order is important (i.e., for example, Spot, Fido, and Rover is a different arrangement than Fido, Rover, and Spot) a permutation is implied. ${}_{36}P_3 = \frac{36!}{(36-3)!} = \mathbf{42{,}840}$ award possibilities.

19. **(a)** $P(7)$ on one roll of two dice $= \frac{6}{36} = \frac{1}{6}$.
$P(7 \textit{ on each roll}) = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \left(\frac{\mathbf{1}}{\mathbf{6}}\right)^{\mathbf{5}} \approx \mathbf{0.00013}$.

(b) $P(7)$ on one roll of two dice $= \frac{1}{6}$; thus $P(\textit{not } 7)$ on one roll of two dice $= 1 - P(7) = 1 - \frac{1}{6} = \frac{5}{6}$. $P(7 \textit{ twice and not-}7 \textit{ three times})$ on five rolls of two dice $= \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3$.

There are ${}_5C_2 = 10$ ways of obtaining 7 twice and not-7 three times in five rolls of two dice.

$P(7 \textit{ exactly twice}) = 10 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 = \frac{1250}{7776}$, or **about 0.161**.

20. If the consonants all must stay in the same place there are 120 arrangements that can be made. If there is only 1 choice for each consonant then there are 5 vowels that can move:
$\underline{e}\,q\,\underline{u}\,\underline{a}\,t\,\underline{i}\,\underline{o}\,n = \underline{5} \cdot 1 \cdot \underline{4} \cdot \underline{3} \cdot 1 \cdot \underline{2} \cdot \underline{1} \cdot 1 = 120$.

21. **(a)** The word "french" has 6 letters. If all letters are used $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ arrangements can be made.

(b) If three letters are used, $6 \cdot 5 \cdot 4 = 120$ arrangements can be made.

(c) If four letters are used, $6 \cdot 5 \cdot 4 \cdot 3 = 360$ arrangements can be made.

22. Nine people can stand in a line $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$ ways.

Mathematical Connections 9-4: Review Problems

13. **(a)** $P(\textit{at least one ace}) = 1 - P(\textit{zero aces}) = 1 - \frac{48}{52} \cdot \frac{47}{51} = 1 - \frac{2256}{2652} = 1 - \frac{188}{221} = \mathbf{\frac{33}{221}}$.

(b) $P(\textit{one card red}) = P(\textit{red and black}) + P(\textit{black and red}) = \frac{26}{52} \cdot \frac{26}{51} + \frac{26}{52} \cdot \frac{26}{51} = \frac{13}{51} + \frac{13}{51} = \mathbf{\frac{26}{51}}$.

14. Since $P(11)$ and $P(12)$ are mutually exclusive events, $P(>10) = P(11) + P(12) = \frac{2}{36} + \frac{1}{36} = \frac{3}{36} = \mathbf{\frac{1}{12}}$.

15. **(*i*)** Possible outcomes: $S = \{(hh),(ht),(th),(tt)\}$. $P(\textit{both heads}) = \frac{1}{4}$; $P(\textit{both tails}) = \frac{1}{4}$; $P(\textit{no match}) = \frac{1}{2}$.

$E = \$5 \cdot \frac{1}{4} + \$3 \cdot \frac{1}{4} + {}^{-}\$4 \cdot \frac{1}{2} = \$\frac{5}{4} + \$\frac{3}{4} + {}^{-}\$\frac{8}{4} = \mathbf{\$0}$.

(*ii*) **Yes**. The expected outcome is 0.

16. **(a)** **No**. $E = \$36 \cdot \frac{1}{38} = \$\frac{36}{38}$, or about $95\cent$. It is not a fair game because expected gain and cost are not the same.

(b) Each time you play you either win \$36 or lose \$1.Then $\$36\left(\frac{1}{38}\right) - \$1\left(\frac{37}{38}\right) = \$\left(\frac{-1}{38}\right)$, or an expected loss of about $2\frac{1}{2}\cent$.

17. We first find the probability that all 40 children have different birthdays and then subtract the result from 1. We get $1-\left(\frac{365-1}{365}\right)\left(\frac{365-2}{365}\right)\ldots\left(\frac{365-39}{365}\right) \approx 0.89$. Thus the probability that the friend wins the bet is approximately 0.89 or 89%. For 50 people, the probability rises to approximately 97%.

Chapter 9 Review

1. **(a)** $S =$ **{(*hhh*), (*hht*), (*hth*), (*htt*), (*ttt*), (*tth*), (*thh*), (*tht*)}**. There are $2^3 =$ eight elements in the sample space.

(b) "At least two" means two or three heads, so $S =$ **{(*hhh*), (*hht*), (*hth*), (*thh*)}**.

(c) $P(\textit{at least two heads}) = \frac{n(\textit{at least two heads})}{n(S)} = \frac{4}{8} = \mathbf{\frac{1}{2}}$.

2. **(a)** $S = \{(\textit{Sunday}), (\textit{Monday}), (\textit{Tuesday}), (\textit{Wednesday}), (\textit{Thursday}), (\textit{Friday}), (\textit{Saturday})\}$.

(b) $S = \{(\textit{Tuesday}), (\textit{Thursday})\}$.

(c) $P(\textit{day starting with T}) = \mathbf{\frac{2}{7}}$.

3. Answers may vary; e.g.,

(*i*) $\frac{4}{5} \cdot 1000 = 800$, so there are 800 blue beans;

(*ii*) $\frac{1}{8} \cdot 1000 = 125$, so there are 125 red beans;

(*iii*) $\frac{4}{5} + \frac{1}{8} < 1$, so there are jelly beans in the jar that are neither red nor blue; or

(*iv*) There are $1000 - 800 - 125 = 75$ jelly beans that are neither red nor blue.

4. **(a)** $P(\textit{vote for Romney}) = \frac{60{,}932{,}152}{60{,}932{,}152+65{,}899{,}660} \approx$ **0.480**.

(b) $P(\textit{vote for Obama}) = 1 - P(\textit{vote for Romney}) \approx 1 - \frac{60{,}932{,}152}{60{,}932{,}152+65{,}899{,}660} \approx$ **0.520**

(c) Odds in favor of not voting for Obama $0.480 : 0.520 \Rightarrow$ about 12:13.

5. **(a)** $P(black) = \frac{n(black)}{n(S)} = \mathbf{\frac{5}{12}}$.

(b) $P(black\ or\ white) = P(black) + P(white) = \frac{5}{12} + \frac{4}{12} = \frac{9}{12} = \mathbf{\frac{3}{4}}$. These are mutually exclusive events.

(c) $P(neither\ red\ nor\ white) = P(black) = \mathbf{\frac{5}{12}}$.

(d) $P(red\ not\ drawn) = P(black\ or\ white) = \mathbf{\frac{3}{4}}$.

(e) $P(black\ and\ white) = \mathbf{0}$. An impossible event.

(f) $P(black\ or\ white\ or\ red) = \mathbf{1}$. A certain event.

6. **(a)** $P(club) = \frac{n(club)}{n(S)} = \frac{13}{52} = \mathbf{\frac{1}{4}}$.

(b) $P(spade\ and\ 5) = \mathbf{\frac{1}{52}}$. There is only one 5 of spades.

(c) $P(heart\ or\ face\ card) = P(heart) + P(face\ card) - P(heart\ and\ face\ card) = \frac{13}{52} + \frac{12}{52} - \frac{13}{52} \cdot \frac{12}{52} = \frac{1}{4} + \frac{3}{13} - \frac{3}{52} = \mathbf{\frac{11}{26}}$.

(d) $P(jack\ not\ drawn) = 1 - P(jack) = 1 - \frac{4}{52} = \frac{48}{52} = \mathbf{\frac{12}{13}}$.

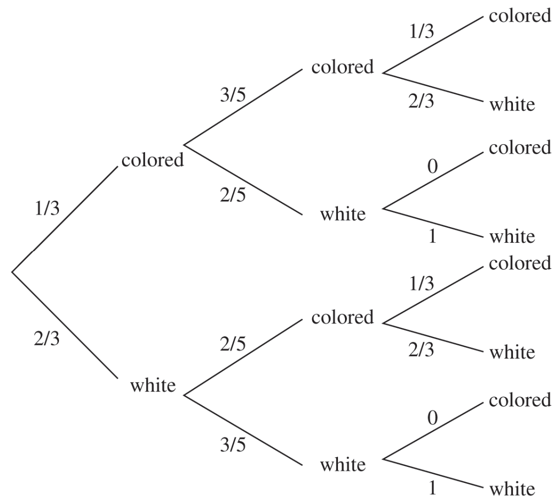
7. **(a)** $P(all\ white) = \frac{4}{9} \cdot \frac{4}{9} \cdot \frac{4}{9} = \mathbf{\frac{64}{729}}$. With replacement the probabilities of each draw are the same.

(b) $P(all\ white) = \frac{4}{9} \cdot \frac{3}{8} \cdot \frac{2}{7} = \frac{24}{504} = \mathbf{\frac{1}{21}}$. Without replacement the probabilities are changed for each draw.

8. Either an L can be drawn from box **1** and an L from box 2 or not an L from box 1 and an L from box 2. $P(L) = \frac{1}{5} \cdot \frac{2}{5} + \frac{4}{5} \cdot \frac{1}{5} = \frac{2}{25} + \frac{4}{25} = \mathbf{\frac{6}{25}}$.

9. $P(A\ from\ any\ box) = P(choosing\ that\ box) \cdot P(A\ from\ chosen\ box)$. The probabilities are added because the probabilities of A from each of the boxes are mutually exclusive events.
$P(A) = \frac{1}{4} \cdot \frac{0}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{5} = 0 + \frac{1}{16} + \frac{1}{16} + \frac{1}{20} = \mathbf{\frac{7}{40}}$.

10. ***(i)***



(ii) $P(colored) = \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{5} \cdot 0 + \frac{2}{3} \cdot \frac{2}{5} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} \cdot 0 = \frac{3}{45} + 0 + \frac{4}{45} + 0 = \mathbf{\frac{7}{45}}$.

11. $P(jack) = \frac{4}{52} = \frac{1}{13}$. Odds in favor $= \frac{P(jack)}{1-P(jack)} = \frac{\left(\frac{1}{13}\right)}{\left(\frac{12}{13}\right)} = \frac{1}{12}$ or **1:12**.

12. $P(prime\ number) = \frac{3}{6} = \frac{1}{2}$. Odds against a prime number $= \frac{1-P(prime\ number)}{P(prime\ number)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{1}$ or **1:1**.

13. $\frac{P(event)}{1-P(event)} = \frac{3}{5} \Rightarrow 5.P(event) = 3 \cdot [1 - P(event)] \Rightarrow 5 \cdot P(event) = 3 - 3 \cdot P(event) \Rightarrow 8 \cdot P(event) = 3 \Rightarrow P(event) = \mathbf{\frac{3}{8}}$.

Note that by Theorem 9-8 if the odds in favor of an event are m to n, the probability of that event is $\frac{m}{m+n}$.

14. $P(double\ 1\text{'s}) = \frac{1}{36}$; $P(double\ 6\text{'s}) = \frac{1}{36}$.
$E = \$7.20 \cdot \frac{1}{36} + \$3.60 \cdot \frac{1}{36} + \$0 \cdot \frac{34}{36} = \$0.30$ or **30¢**.

15. $P(win) = \frac{1}{3000}$. $E = \$1000 \cdot \frac{1}{3000} = \$\frac{1}{3} = 33\frac{1}{3}$¢. There is **no actual value,** to the nearest cent, that would produce a fair game.

16. $S = 10^4 \Rightarrow P(win) = \frac{1}{10{,}000}$. $E = \$15{,}000 \cdot \frac{1}{10{,}000} = \mathbf{\$1.50}$. If the ticket costs \$2, expected earnings are $\$1.50 - \$2.00 = {}^{-}50¢$.

17. There are $9 \cdot 10 \cdot 10 \cdot 1 = \mathbf{900}$ possible different numbers.

18. Order is not distinct, implying a combination of 10 people taken 3 at a time. ${}_{10}C_3 = \frac{10!}{(10-3)!3!} =$ **120** possible ways.

19. Order is distinct, implying a permutation of 10 flags taken 4 at a time. ${}_{10}P_4 = \frac{10!}{(10-4)!} = 10 \cdot 9 \cdot 8 \cdot 7 = \mathbf{5040}$ possible different ways.

20. $P(both\ blue) = \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20} = \mathbf{\frac{1}{10}}$.

21. **(a)** Order is distinct, implying a permutation of 5 finishes taken 3 at a time. ${}_5P_3 = \frac{5!}{(5-3)!} = 5 \cdot 4 \cdot 3 = \mathbf{60}$ possible different finishes.

(b) There are ${}_5P_2 = 20$ possible ways for a first/second place finish; there is only one way for a Deadbeat/Bandy finish. $P(Deadbeat/Bandy\ finish) = \mathbf{\frac{1}{20}}$.

(c) There are ${}_5P_3 = 60$ possible first-, second-, and third-place finishes; there is only one way for a Deadbeat/Egglegs/Cash finish. $P(Deadbeat/Egglegs/Cash\ finish) = \mathbf{\frac{1}{60}}$.

22. If they roll the same number no one wins so half of all possible outcomes minus the ties is how many ways one can win. The possible outcomes of Al's roll and Ruby's roll are: $\{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$, so there are 15 ways for Ruby to roll a higher number than Al. $P(Ruby > Al) = \frac{15}{36} = \mathbf{\frac{5}{12}}$.

23. Order is not distinct, implying a combination of 5 questions taken 3 at a time. There are ${}_5C_3$ ways of selecting three questions out of five; since the number l is not to be selected, there are four options left and three of them will be selected. Thus $P(1\ not\ chosen) = \frac{{}_4C_3}{{}_5C_3} = \frac{4}{10} = \mathbf{\frac{2}{5}}$.

24. Since fourth and last bats are fixed, there are 7 remaining positions to consider. $7! = \mathbf{5040}$ ways.

25. $P(all\ green) = 0.3 \cdot 0.3 \cdot 0.3 = \mathbf{0.027}$.

26. $P(success) = 1 - P(failure)$. $P(second\ stage\ success) = 1 - \frac{1}{8} = \frac{7}{8}$; $P(third\ stage\ success) = 1 - \frac{1}{10} = \frac{9}{10}$. $P(success\ given\ stage\ one) = \frac{7}{8} \cdot \frac{9}{10} = \mathbf{\frac{63}{80}}$.

27. Answers may vary. E.g.,

(a) Randomly select digits 1 through 6 from a random digit table; discard digits 0 and 7 through 9.

(b) Block off twelve two-digit blocks in a random digit table (discarding any other than 01 through 12); randomly select three.

(c) Let random digits 0 through 2 represent red; digits 3 through 5 represent white; digits 6 through 8 represent blue; discard any nines.

28. The events are not equally likely. The sample space consists of eight possibilities, so computing the probabilities yields: $P(3\ heads) = \frac{1}{8}$; $P(2\ heads) = \frac{3}{8}$; $P(1\ head) = \frac{3}{8}$; and $P(0\ heads) = \frac{1}{8}$.

29. Divide the figure into 16 equal triangles:

(a) A would represent two of the triangles. $P(A) = \frac{2}{16} = \mathbf{\frac{1}{8}}$.

(b) B would represent four of the triangles. $P(B) = \frac{4}{16} = \mathbf{\frac{1}{4}}$.

(c) C would represent one of the triangles. $P(C) = \mathbf{\frac{1}{16}}$.

30. $d(\overline{MQ}) = 20$; $d(\overline{NO}) = 8$. $P(between\ N\ and\ O) = \frac{8}{20} = \mathbf{\frac{2}{5}}$.

31. *P(at least one face card)=1-P(no face card)* $= 1 - \frac{{}_{40}C_3}{{}_{52}C_3} = 1 - \frac{\frac{40!}{3!37!}}{\frac{52!}{3!49!}} = \frac{47}{85} \approx 0.553$

32. There are 7 shades of yellow in a box of 64 Crayola crayons. The probability that a crayon chosen at random from the box of crayons is not a shade of yellow

$$P(\text{not a shade of yellow}) = 1 - P(\text{shade of yellow})$$
$$= 1 - \frac{7}{64} = \frac{57}{64}$$

33. The probability of drawing two crayons without replacement that are the most used colors is

P(2 of the most used colors)=

1-P(none of the most used colors)

$$=1-\left(\frac{60}{64}\cdot\frac{59}{63}\right) = 1-\frac{3540}{4032} = \frac{492}{4032} \approx 0.122$$

34. The probability of choosing the United States out of the two countries is $\frac{1}{2}$. The probability of choosing a well-known zoo in the United States is $\frac{1}{350}$. The probability of choosing a well-known zoo in the United States if you choose one of the two countries at random and one of the zoos is chosen at random is $\frac{1}{2}\cdot\frac{1}{350} = \frac{1}{700}$.

35. **(a)** $P(\text{three plums}) = \frac{5}{20}\cdot\frac{1}{20}\cdot\frac{5}{20} = \frac{25}{8000} = \frac{1}{320}$.

(b) $P(\text{three oranges}) = \frac{3}{20}\cdot\frac{6}{20}\cdot\frac{7}{20} = \frac{126}{8000} = \frac{63}{4000}$

(c) $P(\text{three lemons}) = \frac{3}{20}\cdot\frac{0}{20}\cdot\frac{4}{20} = \frac{0}{8000} = 0$

(d) $P(\text{no plums}) = \frac{15}{20}\cdot\frac{19}{20}\cdot\frac{15}{20} = \frac{4275}{8000} = \frac{171}{320}$

36. If the probability that an engine fails is 0.01, then the probability of an engine working is 1- 0.01 = 0.99.

(a) Probability of a successful flight with 2 engines is equal to one or two engines working which is the complement of no engines working.

$$1 - P(\text{no engines working}) = 1 - (0.01 \cdot 0.01)$$
$$= 1 - (0.0001) = 0.9999$$

(b) Probability of a successful flight with 4 engines is two or three or four engines working which is the complement of no engines working or one engine working.

$P(\text{no engines working}) = 0.01\cdot 0.01\cdot 0.01\cdot 0.01 = 0.00000001$

$P(\text{one engine working})=$

$P(\text{1st engine working})=0.99\cdot 0.01\cdot 0.01\cdot 0.01=0.00000099$

$+P(\text{2nd engine working})=0.01\cdot 0.99\cdot 0.01\cdot 0.01=0.00000099$

$+P(\text{3rd engine working})=0.01\cdot 0.01\cdot 0.99\cdot 0.01=0.00000099$

$+P(\text{4th engine working})=0.01\cdot 0.01\cdot 0.01\cdot 0.99=0.00000099$

=0.00000099+0.00000099+0.00000099+0.00000099

=0.00000396

$P(\text{no engines working})$ or $P(\text{one engine working})=$

0.00000001+0.00000396=0.00000397

P(successful flight)=1-0.000000397=0.99999603.

37. The probability of Betty getting the same number of heads as Al is equal to $\frac{3}{8}$. Betty has four possible outcomes (HH, HT, TH, TT), Al has 2 possible outcomes (H or T). Together there are 8 possible outcomes (Al tosses heads then Betty tosses HH, HT, TH, or TT; Al tosses tails and Betty tosses HH, HT, TH, or TT). Out of these 8 possible outcomes 3 have the same number of heads (One head: Al tosses Heads, Betty tosses HT or TH; No head: Al tosses Tails, Betty tosses TT).

CHAPTER 10

DATA ANALYSIS/STATISTICS: AN INTRODUCTION

Assessment 10-1A: Designing Experiments/Collecting Data

This entire assessment is subject to varying answers. Each of the following are representative possibilities.

1. Among the questions the class must determine are:

(i) Do you count the houses all around the block or only on the side of your house?

(ii) Are the houses across the street from you on your block, and do you count them?

(iii) What happens if you are in a new part of town where the blocks are not developed yet?

(iv) Do you count any businesses that might be on your block?

Data to be collected will be determined by the questions asked, but in the second grade will likely be a frequency count.

The frequency count could be shown in a histogram or a bar graph. Any interpretations would be made about the graph.

2. Among the questions the class must determine are:

(i) What is the definition of “active?”

(ii) What are the types of activities; does one group do them more than the other?

(iii) Are these leisure or working activities?

(iv) Does one group sleep more than the other?

(v) What is an “adult?”

Both adults and sixth-graders would be represented in a middle school. Randomization is important but probably could not be accomplished on a large scale in this population, which could include some or all of the sixth-graders and some or all of the adult faculty and administrators.

3. Among the questions the class must determine are:

(i) What is the definition of “classroom?” Does the library count as a classroom? What about auditoriums, the lunch room, the gymnasium, offices, hallways, etc.?

(ii) When will the temperature be measured in each location…same time in every room? Or, will the measure be a maximum temperature for the room during the class day, regardless of time? Will it be measured several times per day in each room?

(iii) What will be used to measure the temperature? How accurate does the measure have to be?

(iv) Is there a temperature baseline?

4. The sample in **(i)** is more likely to be random since it assures that students in different grades are asked. The sample in **(ii)** is likely to be biased because friends sitting together in the library are likely to be in the same grade and may have similar tastes.

5. The question in **(a) is fair** because it does not make any assumptions about favorite subjects. The question in **(b)** is **biased** because it uses words such as “refreshing” and “ice-cold…on a hot day” to suggest a person should like soda. The question in **(c)** is **fair** because it is not making any assumptions about the subject, and is not using any persuasive words.

6. **(a)** Elementary students might simply blindfold the adults and hand them each two different cans to see if they really could tell the difference.

(b) Older and more sophisticated students might pour the sodas into unmarked cups and use a series of taste tests to counter random guessing. A double-blind test might eliminate unintended hints.

7. Among the questions the class must determine are:

(i) What is the definition of “visiting?” An airline layover? Overnight? An extended vacation?

(ii) What is the definition of “a country?” Would Monte Carlo qualify?

(iii) If a country is now separated from another, such as with the former Soviet Union, are visits before the breakup to now-independent states counted?

(iv) If a country was a protectorate of another state when a visit occurred, does it count?

8. A strong criticism of the prediction is that unrepresentative voters were contacted. Not many owned automobiles and/or telephones in 1936.

9. How are "positive" and "negative" comments determined? A better and fairer way would be to list a representative sample of comments on the website.

10. **(a)** These students might arrive early for a specific reason (they all rode the same bus, they need to report to detention or a sport/activity), and might over-represent a certain segment of students.

(b) Students who line up in a row do so oftentimes for a specific reason; they are from one class, or they are a large group of friends, etc. In any case, the potential to over-represent a certain segment of students is high.

(c) Not all students from the school will attend the soccer game (for example, students who work Friday nights, or students who do not have transportation, or students who simply do not like soccer). In any case, the sample would not be random.

(d) Not all students will bother to take the time to fill out the questionnaire and drop it in the drop box. Usually, only students who feel strongly about a topic will take the time to respond to it in this manner. Also, there is no guarantee that each student who puts a questionnaire in the drop box filled out exactly one.

11. It must be determined how accurate the second-grade observations really are. Second graders may not actually see what shoes are worn, but instead use personal knowledge of each other; e.g., "Alphie always wears tennis shoes."

It might be observed that on Tuesday the most popular are tennis shoes and crocs, but an equally valid interpretation might be that the students just prefer soft-soled shoes.

12. **(a)** The population is the soccer fans at the stadium. The sample is the 50 fans surveyed. The sample is representative of the population.

(b) The population consists of the students who attend that particular school. The sample is the 75 students surveyed. The sample is **not** representative of the population, since only those students who show up early are surveyed (they may all come on one bus, for example, or all live near the school). It also misses any student who may enter the school by other means than the main gate.

(c) The population under study are the households in the town. The sample consists of the students who attend that school that day. The sample is **not** representative of the population, since only students are surveyed. This eliminates any household that does not have students attending that particular school.

(d) The population under study is the students at the school. Sample: 5 students from each grade. The sample **may or may not be** representative of the population. For example, if the school had 150 first graders and only 30 sixth graders, taking 5 students from each class, even if done randomly, would not reflect the population at large. However, if the number of students in each grade are approximately the same, then the method would be representative of the population.

Assessment 10-1B

1. Among the questions the class must determine are:

(i) What is the definition of "pets?" Do you count all animals that are at one residence? Just the animals that live inside? Or just the animals owned by the person who is being surveyed?

(ii) What does "owned" mean? Suppose an animal just "visits" each day but isn't technically owned by anyone in that house…does that animal count?

(*iii*) Should pets be counted that recently died? If so, how do you define "recent"; within the last week, last month, last year, other?

(iv) How will the students survey the other grades? Will it be a census, or will they randomly select a representative amount of students from each grade? If they select them randomly, how will they ensure that it is an unbiased survey?

Data to be collected will be determined by the questions asked. Since this is a second grade classroom, a frequency count would be the most logical way to organize the collected data. The frequency could then be displayed as a bar graph or a pictograph. Interpretations and analysis would be based on analyzing the graph.

2. Among the questions the class must determine are:

(*i*) Is this too sensitive a question for accurate answers to be received? How could the questions be worded so as to not use the word "lazy?"

(*ii*) What is the definition of "lazy?" Does it mean that some in the population participate in the same activities, but to a greater or lesser extent?

(*iii*) What is the definition of "adult?"

(*iv*) What populations and sample sizes should be used?

(*v*) Is sleep part of being lazy? Some adults need more sleep than others or than students.

3. Among the questions the class must determine are:

(*i*) What is "affluent?" Would all those questioned have the same definition?

(*ii*) Would affluence be defined by the government or by a yardstick developed in the class?

(*iii*) Who should ask the questions? A neutral observer or someone from the class?

(*iv*) How could the question of affluence be asked without embarrassing some in the population?

4. The sample in question **(i)** is more likely to be random since most students eat lunch and thus a variety of students will be asked. The sample in question **(ii)** is likely to be biased by including mostly very studious people.

5. Question **(a)** is **biased** because it suggests that all exploration is a United States tradition, and that it is "great". It also states only continuing funding, not any alternative. Question **(b)** is **fair** because it makes no assumptions about whether the person being asked has a clothing preference. Question **(c)** is **biased** because the question is leading with information that might skew the opinion of those being surveyed (for example, nothing is mentioned about how much, in dollars, of the school budget goes to the sciences, nor is it mentioned how much funding currently goes to athletics). Question **(d)** is **fair** since it uses no persuasive language or does it make any assumptions about the subjects being surveyed.

6. What kinds of chocolates would be used? Milk chocolate? Dark chocolate? Pure chocolate or that with nuts mixed in?

What are "adults?" Who would choose the test participants?

(a) Elementary students might simply blindfold some adults and give them samples to see if they could in fact identify them.

(b) More sophisticated students would want to choose participants randomly. Perhaps they would want to conduct a double-blind test by disguising the samples.

7. Among the questions the class must determine (similar to those in question 5 of 10-4A) are:

(*i*) What is a "sporting event?" A team sport? One that requires a ticket to attend? An event performed by professionals, or college players, or high school players? What about practices or scrimmages; do they count? How about non-traditional sports, such as disc golf or rodeo or hunting, do they count?

(*ii*) Does attendance count if you are a participant in the event?

8. Many factors could have affected the game; e.g., health of the players, pre-game preparations, effect of playing on a neutral field, Super Bowl hype, etc.

9. Each choice is "good" or better, not a likely reflection of the class. In any group, there would be some negative factors.

10. **(a)** The sample probably would not be representative. In many elementary schools the students eat by grade level (with the younger students oftentimes going first), so the sample would not be unbiased.

(b) This would be representative of the students at the school, since it was a random sample; each student at the school had an equally likely chance of being chosen for the sample.

(c) This would probably not be representative, unless the number of students in each of the five areas of campus were almost the same. If the number of students varied (for example, one area of campus has 100 students, while another area has 15), one area might be underrepresented while another area would be over-represented.

(d) This would probably not be representative. For example, the teachers may not pick those students at random. Even if the students were chosen at random, if there are more than ten teachers at the school, then those students in the classes whose teachers were not asked to provide students for the survey would be left out.

11. It is reasonable for students to talk about the most popular types of shoes worn and the least popular. They likely will begin to identify different people's shoes, but that is added information that they might already know that has nothing to do with the graph. It is important that they recognize what the graph says and what they are adding to the conversation from outside knowledge.

12. **(a)** The population is the students at a particular school. The sample is the 50 students surveyed. This sample is not representative of that population, since only those who separate their recyclables at lunch time are being asked.

(b) The population is the teachers and staff at a particular school. The sample in this case is a census – everyone in the population – so it is definitely representative of the population.

(c) The population is the students at a particular school. The sample are the 50 students surveyed. This is not a representative sample of the population, since only those students who go to the library on Monday will be surveyed.

(d) The population consists of the cars in the city. The sample are the cars that happen to drive through the thoroughfare connecting the largest residential district with downtown. This sample is not representative of the population, since only cars who drive on that route will be tallied; most likely, they are driven by those people who live in only that one residential area. Also, there is no guarantee that all of the cars on that route are from the city; there could be cars who drive on that route that are from outside the city.

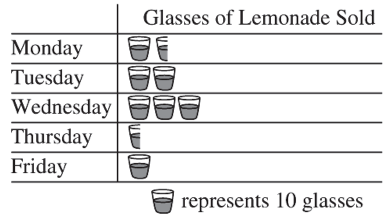
Assessment 10-2A: Displaying Data: Part I

1. **(a)** **Tuesday**.

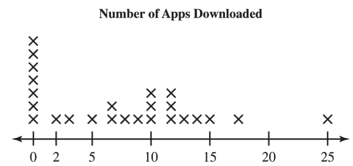
(b) There were approximately 4500 pieces of mail processed on Tuesday and 3000 on Monday. $4500 - 3000 = \mathbf{1500}$.

(c) **Three**.

2. In the following pictograph, each half-glass symbol represents 5 glasses:



3.

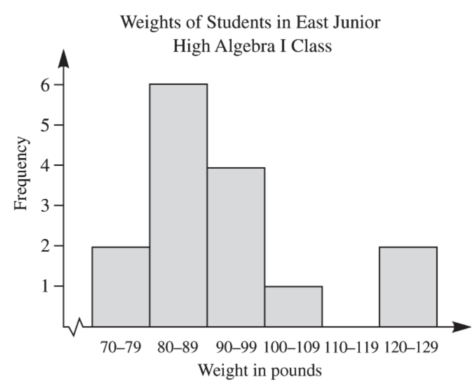


4. **(a)** If 7|24 represents weights of 72 and 74 pounds, weights of the fifteen students are: 72, 74, 81, 81, 82, 85, 87, 88, 92, 94, 97, 98, 103, 123, and 125 pounds.

(b) 72 pounds.

(c) 125 pounds.

5. The histogram below groups weights into 10 pound classes:



6. Data may vary:

(a) A dot plot (line plot) would have six columns of x's, one for each number on a die. Given a fair die, the six columns would be expected to have approximately the same number of x's, with some variation.

(b) A bar graph would have six bars, one for each number on a die. The scale on the vertical axis should be uniform, and be large enough to show the frequencies of all bars. Given a fair die, the six bars would be expected to be approximately the same height, with some variation.

7. (a) **November** (with the highest bar) had the most rain-fall; approximately **30 cm**.

(b) 15 cm (October) + 25 cm (December) + 10 cm (January) = **50 cm**.

8. (a) Ages of HKM employees:

1	889
2	01113333334566679
3	224447
4	1115568
5	2248
6	233

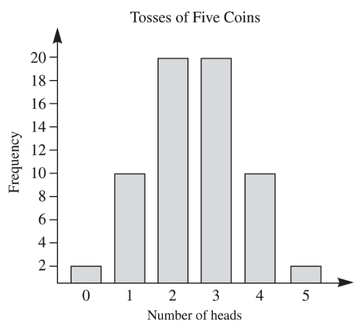
3|4 represents 34 years old

(b) There are more employees in their **40's**. The seven in their 40's are shown by 4|1115568; the four in their 50's are shown by 5|2248.

(c) **20**: three in their teens; seventeen in their 20's.

(d) Seven of the 40 total are 50 or older, $\frac{7}{40} = 0.175 = \mathbf{17.5}\%$.

9. The bar graph below reflects that two or three heads showed up most frequently. Zero or five heads showed up the least number of times.



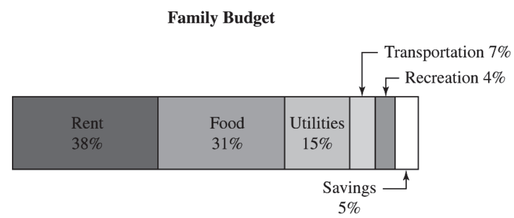
10. Answers may vary slightly. First, determine the approximate percentage of the bar each item should take up.

$2050 for rent $\approx$ 38%;
$1700 for food $\approx$ 31.5%;
$800 for utilities $\approx$ 14.8%;
$400 for transportation $\approx$ 7.4%;
$200 for recreation $\approx$ 3.7%;
and
$250 for savings $\approx$ 4.6%.

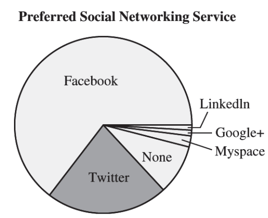
Next, calculate the length of the three-inch rectangle for each percentage:

$2050 for rent $\approx$ 38%; about 1.14 inches
$1700 for food $\approx$ 31.5%; about 0.94 in.
$800 for utilities $\approx$ 14.8%; about 0.44 in.
$400 for transportation $\approx$ 7.4%; about 0.22 in.
$200 for recreation $\approx$ 3.7%; about 0.11 in.
and
$250 for savings $\approx$ 4.6%; about 0.14 in.

Finally, use a ruler to approximate each length on the three inch rectangle and draw the smaller rectangles inside to represent each amount. Note that percentages below are rounded to the nearest whole percentage.



11.



12. **(a)** We need to find out what percentage of the total area (100%) that a sector containing $65°$ would contain. So, $\frac{65°}{360°} = \frac{x}{100}$; solving for x, you get approximately 18%.

(b). Using the proportion $\frac{x°}{360°} = \frac{32}{100}$, by solving for x, the solution is $115.2°$

13. (a) A $\frac{1}{4}$ inch rectangle would be one-eighth of the length of two inches. One-eighth of 100% is 12.5%. Another method of solution is to set up the proportion $\frac{0.25"}{2"} = \frac{x}{100}$ and solve for x.

(b) Using the proportion $\frac{x"}{2"} = \frac{16}{100}$, solving for x yields a solution of 0.32 inches.

14. Answers are rounded to the nearest whole number.

black: 24% of 241 is $\approx$ 58;
navy blue: 22% of 241 is $\approx$ 53;
white: 20% of 241 is $\approx$ 48;
gray: 17% of 241 is $\approx$ 41;
maroon: 12% of 241 is $\approx$ 29;
other: 5% of 241 is $\approx$ 12;

15. **(a)** Answers may vary. To have a population where, by far, more people in their 70's than in any other age group choose the same type books could mean a more homogeneous group. Perhaps they live together. Additionally, to have more people reading in their 90's than in their 80's is cause for suspicion. In any event, it is questionable data and more information is needed in order to make any reasonable conjecture.

(b) The graph only shows the frequency of choice for a book type—not the number of readers. The mode—i.e., the group having the greatest frequency of choice—is the 70's decade.

16. **(a)** Ordered stem-and-leaf plot showing fall textbook costs. Since each cost is rounded to the nearest $10, the "ones" place is the same for each amount. So, the stem will consist of the "hundreds" digit, and the leaf will consist of the "tens" digit:

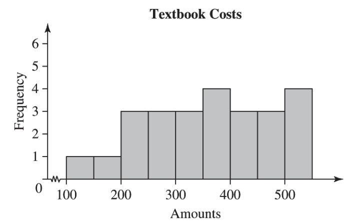
1	3 6
2	2 4 4 5 5 6
3	2 4 4 5 5 5 6
4	2 4 4 5 5 6
5	3 4 4 4

1|3 represents $130

(b) Grouped frequency table showing fall textbook costs:

Classes	Tally	Frequency
$100 –149	|	1
$150–199	|	1
$200–249	|||	3
$250–299	|||	3
$300 –349	|||	3
$350–399	||||	4
$400 –449	|||	3
$450–499	|||	3
$500–549	||||	4

(c) The bar graph below shows the number of students on its vertical axis and the dollar amount paid for their textbooks on the horizontal axis:



17. **(a)** **Double bar graph**

(b) **Women,** from approximately 55 years in 1925 to approximately 79 years in 2005. Men's life expectancy has changed only from about 53 years in 1925 to about 73 years in 2005.

(c) **One to two years** (as closely as can be approximated from the graph).

(d) **Five to six years** (as closely as can be approximated from the graph).

18. Measures are rounded to the nearest tenth of a centimeter.

Savings: 10% of $8\,\text{cm} = \mathbf{0.8\,cm}$.

Rent: 30% of $8\,\text{cm} = \mathbf{2.4\,cm}$.

Food: 12% of $8\,\text{cm} = 0.96\,\text{cm} \approx \mathbf{1.0\,cm}$.

Auto payment: 27% of $8\,\text{cm} = 2.16\,\text{cm} \approx \mathbf{2.2\,cm}$.

Tuition: $100\% - (10 + 30 + 12 + 27)\% = 21\%$ of $8\,\text{cm} = 1.68\,\text{cm} \approx \mathbf{1.7\,cm}$.

19. **(a)** **New Jersey**.

(b) **Montana**.

(c) Around **$34,500**.

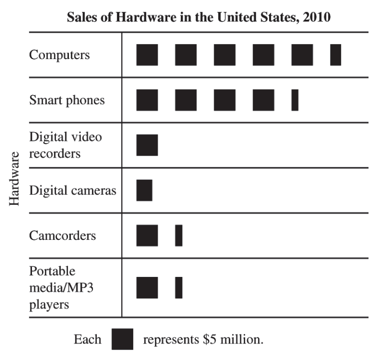
(d) **Answers will vary**, depending upon how the bar graph is read. Based on the bar graph, New Jersey pays about $48,500; Montana pays about $27,500.

$\frac{48{,}500-27{,}500}{27{,}500} = \dfrac{21{,}000}{27{,}500} \approx 0.76363$, so approximately 76%. Note that the actual starting salaries in 2012-13 are: New Jersey $48,631 and Montana $27,274, for an approximately 78% difference.

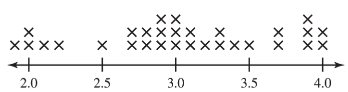
Assessment 10-2B

1. Boys soccer, 4500; Girls soccer, 2250; Boys basketball, 4750; Girls basketball, 4000.

2. Sales of Hardware in United States, 2010

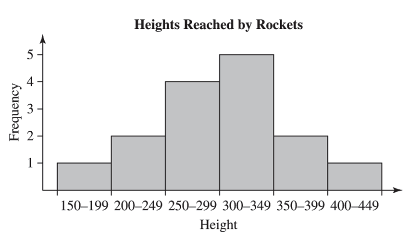


3. GPA's for Washington University's Track Team



4. **(a)** 190 feet.
   (b) 402 feet.
   (c) $\dfrac{8}{15} \approx 53\%$.

5.



6. Answers will vary.
   (a) A dot plot will have two columns of x's, one for each outcome, heads or tails.
   (b) A bar graph will have 2 bars, one for heads and one for tails. The vertical axis will be partitioned to show frequencies of each.

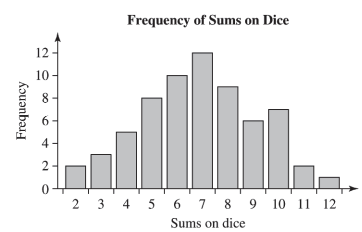
7. **(a)** The length of the Mississippi bar is about $\frac{8}{10}$ of the way between 3000 and 4000, so the Mississippi is **about 3800 km** long.
   (b) The length of the Columbia bar is about $\frac{9}{10}$ of the way between 1000 and 2000, so the Columbia is **about 1900 km** long.

8. **(a)**

1	
2	
3	5
4	8 9
5	0 2 3 3 5 6
6	2 5 6 6 9
7	2 2 5 8 8
8	2

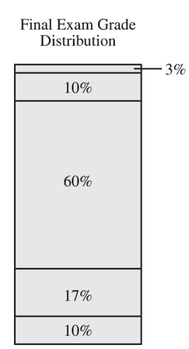
   (b) $\dfrac{6}{20} = \mathbf{30\%}$.

9.

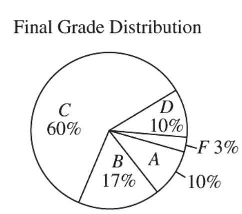


10. Final examination grade distribution, where:

$$A = \frac{6}{60} \approx \mathbf{10}\% \text{ of } 360° = 36°;$$
$$B = \frac{10}{60} \approx \mathbf{17}\% \text{ of } 360° = 60°;$$
$$C = \frac{36}{60} \approx \mathbf{60}\% \text{ of } 360° = 216°;$$
$$D = \frac{6}{60} \approx \mathbf{10}\% \text{ of } 360° = 36°; \text{ and}$$
$$F = \frac{2}{60} \approx \mathbf{3}\% \text{ of } 360° = 12°.$$



11.



12. **(a)** **Asia**, at 30% of the earth's land mass.

(b) **Africa**, at 20% of the earth's mass versus Antarctica's 9.5%.

(c) $\frac{20\% \text{ (Africa)}}{30\% \text{ (Asia)}} \Rightarrow$ Africa is $\mathbf{\frac{2}{3}}$ **as large** as Asia. It could also be said that Asia is 50% larger than Africa.

(d) **Asia and Africa**: 30% (Asia) $+$ 20% (Africa).

(e) $\frac{5\% \text{ (Australia)}}{16\% \text{ (North America)}} = \frac{5}{16}$, or **5:16**.

(f) If Europe's 7% of total area $=$ 4.1 million square miles, then total area $= \frac{4.1 \text{ million}}{0.07}$, or about **58.6 million square miles**.

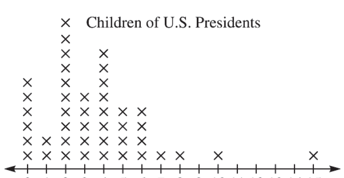
13. **(a)** We need to find out what percentage of the total area (100%) that a sector containing $82°$ would contain. So $\frac{82°}{360°} = \frac{x\%}{100\%}$; solving for x, the answer is **approximately 23%**.

(b) Using the proportion $\frac{x°}{360°} = \frac{0.05\%}{100\%}$, by solving for x the solution is $0.18°$

14. **(a)** If a 5 centimeter rectangle is split into 10 equal pieces of 0.5 cm each, (so each piece is 10% of the rectangle), a 1.5 cm rectangle would be three of the ten pieces. So, that means a 1.5 cm rectangle would be 30% of the entire rectangle. Another method of solution is to set up the proportion $\frac{1.5}{5} = \frac{x\%}{100\%}$ and solve for x.

(b) Based on the work in part (a), we know that a 1 cm rectangle would take up 20% of the rectangle. Using proportions, we would have $\frac{x}{5} = \frac{20\%}{100\%}$; solving for x would arrive at the same answer of 1 cm.

15. **(a)** A dot plot of the number of children of U.S. presidents is shown below:



(b) A frequency table is shown below:

Num. of Children	Tally	Freq	Num. of Children	Tally	Freq
0	𝍸 𝍷	6	9		0
1	𝍷𝍷	2	10	𝍷	1
2	𝍸 𝍸	10	11		0
3	𝍸	5	12		0
4	𝍸 𝍷𝍷𝍷	8	13		0
5	𝍷𝍷𝍷𝍷	4	14		0
6	𝍷𝍷𝍷𝍷	4	15	𝍷	1
7	𝍷	1	Total		42
8	𝍷	1			

(c) Ten presidents had **two children**.

16. Answers vary, for example,

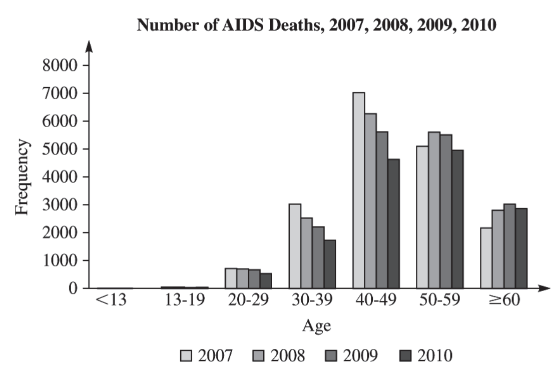
(a) The greatest profit was in the **4th** quarter: \$120,000-\$80,000. The greatest loss was in the **3rd** quarter: \$78,000-\$80,000.

(b) **Profit.** The profit in either of the 1st,2nd, or 4th quarters was greater than the amount of loss in the 3rd.

(c) This is true for the quarters that are shown. We do not know about other quarters.

17. Answers may vary.

(a) The bar graph below graphically illustrates the data.



(b) AIDS deaths appears to be decreasing among all age groups of Americans, although in the four year span, AIDS deaths did inrease for the oldest age group (greater than age 60).

18. **(a)** **5**

(b) **7**

(c) For example, the first bar estimates that 8 people read 1 book; the second, 5 people read 4 books; the third, 7 people read 7 books; and the four, 1 person reads 10 books.
$$\frac{8(1)+5(4)+7(7)+1(10)}{8+5+7+1} \approx 4 \text{ books.}$$

Mathematical Connections 10-2: Review Problems

16. **(a)** This is a **fair** question because there is no hint of the researcher's or others' opinion.
(b) This is a **biased** question because agreeing with most people might seem like a better choice.
(c) This is a **fair** question because there is no hint of the researcher's or others' opinion.
(d) This is a **biased** question because agreeing with the researcher might seem like a better choice.
(e) This is a **biased** question because agreeing with the researcher might seem like a better choice.
(f) This is a **fair** question because there is no hint of the researcher's or others' opinion.

17. Method (i) is biased in that you might only get the opinions of the very satisfied or unsatisfied, as only those with a strong opinion are probably going to mail a postcard back to the restaurant. Method (ii) has similar issues as (i); only those patrons who have a strong opinion will bother visiting the web site to fill out a survey. Method (iii) is probably the least biased of the methods listed here. It is as convenient as possible for patrons to complete and drop off the survey, meaning a better cross-section of patrons will fill it out. Offering the possibility of a prize also encourages all patrons to fill it out. Method (iv) might have some issues, simply because of who is asking the questions. Patrons might be less honest/rate more positively their experience simply because the pollster is working for the restaurant.

Assessment 10-3A Displaying Data: Part II

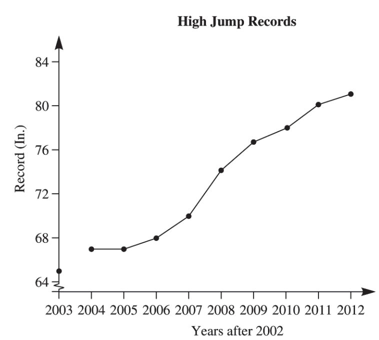
1. **(a)** 70% of $12,000, or **about $8400**.
(b) (100 − 30)% of $20,000 = 70% of $20,000 in depreciation, or **about $14,000**.
(c) 35% of $20,000, **about $7000**.
(d) Right after **two years**. Average trade-in value is about 55% at two years.

2. **(a)** The census that was taken in the year 1800 appears to be just slightly below 6 million people. So, an estimate of when the population was about 6 million people could be **1802**.
(b) 1830 corresponds to about 13, which represents **13,000,000**.
(c) The slope of the line represents change in population over change in years. Since the slope is steeper from **1810 to 1820** the population increased more during this period.

3. **(a)** **January** corresponds to the largest vertical values on both graphs.
(b) **March**.
(c) Although other factors might influence sales, in 4 of the five months illustrated, more snow shovels were sold in **2015**.

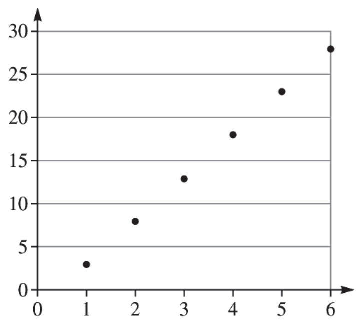
4. **(a)** **July**.
(b) **(i)** is **false**, since in December, the cost of gasoline in New York was higher; **(ii)** is **true**, since the cost of gasoline from September (the beginning of fall) to December (the end of fall) decreases in both states; **(iii)** is **false**, as the price of gasoline in California is higher; and **(iv)** is **false**, as there are several months in the summer where the cost of gasoline in New York is above $3.50 per gallon.

5.

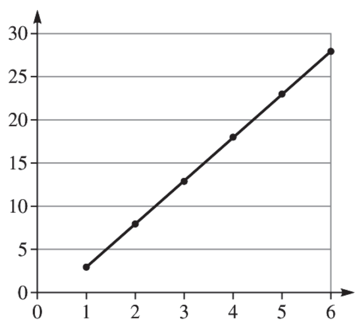


6. **(a)** **Negative**. The trend line slopes down from left to right.
(b) The number of movies point corresponding to age 25 is **about 10**.
(c) The age point corresponding to 16 movies seen is **about 22**.

7. **(a)** The first six terms and their corresponding values [(1,3), (2,8), (3,13), (4,18), (5,23), and (6,28)] are plotted on the following scatterplot, showing each term on the vertical axis (y) with the value of the term on the horizontal axis (x):



(b) Trend line, depicting the scatterplot points as a set of continuous data:



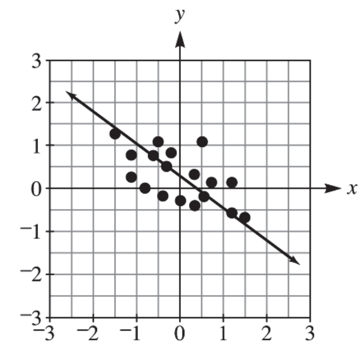
(c) Use a spreadsheet approach, where values of x are in column A and values of y are in column B:

A	B
1	3
2	8
3	13
⋮	⋮

Each time the value of the x-coordinate increases by 1 the value of the y-coordinate increases by $5 \Rightarrow$ the y-coordinates increase at 5 times the rate of increase of the corresponding x-coordinates; i.e., $y = 5x$. The trend line must be lowered by 2, though, since the first term is 3, not 5. Thus the equation is $y = \mathbf{5}x - \mathbf{2}$.

8. **(a)** (*i*) The points generally decrease when moving from left to right, so the association is **negative *x*.**

(*ii*) Answers may vary. A possible trend line is:



(b) There is no apparent trend.

9. Answers may vary.

(a) Two representative sets of coordinates appear to be

x	y
$^{-}2$	$^{-}1$
3	1

When the x-coordinates of these points go from $^{-}2$ to 3, an increase of 5, the y-coordinates go from $^{-}1$ to 1, an increase of 2. So, we have a slope of $\frac{2}{5}$, which means we know that $y = \frac{2}{5}x.$, or $y = 0.4x$. The trend line must be lowered by about 0.2, as nearly as can be determined by looking at the graph; so our equation could be $y = \mathbf{0.4}x - \mathbf{0.2}$.

(b) Two representative sets of coordinates appear to be

x	y
$^{-}1$	1
1	2

When the x-coordinates of these points go from $^{-}1$ to 1, an increase of 2, the y-coordinates go from 1 to 2, in increase of 1. So, the slope is $\frac{1}{2} = 0.5$. The trend line must be raised by about 1.5, as nearly as can be determined from the graph, so the equation would be $y = \mathbf{0.5}x + \mathbf{1.5}$.

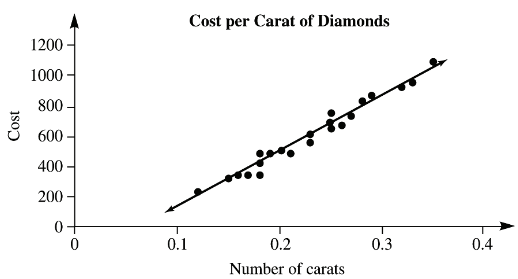
(c) Two representative sets of coordinates appear to be

x	y
$^{-}3$	$^{-}2$
1	1

When the x-coordinates of these points go from $^{-}3$ to 1, an increase of 4, the y-coordinates go from $^{-}2$ to 1, an increase of 3. So the slope is $\frac{3}{4} = 0.75$. The trend line must be raised by about 0.25, as nearly as can be determined from the graph, so the equation would be, $y = 0.75x + 0.25$.

10. Answers may vary.

(***i***)



(***ii***) With so many data points it is difficult to draw a relatively accurate trend line; also, extrapolation (estimation of a data point outside the range of the data) is always fraught with difficulty; but, one could estimate that the cost of a 0.5 carat diamond should be in the neighborhood of \$1600 to \$1700.

11. If the equation is $y = 3.2x - 0.11$:

(a) $y = 3.2(1) - 0.11 = 3.09$

(b) $y = 3.2(0) - 0.11 = -0.11$

(c) $y = 3.2(10) - 0.11 = 31.89$

(d) $y = 3.2(20.3) - 0.11 = 64.85$

12. **Negative**. Values of y will decrease as values of x increase.

Assessment 10-3B

1. **(a)** (***i***) **22**

(***ii***) **20.4**

(***iii***) **25.4**

(b) Between **1940 and 1950**, the slope of the line connecting the ordered pairs corresponding to these years is steepest (in the negative direction) than between any other two ordered pairs.

(c) **1980–1990** for similar reasons as in 1(b).

2. **(a)** The number of workers per beneficiary will drop to only 1.9 workers/beneficiary in 2050, but we do not know anything about the data after 2050.

(b) About **5**.

(c) About **2**.

3. **(a)** Answers will vary; approximately **50,000** more men than women participated in NCAA college sports in 2004–2005.

(b) The slopes of the lines connecting the ordered pairs that correspond to 2000–01 to 2000–02 are both relatively flat, indicating that the participation did not change.

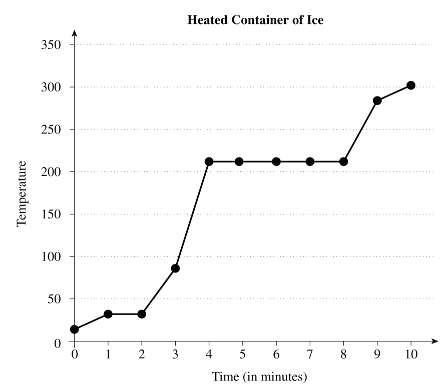
(c) The trend is positive and participation among women appears to be increasing more rapidly than among men.

4. **(a)** In both classes, the number of spelling errors decreases as the number of weeks increases.

(b) That however Mr. DiMaso is teaching spelling, it seems to be working.

(c) The eighth graders will probably make five spelling errors or less; the seventh graders will probably make about 10–15 spelling errors.

5. (***i***)

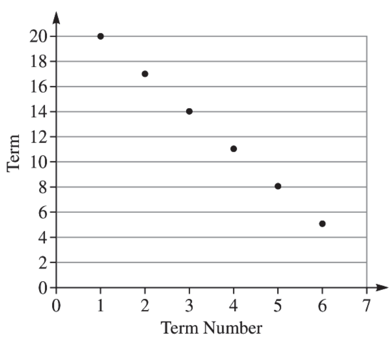


(***ii***) When the ice is heated, the temperature rises until it gets to 32° Fahrenheit. At this point, time is needed for the ice to change states (to melt), so the temperature stays the same at 32°. Once the ice turns into water, the temperature rapidly rises until it reaches 212°, which is the boiling point of water. The temperature stays the same for several minutes until the water changes states again, this time into a gas.

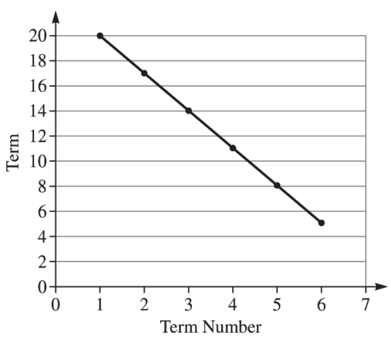
6. **(a)** **Negative association,** as weight of the vehicle increases, the fuel efficiency decreases.

(b) About **32 mpg.**

7. **(a)** Scatterplot, showing each term on the vertical axis (y) with the value of the term on the horizontal axis (x):



(b) Trend line, depicting the scatterplot points as a set of continuous data:



(c) Use a spreadsheet approach, where values of x are in column A and values of y are in column B:

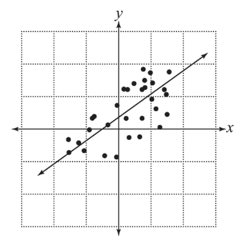
A	B
1	20
2	17
3	14
⋮	⋮

Each time the value of the x-coordinate increases by 1 the value of the y-coordinate decreases by 3 $\Rightarrow$ the y-coordinates increase at 3 times the rate of increase of the corresponding .x-coordinates; i.e., $y = -3x.$, Using this pattern, we can figure out that the y-intercept term will occur when $A = 0$ and $B = 23$. Thus the equation in slope-intercept form is $y = -3x + 23$.

8. Trend lines may vary, but should show an average value through the data points.

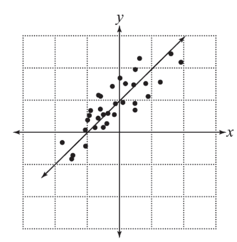
(a) (***i***) The data points generally increase when moving from left to right, so the association is **positive**.

(***ii***)



(b) (***i***) The points generally increase when moving from left to right, so the association is **positive**.

(***ii***)



9. Answers may vary.

(a) Two representative sets of coordinates appear to be:

x	y
$^-3$	$^-1.8$
2	1

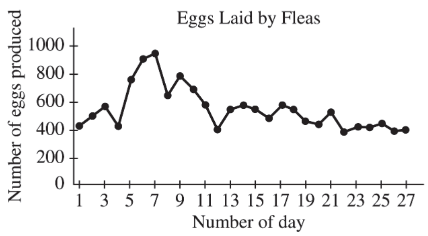
When the x-coordinates of these points go from $^-3$ to 2, an increase of 5, the y-coordinates go from $^-1.8$ to 1, an increase of 2.8. So, the slope is $\frac{2.8}{5} = 0.56$. The trend line must be lowered by about 0.1, as nearly as can be determined from the graph, so the equation would be $\boldsymbol{y = 0.56x - 0.1}$.

(b) Two representative sets of coordinates appear to be:

x	y
$^-2$	1
1	0

When the x-coordinates of these points go from $^{-}2$ to 1, an increase of 3, the y-coordinates go from 1 to 0, a decrease of 1. So the slope is $\frac{-1}{3}$ The trend line must be raised by about $\frac{1}{3}$, as nearly as can be determined from the graph, so the equation would by $\boldsymbol{y = \frac{^{-}1}{3}x + \frac{1}{3}}$.

10. **(a)**



(b) Probably **around 400**. The numbers from day 23 to day 27 are relatively constant, hovering close to 400. A graphing calculator would approximate the data to be about 410 inches.

11. If $y = -3.2x + 0.11$:

(a) If $y = -3.2(10) + 0.11 = -31.89$

(b) If $y = -3.2(0) + 0.11 = 0.11$

(c) If $y = -3.2(4.3) + 0.11 = -13.65$

(d) If $y = -3.2(7.2) + 0.11 = -22.93$

12. **Positive**. Values of y will increase as values of x increase.

13. The data is fairly close to being constant, but it is showing a slight positive trend. As the values of x increase, y will increase very slowly. For example, at $x = 0$, $y = 12$; at $x = 1000$, $y = 12.1$.

14. **(a)** There would be a **positive association**. For people with larger right hands, they should have larger left hands.

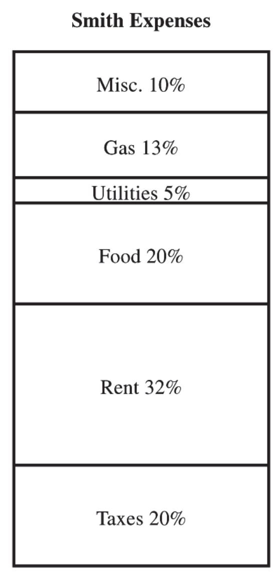
(b) There would be a **positive association**. As the temperature in the summer would increase, one would expect more ice cream to be sold.

(c) There would be a **negative association**. As the number of vaccines given in a city increased, the incidence of illness should decrease.

Mathematical Connections 10-3: Review Problems

8. The data sums to 102%, or more than 360° of their circle graph. The family needs to redo their calculations.

9. Reducing the miscellaneous category to 10% produces the stacked bar graph below:



10. **(a)** Mark McGwire Home runs by season.

0	**399**
1	
2	**29**
3	**22399**
4	**29**
5	**28**
6	**5**
7	**0**

(b) There are three very low totals which could be considered outliers: 3, 9, 9. One would suspect McGwire didn't play many games in those seasons. Most home run totals are between 22 and 49, with four seasons of over 50 home runs. Data is fairly well distributed.

Assessment 10-4A: Measures of Central Tendency and Variation

1. **(a)** Ordering the data: 3, 4, 6, 7, 7, 7, 7, 8, 11.

(*i*) Mean $= \frac{3+4+5+7+7+7+8+11}{8} = \frac{53}{8} = \mathbf{6.625}$.

(*ii*) Median is in the 4.5th position, or between the first 7 and the second 7. So, the median $= \mathbf{7}$; or $\frac{7+7}{2}$.

(*iii*) Mode $= 7$, the value which occurs most frequently.

(b) Ordering the data: 10.5, 11.5, 12.5, 12.5, 12.5, 14.5, 14.5, 16.5, 20.5.

(*i*)

$$\text{Mean} = \frac{10.5+11.5+12.5+12.5+12.5+14.5+14.5+16.5+20.5}{9} = \frac{125.5}{9} = \mathbf{13.9\overline{4}}.$$

(*ii*) Median is in the 5th position, which makes it the last **12.5**

(*iii*) Mode $= \mathbf{12.5}$ the value which occurs most frequently.

(c) Ordering the data: -7.4, -6.2, -2, -1.1, 1.2, 3, 4.2, 8.3.

(*i*)

$$\text{Mean} = \frac{-7.4+(-6.2)+(-2)+(-1.1)+1.2+3+4.2+8.3}{8} = \frac{0}{8} = \mathbf{0}.$$

(*ii*) Median is in the 4.5th position, or between -1.1 and 1.2; so $\frac{-1.1+1.2}{2} = 0.05$.

(*iii*) Mode; there is no mode, since each value in the data set occurs exactly once.

(d) Ordering the data: 0, 79, 80, 82, 85, 85, 93

(i) Mean $= \frac{0+79+80+82+85+85+90}{7} = 72$

(ii) Median is in the 4th position, which makes the median $= \mathbf{82}$

(iii) Mode$=$ **85**, since this is the value that occurs most frequently.

2. **(a)** Answers may vary; one example might be 1, 2, 3, 4, 5, 60, 60.

(b) Answers will vary; one example might be 1, 1, 1, 10, 90, 90, 90

3. **(a)** (*i*) $\frac{3(75)+3(88)}{6} = 81.5$

(*ii*) The median will be found in the 3.5th position, or between the last 75 and the first 88. Average those two middle values: $\frac{75+88}{2} = 81.5$

(*iii*) Bi-modal: **75 and 88**.

(b) Mean $= \frac{\text{sum of test scores}}{\text{number of test scores}}$. So, $75 = \frac{\text{sum of test scores}}{20}$. So, sum $= 75 \cdot 20 = \mathbf{1500}$.

4. Mean of 28 scores being 80 implies that the sum of scores is $28 \cdot 80 = 2240$. Adding scores or 60 and 50 yields a new sum of $2240 + 110 = 2350$. New mean $= \frac{2350}{28+2} = \mathbf{78.\overline{3}}$.

5. Mean $= \frac{4(98)+11(60)}{15} = 70.13$

6. Mean $= \frac{m\cdot 100+n\cdot 50}{m+n}$; i.e., the sum of the data divided by the number of data points.

7. The number of points for each course is the product of the number of credits and the point value of each. 15 points for math, 12 for English, 10 for physics, 3 for German, and 4 for handball gives a total of 44 points. GPA $= \frac{44 \text{ points}}{17 \text{ credits}}$, or **about 2.59**.

8. Total tackle weight is $7 \cdot 230 = 1610$ pounds. Total backfield weight is $4 \cdot 190 = 760$ pounds. Total player weight is $1610 + 760 = 2370$ pounds. Mean $= \frac{2370 \text{ pounds}}{11 \text{ players}}$, or **about 215.5 pounds**.

9. (a) The total salary for the 50 faculty is shown below:

Salary ($)	No. Faculty	Total ($)
18,000	2	36,000
22,000	6	132,000
32,000	24	768,000
48,000	15	720,000
80,000	2	160,000
150,000	1	150,000
Total	50	1,966,000

Mean annual salary $= \frac{\$1{,}966{,}000}{50} = \mathbf{\$39{,}320}$

(b) Ordering the salaries: \$18,000, 18,000, 22,000, ..., 150,000; the median is between the 20th and 21st and all salaries between the 13th and 25th are \$32,000.
Median $=$ **\$32,000**.

(c) **\$32,000**, or the most frequent salary.

10. (a) Salaries are between \$18,000 and \$150,000. The range is thus $\$150{,}000 - \$18{,}000 =$ **\$132,000**.

(b) The mean \$39,320.

Number of Faculty	Salary ($)	Absolute Deviation	No. Faculty × Absolute Deviation
2	18,000	21,320	42,640
6	22,000	17,320	103,920
24	32,000	7,320	175,680
15	48,000	8680	130,200
2	80,000	40,680	81,360
1	150,000	110,680	110,680
50			Sum = 644,480

$MAD = \frac{644{,}480}{50} = \mathbf{\$12{,}889.60}$.

(c) $\overline{x} = \$39{,}320$; $n =$ number of faculty.

x Salary ($)	$x - \overline{x}$	$(x - \overline{x})^2$	$n(x - \overline{x})^2$
18,000	-21,320	454542400	909,084,800
22,000	-17,320	299982400	1,799,894,400
32,000	-7,320	53582400	1,285,977,600
48,000	8,680	75342400	1,130,136,000
80,000	40,680	1,654,862,400	3,309,724,800
150,000	110680	12250,062,400	12,250,062,400
			20,684,880,000

$s = \sqrt{\frac{20684880000}{50}} \approx \$\mathbf{20{,}339.56}$.

(d) Upper extreme $=$ \$150,000;
Upper quartile $=$ \$48,000;
Median $=$ \$32,000;
Lower quartile $=$ \$32,000; and
Lower extreme $=$ \$18,000.
$IQR = \$48{,}000 - \$32{,}000 = \mathbf{\$16{,}000}$.

11. (a) Answers may vary. One possible viewpoint might be that the median/mode of \$32,000, coupled with the IQR of \$16,000 would give a realistic picture.

(b) Answers may vary. Based on the above answer for (a), we know that all but 3 salaries are within \$16,000 of \$32,000.

12. Total trip miles $= 43{,}390 - 42{,}800 = 590$. Total gasoline used $= 12 + 18 = 30$ gallons. Average fuel mileage $= \frac{590}{300} = \mathbf{19\frac{2}{3}}$ **mpg**.

13. $24 + 34 =$ **58 years old**.

14. Let S be the score needed on the fifth exam. Then $\frac{84+95+86+94+S}{5} = 90 \Rightarrow 359 + S = 450 \Rightarrow$ $S = \mathbf{91}$.

15. (a) Theater **A: \$25**; Theater **B: \$50**. The median in a box plot is the middle line through the box; i.e., 25 in the Theater A box and 50 in the Theater B box.

(b) Theater **B**. Range is the difference between upper and lower extremes. Theater A range $= 40 - 15 = 25$; Theater B range $= 80 - 15 = 65$.

(c) **\$80** at Theater B, its upper extreme. Theater A's upper extreme is \$40.

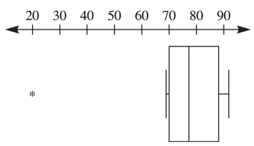
(d) Answers may vary. There is significantly more variation and generally higher prices at Theater B.

16. Ordering the data: 20, 69, 70, 72, 75, 80, 83, 88, 90, 92.
Lower extreme $= 20$; upper extreme $= 92$;
lower quartile $= 70$; upper quartile $= 88$; and
median $= \frac{75+80}{2} = 78.5$
IQR $= 88 - 70 = 18$;
$1.5 \cdot$ **IQR** $= 27$

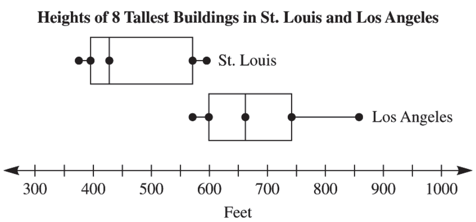
Lower quartile − (1.5 · **IQR**) = 43, so there is a lower outlier of 20.

Upper quartile + (1.5 · **IQR**) = 115, so there is no upper outlier.

Yields the following box plot:



17. (a) *L. A.*: $Q_1 = 599$; $Q_2 = 662$; $Q_3 = 742.5$.
S. L.: $Q_1 = 395$; $Q_2 = 427$; $Q_3 = 564$.



(b) Answers will vary. Of the 8 tallest buildings in each city, more than 50% at the buildings in LA are taller than those in St. Louis; the buildings at the third quartile in St. Louis are shorter than the buildings at the first quartile in L.A.

18. $\bar{x} = \frac{175+182+190+180+192+172+190}{7} = \frac{1281}{7} =$ 183 cm. The summation of $(x - \bar{x})^2$ is tabularized below:

x	$x - \bar{x}$	$(x - \bar{x})^2$
175	$^-8$	64
182	$^-1$	1
190	7	49
180	$^-3$	9
192	9	81
172	$^-11$	121
190	7	49
		Total: 374

$s = \sqrt{\frac{374}{7}} \approx$ **7.3 cm**.

19. (a) Sum of grades = 96 + 71 + 43 + 77 + 75 + 76 + 61 + 83 + 71 + 58 + 97 + 76 + 74 + 91 + 74 + 71 + 77 + 83 + 87 + 93 + 79 = 1613. $\bar{x} = \frac{1613}{21} \approx$ **76.8**.

(b) Ordering the scores: 43, 58, 61, 71, 71, 71, 74, 74, 75, 76, 76, 77, 77, 79, 83, 83, 87, 91, 93, 96, 97. Median = **76** (the 11th score).

(c) The most frequent score is 71. Mode = **71**.

(d) Upper quartile $= \frac{83+87}{2} = 85$; lower quartile $= \frac{71+71}{2} = 71$.
IQR = 85 − 71 = **14**.

(e) $(x - \bar{x})^2$ is tabularized below:

x	$x - \bar{x}$	$(x - \bar{x})^2$
96	19.2	368.64
71	$^-5.8$	33.64
43	$^-33.8$	1142.44
77	0.2	0.04
75	$^-1.8$	3.24
76	$^-0.8$	0.64
61	$^-15.8$	249.64
83	6.2	38.44
71	$^-5.8$	33.64
58	$^-18.8$	353.44
97	20.2	408.04
76	$^-0.8$	0.64
74	$^-2.8$	7.84
91	14.2	201.64
74	$^-2.8$	7.84
71	$^-5.8$	33.64
77	0.2	0.04
83	6.2	38.44
87	10.2	104.04
93	16.2	262.44
79	2.2	4.84
		Total: 3293.24

$v = \frac{3293.24}{21} \approx$ **156.8**.

(f) $s = \sqrt{v} = \sqrt{156.8} \approx$ **12.5**.

(g) The mean, $\bar{x}$, is 76.8.

x	$\lvert x-\bar{x}\rvert$
96	19.2
71	5.8
43	33.8
77	0.2
75	1.8
76	0.8
61	15.8
83	6.2
71	5.8
58	18.8
97	20.2

x	$\lvert x-\bar{x}\rvert$
76	0.8
74	2.8
91	14.2
74	2.8
71	5.8
77	0.2
83	6.2
87	10.2
93	16.2
79	2.2

Total: 189.8

MAD $= \frac{189.8}{21} \approx$ **9.04**.

20. 95% of the area under the normal curve lies with ± 2 standard deviations from the mean. The range corresponding to ± 2 standard deviations is between $65.5 - 2 \cdot 2.5 =$ **60.5** and $65.5 + 2 \cdot 2.5 =$ 7**0.5** in.

21. 1 minute is two standard deviations below the mean. Approximately 95% of the area under the normal curve lies within 2 standard deviations of the mean. So, **100%** − **95%** =**5%** of the data lies outside 2 standard deviations of the mean; since a normal distribution is symmetrical, $\mathbf{\frac{5\%}{2}} = \mathbf{2.5\%}$ of the calls should last less than 1 minute.

22. **(a)** Q_2 represents the number of scores falling below the median, which in the normal distribution equals the mean. Thus $Q_2 =$ **65**.

(b) P_{16} represents the bottom 16% of all scores, which in the normal distribution falls one standard deviation below the mean. Thus $P_{16} = 65 - 12 =$ **53**.

(c) P_{84} represents the value below which are 84% of the scores, which in the normal distribution falls one standard deviation above the mean. Thus $P_{84} = 65 + 12 =$ **77**.

(d) D_5 is the decile ranking, below which is 50% of the scores; i.e., the same as the mean in the normal distribution, or **65**.

23. \$6.75 is one standard deviation below the mean; \$8.25 is one standard deviation above the mean. About 68% of the area under the normal curve lies within ± 1 standard deviation from the mean. Therefore the probability of having a wage between \$6.75 and \$8.25 is **68%** or **0.68**

24. Two standard deviations above the mean is $79 + (2 \cdot 5.5) = 79 + 11 = 90$. Thus a **90** is the lowest score a person could receive and still earn an A.

25. About 68% of the Values 1 is within are standard deviation of the mean an a normal distribution plot. Al's score is exactly equal to the mean plus one standard deviation. Thus, $68\% + \frac{32\%}{2} = 68\% + 16\% = 84\%$ of the students scored below Al, or $.84 \cdot 10{,}000 =$ **8400 students**.

26. If Al is ranked 14th, 31 students rank below him $\frac{31}{45} = 0.6888...$ Thus, Al is in the **69th percentile**.

27. Jack's percentile ranking is $\frac{200-70}{200} = \frac{130}{200} = .65$, meaning the 65th percentile. Thus, **Jill's** standing is higher than Jack's.

Assessment l0-4B

1. **(a)** Ordering the data: 66, 78, 83, 83, 83, 83, 85, 93, 95, 95.

(***i***) Mean $= \frac{66+78+83+83+83+83+85+93+95+95}{10} = \frac{844}{10} =$ **84.4**.

(***ii***) Median $=$ **83**, the midpoint between the fifth and sixth 83.

(***iii***) Mode $=$ **83**, the value which occurs most frequently.

(b) (***i***) Mean $= \frac{2+2+2+2+2+12}{6} = \frac{22}{6} \approx$ **3.67**.

(***ii***) Median $=$ **2**, the midpoint between the third and fourth 2.

(***iii***) Mode $=$ **2**, the value which occurs most frequently.

(c) Ordering the data: $-5.4, -4.2, -4, -3.1, -1.2,\ 5,\ 6.2,\ 6.3$

(i) Mean:

$$\frac{-5.4+(-4.2)+(-4)+(-3.1)+(-1.2)+5+6.2+6.3}{8} = \frac{-0.4}{8} = -0.05$$

(ii) The median is between the 4th and 5th values, or $\frac{-3.1+(-1.2)}{2} = -2.15.$

(iii) Mode; there is no mode, since each value in the data set occurs exactly once.

(d) Ordering the data: 5.99, 7.99, 10.99, 12.99, 18.99, 19.99

(i) Mean:

$$= \frac{5.99+7.99+10.99+12.99+18.99+19.99}{6} = \frac{76.94}{6} = 12.82\overline{3}$$

(ii) The median is between the third and fourth values, or $\frac{10.99+12.99}{2} = 11.99.$

(iii) Mode; there is no mode, since each value in the data set occurs exactly once.

2. Answers vary. For example, 1, 1, 2, 2, 3, 3, 800.

3. (a) If all scores $= 80$:

(*i*) Mean $=$ **80**; i.e., $\frac{80+80+80+80+80+80}{7} = 80.$

(*ii*) Median $=$ **80**, since the midpoint is the fourth 80.

(*iii*) Mode $=$ **80**, since all values are 80.

(b) Answers may vary. One set might be $\{70, 80, 80, 80, 80, 90\}$.

4. If 50 people average $\frac{7500}{50} =$ **150 pounds,** the tram will be at capacity.

5. (a) $\frac{42+36+10+8+4}{5} = \frac{100}{5} =$ **20 years**.

(b) In ten years the family's ages will be 52, 46, 20, 18, and 14.

$= \frac{52+46+20+18+14}{5} = \frac{150}{5} =$ **30 years**.

The family's ages all increase by the same amount, so the average will also increase by that amount.

(c) Since each family member will age by 20 years, we know the average age will also increase by 20 years. So, in 20 years the mean of their ages will be **40 years**.

(d) Means will increase by the same amount as the increase in each individual data point (if those increases are the same for each data point).

6. If h students scored 100, their total points would be $h \cdot 100.$ If there were n students total, then $n - h$ scored 50 and their total points would $(n - h) \cdot 50.$ Thus Mean $= \frac{h \cdot 100 + (n-h) \cdot 50}{n}.$

7. (a) If a total of 100 points could be awarded $(60\% + 25\% + 15\%),$ then the following number could be gained in each category:

Term paper; $0.60 \cdot 85 = 51,$

Homework; $0.25 \cdot 78 = 19.5,$ and

Final; $0.15 \cdot 90 = 13.5.$

The grade would then be $51 + 19.5 + 13.5 =$ **84**.

(b) Let x be the term paper percentage, y be the homework percentage, and z be the final exam percentage. Let t be the term paper score, h be the homework score, and f be the final exam score. Then the overall grade would be $\boldsymbol{(xt + yh + zf)\%}$. Note that $x + y + z$ must equal 100%.

8. 99 people with mean of $\$13,500 = 99 \times 13,500 =$ \$1,336,500 total income. An increase of \$210,000 yields \$1,546,500 new total income. New mean income $= \frac{\$1,546,500}{99+1 \text{ people}} = \$15,465,$ or an **increase of \$1965**.

9. (a) (*i*) Balance beam - **Olga**.

Mean $= \frac{9.5+9.6+9.6+9.6}{4} = 9.575.$

(*ii*) Uneven bars - **Lisa**.

Mean $= \frac{9.8+9.8+9.9+.9.9}{4} = 9.85.$

(*iii*) Floor exercises - **Lisa**.

Mean $= \frac{9.8+9.9+10+10}{4} = 9.925.$

(b) **Lisa**, with 29.20 points.

10. **(a)** Set **A**, with mean $= \frac{24}{4} = 6$ and range $= 9 - 3 = 6$.

(b) Sets **B** and **C**.

Set B: all elements $= 11$, thus mean $=$ median $=$ mode $= 11$.

Set C: mean $= \frac{77}{7} = 11$, median $=$ mode $= 11$.

(c) Set **C**, with mean $= \frac{12}{4} = 3$; median $= \frac{2+4}{2} = 3$; and no value occuring more than once so no mode.

(d) Set **D**, with lower quartile $= 4.5$; median $= 8$; and upper quartile $= 10.5$.

11. Mean of five numbers $= 6$ implies that the sum of the numbers $= 5 \cdot 6 = 30$. Removing a number; $\frac{30 - \text{number}}{5-1} = 7 \Rightarrow 30 - \text{number} = 28$.
Number $=$ **2**.

12. See Chapter 1 of the text to review arithmetic sequences.

(a) **(*i*)** $200 = 2 + (n-1) \cdot 2 \Rightarrow$
$n = 100$ elements.
$S_{100} = \frac{100(200+2)}{2} = 10{,}100$.
$a_{50} = 2 + (50-1) \cdot 2 = 100$.
Mean $= \frac{10{,}100}{100} =$ **101**.
Median $= \frac{100+102}{2} =$ **101**, between the 50th and 51st elements.

(*ii*) $199 = 1 + (n-1) \cdot 2 \Rightarrow$
$n = 100$ elements.
$S_{100} = \frac{100(199+1)}{2} = 10{,}000$.
$a_{50} = 1 + (50-1) \cdot 2 = 99$.
Mean $= \frac{10{,}000}{100} =$ **100**.
Median $= \frac{99+101}{2} =$ **100**, between the 50th and 51st elements.

(*iii*) $157 = 5 + (n-1) \cdot 8 \Rightarrow$
$n = 20$ elements.
$S_{20} = \frac{20(5+157)}{2} = 1{,}620$.
$a_{10} = 5 + (10-1) \cdot 8 = 77$.
Mean $= \frac{1{,}620}{20} =$ **81**.
Median $= \frac{77+85}{2} = \frac{162}{2} =$ **81**, the middle element, between the 10^{th} and 11^{th} elements.

(b) The mean and median of an arithmetic sequence are the same.

13. **(a)** $66 - 21 =$ **45**.

(b) **Not necessarily**, the range can be effected by extreme values.

14. Let the three scores be represented by F (first), S (second), and T (third):
$\frac{F+S+T}{3} = 92$ (mean), or $F + S + T = 276$.
$F - T = 6$ (range), or $F = T + 6$.
$S = 90$ (median).
Then $(T+6) + 90 + T = 276 \Rightarrow T =$ **90**.
$F = 90 + 6 \Rightarrow F =$ **96**.
$S =$ **90**.

15. **(a)** About **23.7**.

(b) About **21**.

(c) The ranges are **about the same**, although the range of women's ages is slightly greater.

(d) In the sample, women tend to marry about 2.7 years younger than men. The spread of the data is about the same.

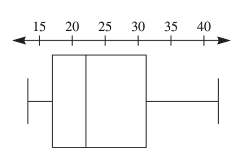
16. Ordering the data: 13, 15, 17, 17, 19, 21, 21, 23, 25, 25, 31, 33, 33, 41, 42.

Lower extreme $= 13$; lower quartile $= 17$; median $= 23$; upper quartile $= 33$; and upper extreme $= 42$.
$\text{IQR} = 33 - 17 = 16$;
$\text{IQR} \times 1.5 = 24$

Lower quartile $- (1.5 \cdot \text{IQR}) = {}^{-}7$, so there is no lower outlier.

Upper quartile $+ 1.5 \cdot \text{IQR} = 57$, so there is no upper outlier.

Yields the following box plot:



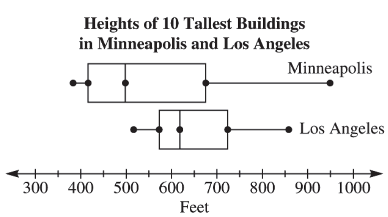
17. **(a)** Minneapolis:

Lower extreme $= 366$;
upper extreme $= 950$;
lower quartile $= 416$;
upper quartile $= 668$;
and median $= \frac{447+561}{2} = 504$.
IQR $= 668 - 416 = 252$.
Lower quartile $- 1.5 \cdot$ **IQR** $= 38$,
so there is no lower outlier.
Upper quartile $+ 1.5 \cdot$ **IQR** $= 1046$,
so there is no upper outlier.

Los Angeles:

Lower extreme $= 516$;
upper extreme $= 858$;
lower quartile $= 571$;
upper quartile $= 735$;
and median $= \frac{620+625}{2} = 622.5$.
IQR $= 735 - 571 = 164$.
Lower quartile $- 1.5 \cdot$ **IQR** $= 325$,
so there is no lower outlier.
Upper quartile $+ 1.5 \cdot$ **IQR** $= 981$,
so there is no upper outlier.

Giving the following box plot:



(b) **No**. In neither case are there values below the lower quartile minus 1.5 times the interquartile range (IQR) nor above the upper quartile plus 1.5 times the IQR.

(c) Answers may vary. Minneapolis has more spread in their ten tallest buildings and has at least one building higher than any in Los Angles. All the tallest buildings in Los Angeles are higher than over 50% of the tallest buildings in Minneapolis. All the tallest buildings in Los Angeles are between 516 and 858 feet; the highest building in either group is in Minneapolis and is 950 feet high.

18. **(a)** $s = \mathbf{0}$ because there are no deviations from the mean.

(b) **Yes**. Otherwise there would be some nonzero differences from the mean. Since these difference terms are squared some would not be zero. The standard deviation would therefore not be zero.

19. **(a)** (*i*) New mean

$$= \frac{\text{Total of old salaries} + 1000 \cdot num. \text{ of teachers}}{\text{Num. of teachers}}$$

$=$ Old mean $+$ \$1000. The mean **increases by \$1000**.

(*ii*) The median **increases by \$1000**.

(*iii*) The extremes **increase by \$1000**.

(*iv*) The quartiles **increase by \$1000**.

(*v*) The standard deviation is **unchanged**.

(*vi*) The IQR is **unchanged**.

(b) (*i*) The mean **increases by 5%**.

(*ii*) Variations about the mean would be unchanged, so the standard deviation **increases by 5%**.

20. **(a)** 85 is one standard deviation below the mean; 115 is one standard deviation above the mean. 68% of the area under the normal curve lies within ± 1 standard deviation from the mean.

68% of the students should have IQ's between 85 and 115; 68% of 1500 is **1020 students**.

(b) 70 is two standard deviations below the mean; 130 is two standard deviations above the mean. 95% of the area under the normal curve lies within ± 2 standard deviations from the mean.

95% of the students should have IQ's between 70 and 130; 95% of 1500 is **1425 students**.

(c) 145 is three standard deviations above the mean. 0.1% of the area under the normal curve is more than three standard deviations from the mean.

0.1% of the students should have IQ's above 145; 0.1% of 1500 is 1.5, or **one or two students**.

21. 16 ounces is two standard deviations below the mean; $13.5\% + 34\% + 50\% = 97.5\%$ of the area under the normal curve is above 2 standard deviations below the mean. **97.5%** of the boxes should contain more than 16 ounces.

22. **(a)** 130 is two standard deviations above the mean. $34\% + 13.5\% = 47.5\%$ of the area under the normal curve lies between the mean and 2 standard deviations above the mean. Thus **47.5%** of the population should have IQ's between 100 and 130.

(b) 85 is one standard deviation below the mean. $0.1\% + 2.4\% + 13.5\% = 16\%$ of the area under the normal curve lies below 1 standard deviation below the mean. Thus **16%** of the population should have IQ's below 85.

23. 108 ounces is one standard deviation above the mean. $50\% + 34\% = 84\%$ of the area under the normal curve lies below 1 standard deviation above the mean. Thus the probability a weight more than 108 oz $=$ **0.16**.

24. 49,000 miles is two standard deviations below the mean. 2.5% of the area under the normal curve lies below 2 standard deviations below the mean. Thus 2.5% of 1500 tires is **37 to 38 tires**.

25. A score of 775 is one standard deviation below the mean. $0.1\% + 2.4\% + 13.5\% = 16\%$ of the area under the normal curve lies below 1 standard deviation below the mean. Thus 16% of 14,000 is **2240 students**.

26. **0th percentile**; i.e., there are 0 students ranking below Steve.

27. Jill has a percentile rank of $\frac{200-80}{200} = 60$. Thus **Nathan** with a percentile rank of 80 has a higher standing.

Mathematical Connections 10-4: Review Problems

24. Small: 28% of 360° ≈ **101°**.

Midsize: 48% of 360° ≈ **173°**.

Large: 7% of 360° ≈ **25°**.

Luxury: 17% of 360° ≈ **61°**.

25. **(a)** **Mount Everest** is the highest mountain, at about **8500** meters.

(b) Mounts **Aconcagus, Everest, and McKinley**.

26. **(a)** History Test Scores:

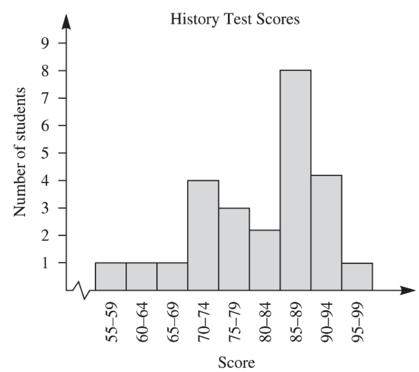
5	5
6	48
7	2334679
8	0255567889
9	00346

7 | 2 represents a score of 72

(b) History Test Scores:

Classes	Tally	Frequency
55–59	|	1
60–64	|	1
65–69	|	1
70–74	||||	4
75–79	|||	3
80–84	||	2
85–89	~~||||~~ |||	8
90–94	||||	4
95–99	|	1

(c)



(d) $\frac{8}{25} \cdot 360° \approx$ **115°**.

Assessment 10-5A: Abuses of Statistics

This entire assessment is subject to varying answers. Each the following are representative possibilities.

1. **(a)** It is not known what average was used, or how it was calculated (i.e., earn an average of \$500 a week if you work 80 hours).

(b) It's possible that as the respondents answer the questions, they become less and less accurate or interested as they work through the questionnaire. Therefore, the last few questions might be answered more briefly, or not at all.

(c) It is possible they only asked four dentists, or they continued to ask sets of four dentists until they achieved the desired results.

(d) Only those truly interested in the education topic will respond. Also, you have to have a device that is capable of sending text messages in order to respond to this question.

(e) By adding the words "much needed", the question could be biasing the responses from the participants.

2. One possibility is that people might think the temperature is always 82°. Also, it is not known if this is an average temperature, or an average high temperature.

3. **No**. It could very well be that most of the pickups sold in the past ten years were actually sold during the previous two years. In such case most of the pickups would have been on the road for only two years and therefore the given information would not imply that the average life of a pickup is around ten years.

4. The horizontal axis has non-uniform intervals, and neither the horizontal axis nor the graph are labeled.

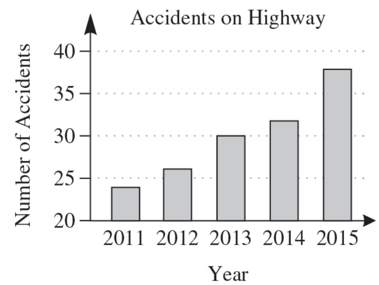
5. The three-dimensional drawing distorts the graph. The result of doubling the radius and the height of the can is to increase the volume by a factor of 8, whereas sales only doubled.

6. More information is necessary; e.g., is the graph referring to percentages of those who drive, or to actual numbers?

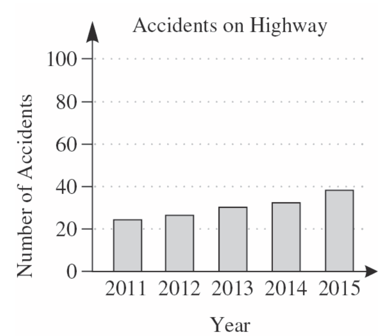
7. No. A mean can be affected greatly by a few outliers.

8. The first graph, as it makes the increase look much steeper.

9. (a) This bar graph would have perhaps 20 accidents as the baseline and perhaps 40 or 50 as its maximum. Then 24 in 2011 would be 4 units above the baseline while 38 in 2015 would be 18 units above the baseline, and appear as 3.5 times or 350% taller than 24 of 2011, when in fact it is just 58% higher. Below is an example of one possible solution.



(b) This bar graph would have zero accidents as its baseline and some much larger number (e.g., 100) as its maximum value. The effect would be to show values from 24 to 38 as lying a line with a small positive slope. Below is an example of one possible solution.



10. One such example would have scores of 5, 5, 5, 5, 5, 5, 100, and 100; the mean would be 28.75 and the median would be 5. Neither would be representative.

11. The mode might be representative if enough randomly-selected spots were reported—although it is possible that a mode might not exist. A median or mean might be misleading, depending upon the number of data points selected. A report of mean, median, mode, and standard deviation would probably be the most helpful.

12. The student is incorrectly mixing percentages of "effectiveness" with percentages of "times taking the drug." Further, "up to" 92% effective could mean something much lower as the norm. Finally, an 8% increase on 92% does not yield 100%; it would result in 99.36%.

13. (a) Probably not. A representative sample must be taken at random from the population as a whole, and simply taking opinions such as this is not so. This is known as a convenience sample.

(b) Probably not. Students eating in the cafeteria would probably be biased since only certain grades might eat their lunch around the noon hour.

(c) One way would be to assign students numbers from one to *n* (where there *n* students). Then choose the sample using random numbers, and make sure the sample is of sufficient size to be statistically significant.

14. The data are simply an indicator of passenger complaints and not a overall airline rating. Larger airlines might also have more total complaints but a lower percentage per passenger-mile. Also, not all airlines are included in this chart. Airlines such as Virgin Airlines, Jet Blue, etc. are all missing.

15. An erroneous conclusion could be that less spending produces higher SAT scores.

16. Only part (b) might not be helpful, even though it could have some bearing on insurance cost. All the others have some bearing on the true cost of operating an automobile.

Assessment 10-5B

1. (a) It's possible that a vast majority of the motorcycles sold in Texas by this manufacturer were sold within only the last year (perhaps they never had dealerships in Texas until very recently).

(b) "Up to" is quite indefinite (6 mpg is "up to" 30 mpg). The conditions under which 30 mpg may have been realized are not stated (the car may have been rolling downhill!).

(c) Reduce by how much? That is not given. The change might not be enough to be considered worth taking the product. Also, "some" people see reduced cholesterol. This means that, perhaps, most people do not see a reduction in their cholesterol level.

(d) Fresher than what? Other bread? Stale gym socks? It isn't stated. Also "40%" is used. How exactly is "freshness" measured? In other words, 40% of what?

(e) Is there another major airline flying to the city? Perhaps not.

(f) Since most people spend a vast majority of their time inside or around their homes, it would make sense that more accidents occur there.

2. General Cooster was assuming that there were no deep holes in the river where he was crossing.

3. We don't know who was surveyed, how many were surveyed, or how the question was phrased. The question might have been biased. For example, the survey might have asked "Don't you agree that 2 hours of homework per night is needed?"

4. The horizontal axis is not labeled. Also, the vertical axis does not have uniform spacing.

5. The three-dimensional graph distorts the data. It makes it appear as though more cars were sold in 2012 than in 2014. However, the two-dimensional graph demonstrates that the quantities were the same. Worsening the distortion is the fact that no scale is shown on the vertical axis in the three-dimensional graph.

6. When the radius of a circle is doubled the area is quadrupled. These graphs are misleading because the number of elementary teaching majors has only doubled.

7. She could have taken a different number of quizzes during the first part of the quarter than during the second part.

8. There are no labels by which to compare actual sales, and there is no scale on the vertical axis.

9. (a) Answers will vary. The survey shows that a majority of teachers rate their textbooks as good to excellent. The national "experts", on the other hand, seem not to agree.

(b) **No.** The teachers and the "experts", who are not identified, obviously disagree. It may be that the teachers are actually in the classroom and thus have a more intimate knowledge of student reaction to the texts.

10. What was the class average? What was the highest obtainable score?

11. (a) In grades K – 4 homework of less than one hour might be justified because the children are younger and 75% of the teachers surveyed fall within this range.

In grades 5 – 8 the spread is wider; it may be that older students are assigned more homework than those who are younger. 71% of the teachers surveyed assign between 31 and 120 minutes.

(b) A potential misuse in grades 5 – 8 might note that of the teachers who assign at least 91 minutes of homework, about half assign at least two hours.

Based on the survey data, it would be difficult to justify at least two hours at the K-4 level, although you could make an incorrect assumption that these schools (in the 4%) must be the best schools.

12. **No**. Many data sets can demonstrate this. For example, a data set of 1, 1, 1, 1, 1, 10, 11, 11, 20, 24, 29 has a mean and median of 10.

13. Answers will vary. For example, 1, 2, 3, 4, 100, 200, 300, 400.

14. Yes. For example, a country like the Netherlands, where dikes keep out the water of the land that is below sea level.

15. The sample should consist of approximately the same percentages that the population has of certain demographics (such as age, gender, race, etc.)

16. **(a)** The answer depends upon what type of reporter you are. A tabloid reporter would probably use the first headline, as it would grab the attention of the reader more quickly. A mainstream media reporter, on the other hand, might use the second as a more accurate representation of the facts.

(b) One reading depth would feel the greater impact from the second headline, since it provides more baseline information.

(c) The first headline has no baseline, so an educated reader would find more information in the second.

17. **(a)** 647 to 649, or $\frac{647}{649} \approx \mathbf{99.7\%}$.

(b) 622 to 649, or $\frac{622}{649} \approx \mathbf{95.8\%}$.

(c) **(*i*)** $P(\textit{lung cancer and smoker}|\textit{smoker}) = \frac{647}{1269} \approx \mathbf{51\%}$.

(*ii*) $P(\textit{lung cancer and non-smoker}|\textit{non-smoker}) = \frac{2}{29} \approx 7\%$.

(d) There is a high association between smoking and lung cancer because of the disparate probabilities. However, there is insufficient data to arrive at this conclusion, since correlation does not imply causation.

18. That conclusion could not automatically be drawn. Based on the westward movement of the mean center of population, there would be a strong suspicion that is indeed the case, but it is possible that either the population of the coastal east has decreased or that of the mountain west has increased.

19. That fish outsells dairy, since the length of the "fish bar" is longer than the length of the "dairy bar."

20. In the first graph, it appears that monthly sales have increased by quite a bit, since the artist started the y-axis value at 3000. In the second graph, the increase appears to be much less dramatic; monthly sales are increasing, but slowly.

Mathematical Connections 10-5: Review Problems

3. **(a)** $\overline{x} = \frac{43+91+73+65+56+77+84+91+82+65+98+65}{12} = \frac{890}{12} \approx \mathbf{74.17}$.

(b) Ordering the data: 43, 56, 65, 65, 65, 73, 77, 82, 84, 91, 91, 98. Median $= \frac{73+77}{2} = \mathbf{75}$.

(c) Mode $= \mathbf{65}$, the largest number of scores.

(d) The sum of the $(x - \overline{x})^2$ terms $= 2855.68$. $v = \frac{2855.68}{12} \approx \mathbf{237.97}$.

(e) $s = \sqrt{v} = \sqrt{237.97} \approx \mathbf{15.43}$.

(f) The sum of the $|x_n - \overline{x}|$ terms $= 156$. $\text{MAD} = \frac{156}{12} = \mathbf{13}$.

4. If the mean is 27 then the total of scores $= 36 \cdot 27 = 972$. Then adding scores of 40 and 42: $\overline{x} = \frac{972+40+42}{36+2} = \frac{1054}{38} \approx \mathbf{27.74}$.

5. First ten-paper mean $= 70$ so $10 \cdot 70 = 700$ points. Second 20-paper mean $= 80$ so $20 \cdot 80 = 1600$ points. Combined mean $= \frac{700+1600}{10+20} = \frac{2300}{30} = \mathbf{76.\overline{6}}$.

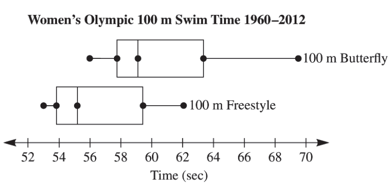
6. Men's Olympic 100-meter Run Times 1896–2012.

9	6 7 7 8 8 9 9
10	0 0 0 1 1 2 3 3 3 4 5 6 8 8 8 8
11	0 0
12	0

10|0 represents 10.0 seconds

7. Freestyle: Lower extreme $=$ 53.00; Lower quartile $=$ 53.84; Median $=$ 54.86; Upper quartile $=$ 58.59; and Upper extreme $=$ 61.20.

Butterfly: Lower extreme $=$ 55.98; Lower quartile $=$ 57.72; Median $=$ 59.195; Upper quartile $=$ 63.34; and Upper extreme $=$ 69.50.



The times for the 100-m butterfly are (relatively) much greater than those for the 100-m freestyle.

Chapter 10 Review

1. **(*i*)** If the average is reported as 2.41 children, the mean is probably being used. The mode would not have a decimal number, and the median would be either a whole number or one ending in .5.

(*ii*) If the average is 2.5, then either the mean or the median might have been used.

2. Ten part-timers with average salary of $100 $=$ a total part-time payroll of $1000. $1000 leaves $3000 for full timers.

$\frac{\$3000 \text{ payroll}}{\$250 \text{ per worker}} = \textbf{12 full-time workers}$.

(Assuming that the meaning of the term "average" as used in this question for part time employees is the same as in the term "mean".)

3. **(a)** **(*i*)** Mean $= \frac{10+50+30+40+10+60+10}{7} = \mathbf{30}$.

(*ii*) Ordering the data: 10, 10, 10, 30, 40, 50, 60. Median $=$ **30**, the middle data point.

(*iii*) Mode $=$ **10**, the most frequent data point.

(b) **(*i*)** Mean $= \frac{5+8+6+3+5+4+3+6+1+9}{10} = \mathbf{5}$.

(*ii*) Ordering the data: 1, 3, 3, 4, 5, 5, 6, 6, 8, 9. Median $= \frac{5+5}{2} = \mathbf{5}$.

(*iii*) Modes $=$ **3**, **5**, and **6**.

(c) ***(i)*** Mean=**150**. This can be found using the formula for the mean $\frac{4(100)+4(200)}{8} = 150$, but it is also easy to calculate by realizing that each 100 pairs with exactly one 200, and the mean of those two values is 150.

(ii) Median = **150**. The median is between the fourth and fifth values, or the mean of the last 100 and the first 200.

(iii) Mode: two modes, 100 and 200 (bimodal).

(d) ***(i)*** Mean $= \frac{208.2}{16} = \mathbf{13.0125}$.

(ii) Median: Ordering the data: 10; 10.1; 10.7; 10.9; 11.1; 11.5; 11.9; 12.1; 12.2; 12.3; 12.9; 13.2; 13.5; 13.6; 13.7; 28.5. Median is between the 8th and 9th values, so median = **12.15**.

(iii) Mode: there is no mode, since each value in the data set occurs exactly once.

4. **(a)** **(*i*)** Range $= 60 - 10 = \mathbf{50}$.

(*ii*) Sum of $|x_n - \overline{x}| = 120$. MAD $= \frac{120}{7} \approx \mathbf{17.1}$.

(*iii*) Upper quartile $=$ 50; lower quartile $=$ 10. IQR $= 50 - 10 = \mathbf{40}$.

(*iv*) Sum of $(x - \overline{x})^2 = 2600$. $v = \frac{2600}{7} \approx$ **371.4**.

(*v*) $s = \sqrt{v} = \sqrt{\frac{2600}{7}} \approx \mathbf{19.3}$.

(b) **(*i*)** Range $= 9 - 1 = \mathbf{8}$.

(*ii*) Sum of $|x_n - \overline{x}| = 18$. MAD $= \frac{18}{10} = \mathbf{1.8}$.

(*iii*) Upper quartile $=$ 6; lower quartile $=$ 3. IQR $= 6 - 3 = \mathbf{3}$.

(*iv*) Sum of $(x - \overline{x})^2 = 52$.
$v = \frac{52}{10} = \mathbf{5.2}$.

(*v*) $s = \sqrt{v} = \sqrt{5.2} \approx \mathbf{2.28}$.

(c) *(i)* Range $= 200 - 100 = \mathbf{100}$.

(ii) Sum of $|x_n - \overline{x}| = 400$. MAD $= \frac{400}{8} = \mathbf{50}$.

(iii) Upper Quartile = 200;
Lower Quartile = 100.
IQR $= 200 - 100 = \mathbf{100}$.

(iv) $(x - \overline{x})^2 = 20{,}000$
$v = \frac{20000}{8} = \mathbf{2500}$.

(v) $s = \sqrt{v} = \sqrt{2500}. = \mathbf{50}$.

(d) *(i)* Range $= 28.5 - 10 = \mathbf{18.5}$.

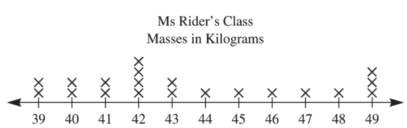
(ii) Using a mean of 13.0125, Sum of $|x_n - \overline{x}| = 34.875$. MAD $= \frac{34.875}{16} \approx \mathbf{2.18}$.

(iii) Upper quartile = 13.35; lower quartile = 11; IQR $= 13.35 - 11 = \mathbf{2.35}$.

(iv) Using a mean of 13.0125,
$(x - \overline{x})^2 = 277.5175$. So,
$v = \frac{277.5175}{16} \approx \mathbf{17.34}$.

(v) $s = \sqrt{v} = \sqrt{17.34} \approx \mathbf{4.16}$.

5. (a) Dot plot of Miss Rider's class (masses in kilograms):



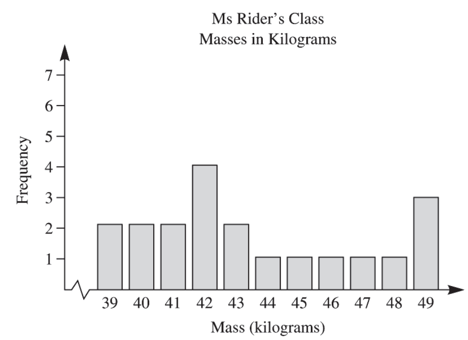
(b) Ordered stem-and-leaf plot of Miss Rider's class (masses in kilograms):

3 | 99
4 | 001122223345678999   4 | 0 represents 40 kg

(c) Frequency table of Miss Rider's class (masses in kilograms):

<table>
<tr><th>Mass</th><th>Tally</th><th>Frequency</th></tr>
<tr><td>39</td><td>||</td><td>2</td></tr>
<tr><td>40</td><td>||</td><td>2</td></tr>
<tr><td>41</td><td>||</td><td>2</td></tr>
<tr><td>42</td><td>||||</td><td>4</td></tr>
<tr><td>43</td><td>||</td><td>2</td></tr>
<tr><td>44</td><td>|</td><td>1</td></tr>
<tr><td>45</td><td>|</td><td>1</td></tr>
<tr><td>46</td><td>|</td><td>1</td></tr>
<tr><td>47</td><td>|</td><td>1</td></tr>
<tr><td>48</td><td>|</td><td>1</td></tr>
<tr><td>49</td><td>|||</td><td>3</td></tr>
<tr><td></td><td></td><td>20</td></tr>
</table>

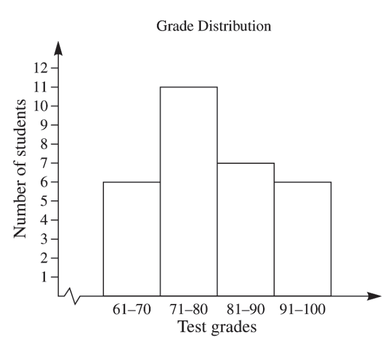
(d) Bar graph of Miss Rider's class (masses in kilograms):



6. (a) Test grade grouped frequency table:

<table>
<tr><th>Classes</th><th>Talley</th><th>Frequency</th></tr>
<tr><td>61–70</td><td>~~||||~~ |</td><td>6</td></tr>
<tr><td>71–80</td><td>~~||||~~ ~~||||~~ |</td><td>11</td></tr>
<tr><td>81–90</td><td>~~||||~~ ||</td><td>7</td></tr>
<tr><td>91–100</td><td>~~||||~~ |</td><td>6</td></tr>
<tr><td></td><td></td><td>30</td></tr>
</table>

(b) Test grade histogram:



7. Wegetum expenditures, where:

Bribes $= \frac{\$600{,}000}{\$2{,}000{,}000} = 0.30$; **30%** of $360° = 108°$.

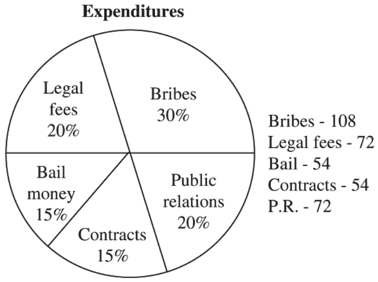
Legal fees $= \frac{\$400.000}{\$2{,}000{,}000} = 0.20$; **20%** of $360° = 72°$.

Public relations $= \frac{\$400{,}000}{\$2{,}000{,}000} = 0.20$; **20%** of $360° = 72°$.

Bail money $= \frac{\$300{,}000}{\$2{,}000{,}000} = 0.15$; **15%** of $360° = 54°$.

Contracts $= \frac{\$300{,}000}{\$2{,}000{,}000} = 0.15$; **15%** of $360° = 54°$.

Wegetem Expenditures



8. The widths of the bars are not uniform. The graph is also missing any labels.

9. Total salary $= 24 \cdot 9000 = \$216{,}000$. New mean $= \frac{\$216{,}000+\$80{,}000}{24+1} = \frac{296{,}000}{25} = \$11{,}840$. The mean was increased by $\$11{,}840 - \$9000 =$ **\$2840**.

10. Total Gold medals = 78

Russian Federation $= \frac{13}{78} = \frac{1}{6} = 0.1\overline{6}$; $\frac{1}{6}$ of $360° = 60°$.

Norway $= \frac{11}{78} \approx 0.141$; 14.1% of $360° \approx 51°$.

Canada $= \frac{10}{78} \approx 0.128$; 12.8% of $360° \approx 46°$.

United States $= \frac{9}{78} \approx 0.115$; 11.5% of $360° \approx 42°$.

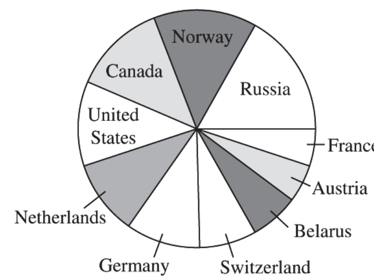
Netherlands and Germany $= \frac{8}{78} \approx 0.103$; 10.3% of $360° \approx 37°$.

Switzerland $= \frac{6}{78} = \frac{1}{13} \approx 0.077$; 7.7% of $360° \approx 28°$.

Belarus $= \frac{5}{78} \approx 0.064$; 6.4% of $360° \approx 23°$.

Austria and France $= \frac{4}{78} \approx 0.051$; 5.1% of $360° \approx 18°$.

Gold medal counts, 2014 Winter Olympics



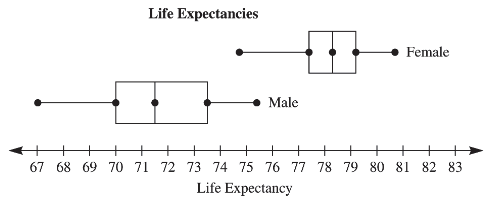
11. (a) Life expectancies at birth of males and females (from 1970 through 2011)

Females		Males
	67	1446
	68	28
	69	156
	70	0049
	71	02235578
	72	01145
	73	1689
7	74	123599
9310	75	146
86	76	023
88532	77	
9999854332211	78	
99655443210	79	
9642	80	
10	81	
7|74| represents 74.7 years old		|67| 1 represents 67.1 years old

(b) Males: Lower extreme: 67.1
Lower quartile: 70.0
Median: 71.9
Upper quartile: 74.2
Upper extreme: 76.3

Females: Lower extreme: 74.7
Lower quartile: 77.8
Median: 78.85
Upper quartile: 79.5
Upper extreme: 81.1

Life expectancies at birth of males and females (from 1970 through 2011):



12. **Larry**. Larry's GPA $= \frac{4\cdot4+4\cdot4+3\cdot3+3\cdot2+1\cdot2}{4+4+3+3+1} =$ $3.2\overline{6}$;

Marc's GPA $= \frac{4\cdot2+4\cdot2+3\cdot3+3\cdot4+1\cdot4}{4+4+3+3+1} = 2.7\overline{3}$.

So Larry's claim was correct.

13. **(a)** Ordering the data: 160, 180, 330, 350, 360, 380, 450, 460, 480. Median $=$ **360** yards, the middle data point.

(b) There is **no mode**, since no two or more lengths are the same.

(c) $\overline{x} = \frac{160+180+330+350+360+380+450+460+480}{9} =$ **350** yards.

(d) The sum of the $(x - \overline{x})^2$ terms $= 105{,}400$.

$s = \sqrt{\frac{105{,}400}{9}} \approx$ **108.2** yards.

(e) Range $= 480 - 160 =$ **320** yards.

(f) Lower quartile $= \frac{180+330}{2} = 255$;

upper quartile $= \frac{450+460}{2} = 455$.

IQR $= 455 - 255 =$ **200** yards.

(g) Variance $=$ the sum of the $(x - \overline{x})^2$ terms divided by 9 $= \frac{105{,}400}{9} = \mathbf{11711.\overline{1}\ yd^2}$.

(h) The sum of the $|x_n - \overline{x}|$ terms $= 760$.

MAD $= \frac{760}{9} = \mathbf{84.\overline{4}}$ yards.

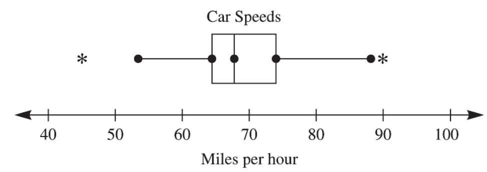
14. **(a)** Ordering the data: 45, 54, 56, 58, 58, 60, 62, 64, 64, 64, 65, 65, 65, 66, 67, 67, 67, 67, 68, 68, 69, 72, 74, 74, 75, 75, 82, 86, 88, 90.

Median $= \frac{67+67}{2} =$ **67** mph.

(b) Lower quartile $=$ **64**, the 8th data point.
Upper quartile $=$ **74**, the 23rd data point.

(c) Lower extreme $= 45$; upper extreme $= 90$.
Inter-quartile range (IQR) $= 74 - 64 = 10$.
Lower quartile $-1.5 \cdot \text{IQR} = 49 \Rightarrow$ outlier at 45.
Upper quartile $+1.5 \cdot \text{IQR} = 89 \Rightarrow$ outlier at 90.

Box plot of car speeds:



(d) Nine drivers exceeded 70 mph.
$\frac{9}{30} = 0.30 = \mathbf{30\%}$ received tickets.

(e) There are fewer speeds close to 67 mph (the median) in the upper quartile than in the lower. About 50% of the people drove between 64 and 74 mph. At least 25% speed.

(f) $\mathbf{P_{25} = 64; P_{75} = 74; D_5 = 67.}$

15. **(a)** **Positive** association. Weights tend to increase as heights increase.

(b) **170 pounds**; i.e., the data point corresponding to 72 inches on the horizontal axis.

(c) **67 inches**; i.e., the data point corresponding to 145 pounds on the vertical axis.

(d) **64 inches**. There are four girls 64 inches tall; more than for any other height.

(e) 170 pounds $-$120 pounds $=$ **50 pounds**.

16. **(a)** **Collette**. There is less deviation from the mean in her scoring.

(b) **(*i*)** Collette: From $24 - 6 =$ **18 to** $24 + 6 =$ **30**.

(*ii*) Rudy: From $24 - 14 =$ **10 to** $24 + 14 =$ **38**.

(c) **Collette** scored more than her average upper range value of 30; Rudy scored less than his average upper range value of 38.

17. 2.5% of the area under the standard normal curve lies more than two standard deviations above the mean. Thus the probability that a student would score more than 2 standard deviations above the mean $=$ **0.025**.

18. **(a)** The mean would be the common score.

(b) The median would be the common score.

(c) The mode would be the common score.

(d) The standard deviation would be **0**.

(e) The mean absolute deviation would be **0**.

19. The length of the columns in the bar graph should be approximately the same.

20. **(a)** $175 - 26 \cdot 6 = \mathbf{19}$ **pounds**.

(b) **No**. This is an example of using a mathematical model where it is no longer valid.

(c) The original statement reads "up to." Any average loss between 0 and 6 pounds would fit that description.

21. The bar graph is more appropriate. Line graphs are used to show change over time. Moreover, the points on the line graph should not be connected since there are no values "between" colors.

22. **Probably not**, unless the diving pool is about the same depth as the swimming pool.

23. Answers may vary. One method might be to pull boxes at random, weigh them, and then determine whether some predetermined percentage of the boxes fall within a predetermined weight range.

24. **(a)** Answers may vary. A high school graduate might want to be with other students of comparable age, and would thus choose the first.

(b) **The first**. Age 23 is further from the median of that box plot, assuming the student wants to go to school with that age group. If the student would want to go to school with those more mature, though, the student would choose **the second**.

25. Answers may vary. The area of both sections is about the same size, so you might conclude you need to eat as much of both; also, with the "sweets" on top, you might conclude it is more important than dairy products.

26. Answers may vary; e.g.,

(a) One way would be to leave the television set on even if no one is watching.

(b) The networks air very popular shows during "rating sweeps" periods.

27. Answers may vary. Graphs may show area or volume instead of relative size. Another way is to select a horizontal or vertical baseline that will support the point trying to be made.

28. **(a)** About 3 billion.

(b) About 6 billion.

(c) The population doubled.

(d) About 9.3 billion.

(e) About 55% percent. Comparing to (c), the growth factor is about 1.55.

29. **(a)** 68 inches is two standard deviations above the mean, and 2.5% of the area under the normal curve lies above 2 standard deviations above the mean. 2.5% of 1000 is **25 girls**.

(b) 60 inches is two standard deviations below the mean. $13.5\% + 34\% = 47.5\%$ of the area under the normal curve lies between the mean and 2 standard deviations below the mean. 47.5% of 1000 is **475 girls**.

(c) 66 inches is one standard deviation above the mean. 50% of the data lies below the mean and 34% lies between the mean and one standard deviation above the mean. This is $50\% + 34\% = \mathbf{84\%}$ of the data, so $100\% - 84\% = \mathbf{16\%}$ is the probability that a girl will be over 66 inches tall.

30. 750 is two standard deviations above the mean. $34\% + 13.5\% = 47.5\%$ of the area under the normal curve lies between the mean and 2 standard deviations above the mean. 47.5% of 1000 is **475 students**.

31. **(a)** P_{16} represents the bottom 16% of all scores, which in the normal distribution falls one standard deviation below the mean. Thus $P_{16} = 600 - 75 = \mathbf{525}$.

(b) D_5 is the decile ranking below which is 50% of the data; i.e., the mean, or **600**.

(c) P_{84} represents the value below which are 84% of the scores, which in the normal distribution falls one standard deviation above the mean. Thus $P_{84} = 600 + 75 =$ **675**.

32. The vertical axis is not starting at zero. It appears as if it is starting at 5,400,000. This gives the impression that the difference between the two values is great. The March 31st goal looks about 3 times larger than the value on March 27th, when in fact it is not.

33. The first person with the calculator types in a large random number (such as a typical annual salary, but not his/hers). The first person then memorizes or records that number, then adds his/her salary to that number on the calculator, passing it to the next person when she/he is done. That person adds her/his salary to the total, and passes it to the next person. They continue in this manner until all 5 people have entered their salaries. When the calculator is returned to the first person, he/she subtracts the random number that was entered, then divides by 5 to obtain the mean salary.

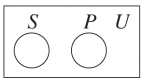
CHAPTER 11

INTRODUCTORY GEOMETRY

Assessment 11-1A: Basic Notions

1. **(a)** Answers vary, $\overrightarrow{BC}, \overrightarrow{AC}, \overrightarrow{CB}, \overrightarrow{CA}$ are examples of rays that contain $\overline{BC}$.

   (b) Answers vary, $\overrightarrow{EG}, \overrightarrow{CG}, \overrightarrow{GE}, \overrightarrow{GC}$ are examples of rays that contain $\overline{EG}$.

2. 

3. **(a)** **True;** Two lines in three-dimensional space are coplanar if there is a plane that includes them both. This is true if the lines are parallel or if they intersect each other. Two lines that are not coplanar are called skew lines.

   (b) **True;** Three points are always coplanar, and if the points are distinct and non-collinear, the plane they determine is unique.

   (c) **False;** they intersect in a line, the empty set, or a plane (if two planes are the same plane).

   (d) **True;** for any two points on any line, you can always find a point between them.

   (e) **False;** for example, the union of two rays could form an angle.

4. **(a)** Different order implies different names; i.e., $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{CD}$, $\overrightarrow{BA}$, $\overrightarrow{CA}$, $\overrightarrow{DA}$, $\overrightarrow{CB}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$, which is the permutation of four points taken two at a time.
   $_4P_2 = \frac{4!}{(4-2)!} =$ **12 ways**.

   (b) Different order implies different names; i.e., a plane is determined by three non-collinear points. The number of different names $=$ $_5P_3 = \frac{5!}{(5-3)!} =$ **60 ways**.

5. **(a)** Skew lines are two lines that cannot be contained in the same plane; e.g., $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$, $\overleftrightarrow{AD}$ and $\overleftrightarrow{GH}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{DH}$, $\overleftrightarrow{AE}$ and $\overleftrightarrow{BD}$, etc.

   (b) **Parallel**. *BDHF* is a rectangle.

   (c) **No**. They are non-coplanar; i.e., skew.

   (d) **None**. $\overleftrightarrow{BD}$ is parallel to plane *EFG*, so there are no points in common.

   (e) $\overleftrightarrow{FH}$ is the line of intersection of the two planes.
   $\overleftrightarrow{DH}$ is perpendicular to plane *FGH* because it is perpendicular to $\overleftrightarrow{HG}$ and $\overleftrightarrow{EH}$ in that plane. Thus $\overleftrightarrow{DH}$ is perpendicular to any line through *H* in the plane $FGH \Rightarrow \overleftrightarrow{DH} \perp \overleftrightarrow{FH}$.

   If a line ℓ through *H* is constructed in the plane *FGH* and perpendicular to $\overleftrightarrow{FH}$, $\overleftrightarrow{DH}$ will be perpendicular to ℓ Then one of the dihedral angles created by the two planes *BDH* and *FHG* measures 90°.

   Thus the planes are perpendicular because a dihedral angle measures 90°.

6. **(a)** **None**. The lines are parallel.

   (b) **Point *C***. Three distinct planes intersecting at one common point.

   (c) **Point *A***. The intersection of two non-parallel lines.

   (d) Answers may vary. $\overleftrightarrow{AC}$ **and** $\overleftrightarrow{BE}$ or $\overleftrightarrow{AB}$ **and** $\overleftrightarrow{CE}$ are two sets.

   (e) $\overleftrightarrow{AC}$ **and** $\overleftrightarrow{DE}$ or $\overleftrightarrow{AD}$ **and** $\overleftrightarrow{CE}$ are parallel; i.e., opposite sides of a rectangle.

   (f) **Planes *BCD*** or ***BEA***; i.e., planes bisecting the pyramid.

7. **14 pairs**; angles *AOB*, *AOC*, *AOD*, *AOE*, *BOC*, *BOD*, *BOE*, *BOF*, *COD*, *COE*, *COF*, *DOE*, *DOF*, and *EOF*. This is the combination of six lines taken two at a time ($_6C_2 = 15$), less the right angle *AOF*.

8. Answers may vary.

   (a) Edges of a room; vertical and horizontal parts of a window frame; intersecting crossroads.

   (b) Branches in a tree; clock hands at 7:30; angle on a yield sign.

   (c) The top of a coat hanger; clock hands at 7:15.

9. As measured on the protractor:

   (a) **110°**.

   (b) $110° - 70° =$ **40°**.

   (c) $180° - 160° =$ **20°**.

   (d) $160° - 30° =$ **130°**.

10. **(a)** **(*i*)** $18°35'29'' + 22°55'41'' = 40°90'70'' = 40°91'10'' = \mathbf{41°31'10''}$.

(*ii*) $15°29' - 3°45' = 14°89' - 3°45' = \mathbf{11°44'}$

(b) **(*i*)** $0.9° = (0.9 \cdot 60)' = \mathbf{0°54'00''}$.

(*ii*) $15.13° = 15° + (0.13 \cdot 60)' = 15°7.8' = 15°7' + (0.8 \cdot 60)'' = \mathbf{15°7'48''}$.

11. **(a)** **(*i*)** The hour hand will be pointed directly at the 3, so it will have moved $\frac{1}{4}$ of $360° = \mathbf{90°}$.

(*ii*) The hand will be $\frac{25}{60}$ of the way from the 12 to the 1, or $\frac{25}{60}$ of the $30°$ between numbers $= \mathbf{12.5° = 12°30'}$.

(*iii*) The hand will have moved 6 whole spaces at $30°$ each plus $\frac{50}{60}$ of another $30°$, or $180° + 25° = \mathbf{205°}$.

(b) Each minute moves the hour hand $\frac{1}{60}$ of $30° = 0.5°$. 1:15 P.M. is 75 minutes, so the hour hand will have moved $0.5 \cdot 75 = 37.5°$ past 12.

In 15 minutes the minute hand moves to $90°$ past 12. $90° - 37.5° = \mathbf{52.5° = 52°30'}$ between the hands.

12. **(a)** If $m(\angle AOB) = \frac{1}{3}m(\angle COD)$, then $m(\angle COD) = 3m(\angle AOB)$.

Let $x = m(\angle AOB)$. Then $x + 90° + 3x = 180°$ $\Rightarrow 4x = 90° \Rightarrow x = 22.5°$.
$m(\angle AOB) = \mathbf{22.5°}$; $m(\angle COD) = 3(22.5°) = \mathbf{67.5°}$.

(b) Let $x = m(\angle BOC)$. Then $x + (3x - 35°) = 90°$ $\Rightarrow 4x = 125° \Rightarrow x = 31.25°$.

$m(\angle BOC) = \mathbf{31.25°}$.

$m(\angle AOB) = 3(31.25°) - 35° = \mathbf{58.75°}$.

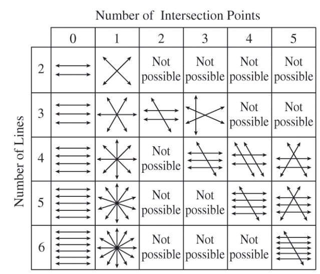
(c) Assume that if the position of $\overline{OE}$ is changed, the other rays will be adjusted so that all x's are congruent and all y's are congruent.
$3x + 3y$ will have different values, and thus x and y will have different values. $3x + 3y$, though, will remain $180°$
$\Rightarrow 3(x + y) = 180°$
$\Rightarrow x + y = 60° \Rightarrow m(\angle BOC) = \mathbf{60°}$.

13. **(a)** Draw a line with four points labeled A, B, C, and D. There are **six rays**: $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DC}$, $\overrightarrow{CB}$, and $\overrightarrow{BA}$.

(b) Five collinear points A, B, C, D, and E result in **eight rays**: $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{ED}$, $\overrightarrow{DC}$, $\overrightarrow{CB}$, and $\overrightarrow{BA}$.

(c) There are four, six, and eight rays for three, four, and five points, respectively. This is an arithmetic sequence with $a_3 = 4$ (i.e., one point would yield 0 rays and two points would yield 2 rays) and $d = 2$. The general term, then, for the number of rays given n points is $a_n = 0 + (n - 1)2 = \mathbf{2(n - 1)}$.

14. **(a)**

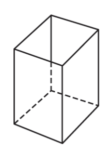


(b) The maximum number of intersections is the combination of n lines taken two at a time, or

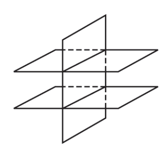
$$_nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)(n-3)\cdots}{2(n-2)(n-3)\cdots} = \frac{n(n-1)}{2}.$$

15. Perspectives may vary; e.g.,

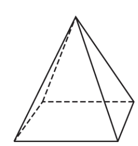
(a)



(b)

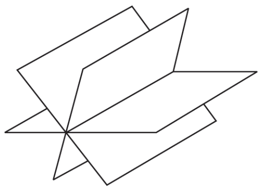


(c)

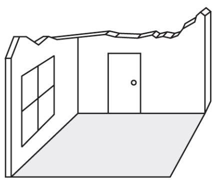


16. Answers may vary; e.g.,

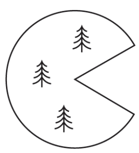
(a) A real-world example would be a paddle wheel:



(b) A real-world example would be the intersection of floor and two adjacent walls of a house:



(c) A real-world example would be a field irrigated with a central pivot:



Field Irrigates with a Central Pivot

17. **(a)** $\overset{\frown}{BC}$ is the arc associated with the central angle $\angle BOC$, whose measure is 70°. Thus, $m(\overset{\frown}{BC}) = \mathbf{70°}$.

(b) $m(\overset{\frown}{CD}) = m(\angle DOC) = \mathbf{110°}$.

(c) $m(\overset{\frown}{AD}) = m(\angle AOD) = \mathbf{70°}$.

(d) $m(\overset{\frown}{AB}) = \mathbf{110°}$.

(e) $m(\overset{\frown}{ADC}) = 80°$.

18. $\angle A = 30^\circ + 4\cdot\angle B$ and $\angle A + \angle B = 170^\circ$.

Substitute for $\angle A$: $30^\circ + 4\cdot\angle B + \angle B = 170^\circ$

Solve for $\angle B$: $5\angle B = 140^\circ \Rightarrow \angle B = 28^\circ$.

The solve for ∠A: $\angle A + 28^\circ = 170^\circ \Rightarrow \angle A = 142^\circ$.

19. **(a)** (plane AFD) $\cap$ (plane XYE) $= \overleftrightarrow{AE}, \overleftrightarrow{AD}$, or $\overleftrightarrow{DE}$.

(b) (Plane XYE) $\cap \overleftrightarrow{AE} = \overleftrightarrow{AE}$

(c) $\overline{BE} \cap \overline{CE} = \{\mathbf{E}\}$

(d) $\overline{CE} \cap \angle BEC = \overline{CE}$

(e) $\overleftrightarrow{AE} \cap {}^{\circ}\overline{DE}^{\circ} = {}^{\circ}\overline{DE}^{\circ}$

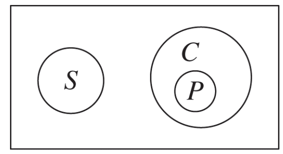
(f) $\overleftrightarrow{EB} \cap \overleftrightarrow{EC} = \{\mathbf{E}\}$

(g) **interior** $\angle BEC \cap \overleftrightarrow{BE}$ $\varnothing$

Assessment 11-1B

1.Answers vary; for example, $A, B, E; B, C, E; C, E, D$.

2.



3. **(a)** False; they could be skew.

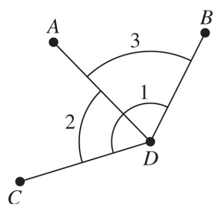
(b) False; the line can pass through and intersect at 1 point.

(c) False; the end points differ.

(d) False; a ray has 1 endpoint.

(e) False; the definition states that skew lines cannot be contained in the same plane.

4. Each point can serve as a vertex for 6 angles, 3 angles whose measures are less than 180°.



So there are 4 possible vertices times 3 angles for each vertex, and thus **12** distinct angles.

Normal: $\angle ADC$, $\angle BDC$, $\angle ADB$, $\angle ACB$, $\angle ACD$, $\angle BCD$, $\angle ABD$, $\angle ABC$, $\angle BAC$, $\angle BAD$, $\angle CAD$.

5. **(a)** **Point *H***. $\overleftrightarrow{BH}$ is a diagonal intersecting plane DCG at point H.

(b) Any **two planes** containing a 90° dihedral angle, e.g., $\overleftrightarrow{EH}$ and $\overleftrightarrow{DH} \Rightarrow AEH$ and CDH, $\overleftrightarrow{DH}$ and $\overleftrightarrow{FH} \Rightarrow CDH$ and EFG, etc.

(c) $\overleftrightarrow{\mathbf{AE}}, \overleftrightarrow{\mathbf{BF}}, \overleftrightarrow{\mathbf{CG}}$, and $\overleftrightarrow{\mathbf{DH}}$ are four such lines.

(d) **90°**. The planes are perpendicular; the measure of any of the four dihedral angles created by the planes is 90°.

6. **(a)** $\overline{AD}$ and $\overline{BE}$ do not intersect. Their intersection is $\boldsymbol{\emptyset}$.

(b) All points lying on $\overline{AD}$ are in this intersection. Since the planes determined are infinite, $\overleftrightarrow{AD}$ represent their intersection.

(c) $\{C\}$, a single point.

(d) Answers vary. $\overleftrightarrow{BE}$ and $\overleftrightarrow{CD}$ is one example.

(e) None of the lines in the drawing are parallel.

(f) Answers vary. The plane determined by *BED* is one example.

7. **6**, $\angle EOC$, $\angle EOB$, $\angle EOA$, $\angle FOD$, $\angle FOC$, $\angle FOB$.

8. Answers may vary.

(a) Railroad tracks.

(b) The corners of walls in your home.

(c) Two opposite facing walls in your home.

9. **(a)** 25°

(b) 135°

(c) 135°

(d) 245°

10. **(a)** **(*i*)** $21°35'31'' + 49°51'32'' = 70°86'63''$
$= 70°87'3''$
$= \mathbf{79°27'3''}$

(*ii*) $93°38'14'' - 13°49'27'' = 93°37'$
$(14+60)'' - 13°49'27'' = 92°(37+60)'$
$74'' - 13°49'27'' = \mathbf{79°48'47''}$.

(b) **(*i*)** $10.3° = 10° + (0.3 \cdot 60)' = \mathbf{10°18'}$.

(*ii*) $15.14° = 15° + (0.14 \cdot 60)'$
$= 15°8.4' = 15°8' + (0.4 \cdot 60)''$
$= \mathbf{15°8'24''}$

11. At noon, the angle between hands is 0°. There will be a 180° angle between hands when the minute hand has moved 180° more than hour hand.

Each minute, the minute hand moves $\frac{1}{60}$ of $360° = 6°$; the hour hand moves $\frac{1}{60}$ of $30° = 0.5°$.

After x minutes, the hands will have moved $6x°$ and $0.5x°$, respectively. $6x - 0.5x = 180 \Rightarrow x = 32.\overline{72}$ minutes $= 32$ minutes $+ 0.\overline{72} \cdot 60$ seconds $= 32$ minutes $+ \frac{72}{99} \cdot 60$ seconds. There will be a 180° angle in 32 minutes $43\frac{7}{11}$ seconds, or at about **12:32:44**.

12. **(a)** Let $x = m(\angle BOA)$. Since

$m(\angle DOC) = \frac{3}{4}\, m(\angle BOA)$ then

$x + 120° + \frac{3}{4}x = 180° \Rightarrow$

$\frac{7}{4}x = 60° \Rightarrow x = 34\frac{2}{7}°$.

$m(\angle DOC) = \frac{3}{4}\left(34\frac{2}{7}°\right) = \mathbf{25\frac{5}{7}°} \approx 25.71°$.

$m(\angle BOA) = \mathbf{34\frac{2}{7}°} \approx 34.29°$.

(b) Let $x = m(\angle BOC)$. Then $x + (2x - 30°) = 90° \Rightarrow 3x = 120° \Rightarrow x = 40°$.

$m(\angle BOC) = \mathbf{40°}$.

$m(\angle AOB) = 90° - 40° = \mathbf{50°}$.

(c) Let $x = m(\angle AOB)$ and $y = m(\angle BOC)$. Then

$$\begin{aligned} x - y &= 50 \\ \underline{x + y} &\underline{= 180} \\ 2x &= 230 \end{aligned}$$

or $x = 115$; $y = 180 - 115 = 65$.

$m(\angle AOB) = \mathbf{115°}$

$m(\angle BOC) = \mathbf{65°}$.

13. **(a)** **One**. Any plane is determined by three non-collinear points.

(b) **Four**. The number of planes is determined by the number of combinations of four points taken three at a time, or ${}_4C_3 = 4$.

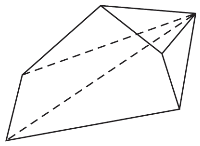
(c) The number of combinations of n points taken three at a time, or ${}_nC_3 = \frac{n!}{(n-3)!3!} = \frac{n(n-1)(n-2)}{3!} = \mathbf{\frac{n(n-1)(n-2)}{6}}$.

14. **(a)** **No**. $\overrightarrow{AB}$ contains all points on the *B* side of point *A*, while $\overrightarrow{BA}$ contains all points on the *A* side of point *B*.

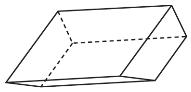
(b) **Yes**. Both $\overline{AB}$ and $\overline{BA}$ contain all points between *A* and *B*.

(c) **Yes**. Both $\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$ extend forever in both directions through points *A* and *B*.

15. **(a)**



(b)



16. Six parallel lines can be drawn.

17. **(a)** $\angle ACO$ is an angle in an isosceles triangle AOC. Its measure is equivalent to $\angle CAO$, which is **35°**.

(b) $m(\overset{\frown}{AC}) = m(\angle AOC) = \mathbf{110°}$.

(c) $m(\overset{\frown}{CB}) = m(\angle COB) = \mathbf{70°}$.

(d) $m(\overset{\frown}{AB}) = m(\angle AOB) = \mathbf{180°}$.

(e) $m(\overset{\frown}{CBA}) = m(\angle COA) = 360° - 110°$.
$= 250°$.

18. $\angle A = 30^\circ + 2 \cdot \angle B$ and $\angle A + \angle B = 180^\circ$.

Substitute for $\angle A$: $30^\circ + 2 \cdot \angle B + \angle B = 180^\circ$

Solve for $\angle B$: $3\angle B = 150^\circ \Rightarrow \angle B = 50^\circ$.

The solve for $\angle A$: $\angle A + 50^\circ = 180^\circ \Rightarrow \angle A = 130^\circ$.

19. **(a)** $\alpha \cap \beta = \overleftrightarrow{YZ}$

(b) $\angle ADF \cap \overrightarrow{BE} = \{\mathbf{B}\}$

(c) $\overleftrightarrow{AF} \cap \overleftrightarrow{BE} = \{\mathbf{E}\}$

(d) $\angle CEF \cap \overleftrightarrow{CF} = \{\mathbf{C, F}\}$

(e) $\overline{AE} \cup \overleftrightarrow{FE} = \overleftrightarrow{FE}$ or $\overleftrightarrow{EF}$

(f) $\overline{BD}^{\circ} \cup \overline{BA} = \overline{AD}^{\circ}$

(g) ${}^{\circ}\overline{AE}^{\circ} \cap \overleftrightarrow{AF} = {}^{\circ}\overline{AE}^{\circ}$

(h) $\overline{AE} \cup \overline{EF} = \overline{AF}$ or $\overline{FA}$

Assessment 11-2A: Curves, Polygons, and Symmetry

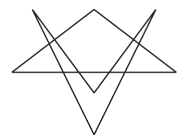
1. **(a)** **1, 4, 6, 7, 8**. A simple closed curve does not cross itself and begins and ends at the same point.

(b) **1, 6, 7, 8**. Polygons are polygonal curves (i.e., made up entirely of line segments) which are both simple and closed.

(c) **6, 7**. All segments connecting any two points of the polygon are inside the polygon; i.e., the region is nowhere dented inwards.

(d) **1, 8**. It is possible to draw a segment between two points of the polygon such that part of the segment lies outside the polygon.

2. A segment can pass through at most two sides of a triangle. If each side of the quadrilateral passes through two sides of the triangle there will then be **eight intersections**. One such figure might be:



3. A **concave polygon**. In a concave polygon it is possible for a line segment connecting two interior points to contain a point or points outside the interior of the polygon..

4. **(a)** **Convex**. A segment connecting any two points would lie fully inside the figure.

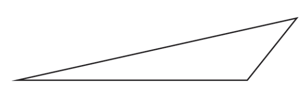
(b) **Concave**. It is possible to connect two points of the figure with a segment that lies partially or fully outside the figure.

(c) **Convex**.

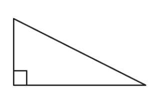
(d) **Concave**.

5. **Squares**. For a shape to be in the shaded region it must be both a rectangle and a kite. Squares are rectangles with two adjacent sides congruent, which is a property of a kite.

6. **(a)** **Possible**; three sides of different lengths with one obtuse (i.e., greater than 90°) angle.



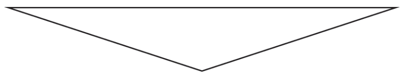
(b) **Possible**; three sides of different lengths with a right angle.



(c) **Impossible**. An equilateral triangle has three 60° angles.

(d) **Impossible**. An equilateral triangle has three 60° angles.

(e) **Possible**; two sides of equal length forming an obtuse angle.



7. (a) The number of ways that all the vertices in a pentagon can be connected two at a time is the number of combinations of five vertices chosen two at a time, or ${}_5C_2 = 10$. This number of segments includes both diagonals and sides. Subtracting the number of sides, 5, from 10 yields **5 diagonals** in a pentagon.

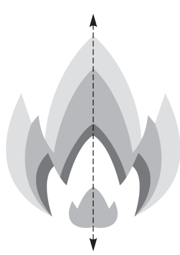
(b) The number of ways that all the vertices in a decagon can be connected two at a time is ${}_{10}C_2 = 45$. Subtracting the number of sides, 10, from 45 yields **35 diagonals** in a decagon.

(c) The number of ways that the vertices in a 20-gon can be connected two at a time is ${}_{20}C_2 = 190$. Subtracting the number of sides yields **170 diagonals** in a 20-gon.

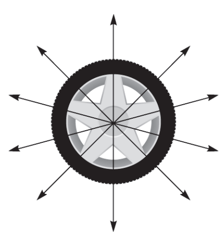
(d) The number of ways that all the vertices in an n-gon can be connected two at a time is the number of combinations of n vertices chosen two at a time, or ${}_nC_2 = \frac{n!}{(n-2)!2!} = \boldsymbol{\frac{n(n-1)}{2}}$.

This number of segments includes both the number of diagonals and the number of sides. Subtracting the number of sides n from $\frac{n(n-1)}{2}$ yields $\frac{n(n-1)}{2} - n = \boldsymbol{\frac{n(n-3)}{2}}$.

8. (a)



(b)



9. (a) **Equilateral** and **isosceles**; three congruent sides.

(b) **Isosceles**; two congruent sides.

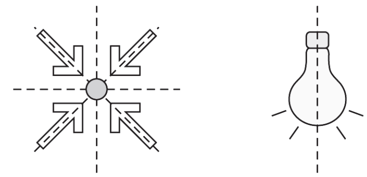
(c) **Scalene**; no sides congruent.

10. (*i*) Line symmetry: (*a*) and (*b*) both have line symmetry since they each can be reflected about a vertical line without changing their appearance.

(*ii*) Turn symmetry: (*a*) has turn symmetry since it can be "turned" 90° without changing its appearance.

(*iii*) Point symmetry: (*a*) also has point symmetry since it can be "turned" or rotated 180° without changing its appearance.

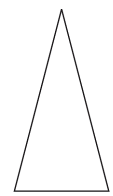
11.



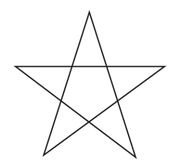
(a) (b)

12. Answers vary.

(a)

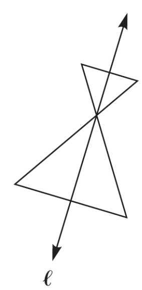


(b)

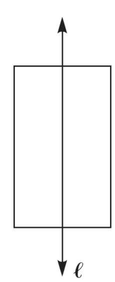


(c) The letter **N** has 180° rotational symmetry and no line of symmetry.

13. (a)



(b)



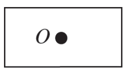
14. **(a)** 0 has vertical and horizontal line symmetry; 3 has horizontal line symmetry. 1 could have line symmetry and point symmetry depending on how it is written.

(b) 0 and 1 depending on how they are written.

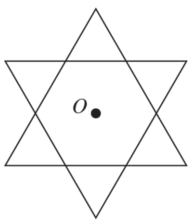
15. **(a)** There is one angle for turn symmetry: 180°.

(b) There are nine sides to this regular polygon. In a full rotation of 360°, there would be turn symmetry at $40^\circ, 80^\circ, 120^\circ, 160^\circ, 200^\circ, 240^\circ, 280^\circ, 320^\circ$.

16. **(a)**



(b)



17. **(a)** A square has exactly 4 lines of symmetry.

(b) A kite has a line of symmetry through a pair of opposite vertices.

(c) A parallelogram has 180 degree rotational symmetry.

Assessment 11-2B

1. **(a)** **1** and **2**. Both are triangles with at least two congruent sides.

(b) **1**. 2 is isosceles but is also equilateral.

(c) **None**. By definition, if a triangle is equilateral it has three congruent sides; an isosceles triangle has at least two congruent sides.

(d) **None**. A parallelogram has two pairs of parallel sides; a trapezoid has at least one pair of parallel sides.

(e) **5**. The figure does not have two pairs of parallel sides.

(f) **7**. A square has four congruent sides.

(g) **None**. A square has four right angles, the definition of a rectangle.

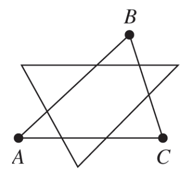
(h) **None**. A square has two pairs of parallel sides; a trapezoid has at least one pair of parallel sides.

(i) **None**. A rhombus has four congruent sides; a kite has two adjacent congruent sides with the other two sides also congruent.

(j) **4** and **6**. Both have four congruent sides.

(k) **4, 6, 8,** and **9**. All have two adjacent congruent sides with the other two sides also congruent.

2. Let *ABC* be a triangle. Let *L* be a line segment that does not contain any of the sides of triangle *ABC*. *L* can intersect the triangle at no more than 2 points. Since a second triangle is constructed of three line segments, this second triangle can intersect *ABC* at no more than 6 points. The follow diagram shows that a triangle that intersects *ABC* at 6 places can be constructed.



3. **(a)** There are three of the six vertices from which a diagonal cannot be drawn to any given vertex; i.e., itself and the vertices at the ends of adjacent sides. Thus **three** can be drawn.

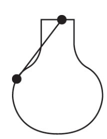
(b) There are three of the ten vertices from which a diagonal cannot be drawn, thus **seven** can be drawn.

(c) There are three of the twenty vertices from which a diagonal cannot be drawn, thus **17** can be drawn.

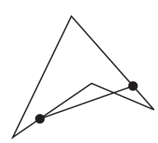
(d) In any *n*-gon there will be three diagonals which cannot be drawn. Thus three will be $\boldsymbol{n-3}$ which can.

4. In both (a) and (c) points can be found so that line segments connecting these points lie partially outside the figures.

(a)



(b)

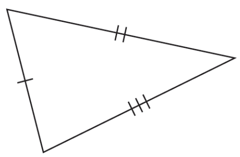


Thus, (a) and (c) are concave. This is not possible for (b) and (d), so (b) and (d) are convex.

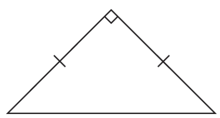
5. Rectangles have four right angles while rhombuses have four congruent sides; both are parallelograms. Their intersection is therefore a **square**.

6. Answers vary. Below are possible responses.

(a)

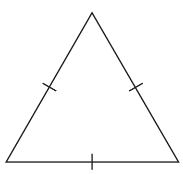


(b)



(c) **Not possible**. A scalene triangle has no congruent sides and an equiangular triangle is an equilateral triangle, which has three congruent sides.

(d) Equilateral triangles are equiangular triangles.



(e) The triangle in (d) is one example.

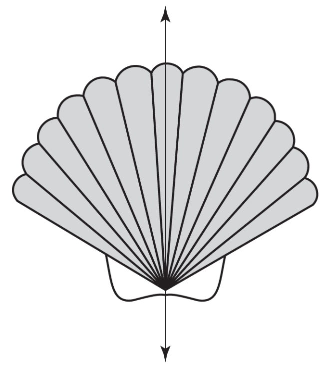
7. **(a)** A hexagon has six vertices and each vertex is contained in three distinct diagonals. (The vertex itself and the two consecutive vertices are excluded from forming diagonals.) This method has counted each diagonal twice, so we must divide by 2. Thus, the number of diagonals in a hexagon is $\frac{6 \cdot 3}{2} = \mathbf{9}$.

(b) An 11-gon has 11 vertices, each of which can be connected to 8 non-consecutive vertices by diagonals. Again, we have counted each diagonal twice. Thus, we have $\frac{11 \cdot 8}{2} = \mathbf{44}$ diagonals.

(c) The key idea above is that n vertices can be connected to $n - 3$ non-consecutive vertices to form diagonals and that this method counts each vertex twice. Thus, in general, an n-gon has $\frac{n(n-3)}{2}$ diagonals. Therefore, a 18-gon has $\frac{18 \cdot 15}{2} = \mathbf{135}$ diagonals.

8. **(a)**

(b)



9. **(a)** equiangular and acute.

(b) obtuse

(c) right

10. **(a)** This figure has line symmetry.

(b) This figure has line symmetry.

11. **(a)** 4 lines of symmetry; $90°$ and $270°$ turn symmetry; and point symmetry, which is also $180°$ turn symmetry.

(b) Point symmetry, which is also $180°$ turn symmetry.

(c) Point symmetry; $90°, 180°$, and $270°$ turn symmetry.

(d) 4 lines of symmetry; $90°, 180°$, and $270°$ turn symmetry; and point symmetry.

12. **(a)** **One** line of symmetry, drawn from the vertex of the two congruent sides to the midpoint of the base.

(b) **Three** lines of symmetry, each drawn from a vertex to the midpoint of the opposite base.

(c) **Two** lines of symmetry, both drawn from a midpoint of a side to the midpoint of the opposite side.

(d) **One** line of symmetry, drawn from the intersection of the two congruent sets of sides.

(e) **Two** lines of symmetry, each drawn from a vertex to the opposite vertex.

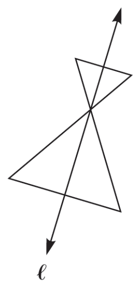
(f) **One** line of symmetry, drawn from the midpoints of the two parallel sides.

(g) **None**; the depicted parallelogram is not a rhombus.

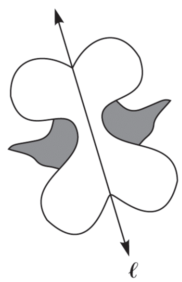
(h) **None**.

(i) **One** line of symmetry, the line containing the angle bisector.

13. **(a)**



(b)



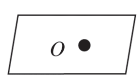
14. **(a)** 8 has vertical and horizontal line symmetry.

(b) 8 has point symmetry.

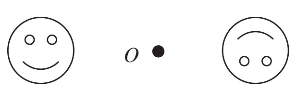
15. **(a)** The shape has turn symmetry for $72^\circ, 144^\circ, 216^\circ$, and 280° **.**

(b) The shape has turn symmetry for $90^\circ, 180^\circ$, and 270° **.**

16. **(a)**



(b)



17. **(a)** A rhombus has two lines of symmetry through a pair of opposite vertices.

(b) A rectangle has two lines of symmetry through a pair of midpoints of opposite sides.

(c) An isosceles trapezoid has a line of symmetry through a pair of midpoints of opposite sides.

Mathematical Connections 11-2: Review Problems

20. Angles are formed by any two rays. The number of angles is the number of pairs of rays; i.e., the number of combinations of all rays taken two at a time.

(a) Ten rays implies $_{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10.9}{2\cdot1} =$ **45 angles**.

(b) $$_nC_2 = \frac{n(n-1)(n-2)\cdots(1)}{[(n-2)(n-2)\cdots(1)]\cdot 2\cdot 1} = \frac{\boldsymbol{n(n-1)}}{\mathbf{2}} \textbf{ angles}.$$

21. **∅** if no intersection. **One**, if a ray of an angle is contained in the line. **Two**, if the line intersects both sides.
A ray, if the line starts at the vertex.
A line if the angle is straight and the line is collinear with it.

22. **(a)** **False**. A ray has one endpoint, continuing forever from that point.

(b) **True**. Both lines continue forever in each direction through both points.

(c) **False**. Skew lines cannot be contained in a single plane.

(d) **False**. $\overrightarrow{MN}$ has endpoint M and extends in the direction of point N; $\overrightarrow{NM}$ has endpoint N and extends in the direction of point M.

(e) **True**.

(f) **False**. Their intersection is a line.

Assessment 11-3A: More About Angles

1. Every pair of lines forms two pairs of vertical angles. Two times the number of combinations of three lines taken two at a time $= 2\cdot {_3C_2} = 2\cdot\frac{3!}{(3-2)!2!} =$ $2\cdot\frac{3\cdot2\cdot1}{1\cdot(2\cdot1)} = 2\cdot 3 =$ **6 pairs** of vertical angles.

2. The measures of the interior angles of every triangle add to $180°$. Subtract the sum of the given angles from $180°$ to find the third angle.

(a) $180° - (70° + 50°) = \mathbf{60°}$.

(b) $180° - (90° + 45°) = \mathbf{45°}$.

(c) $180° - (90° + 30°) = \mathbf{60°}$.

(d) $180° - (60° + 60°) = \mathbf{60°}$.

3.(a) (i) **Complementary** angle:
$90^\circ - m(\angle A) = 90^\circ - 30^\circ = 60^\circ$

(ii) Supplementary angle:
$180^\circ - m(\angle A) = 180^\circ - 30^\circ = 150^\circ$

(iii) Vertical angle: 30° **.**

(b) **(i)** **Complementary** angle:

$90^\circ - m(\angle B) = 90^\circ - x^\circ$

(ii) Supplementary angle:

$180^\circ - m(\angle B) = 180^\circ - x^\circ$

(iii) Vertical angle: x°**.**

4. $m(\angle B) = 3 \cdot m(\angle A)$ and $m(\angle C) = 2 \cdot m(\angle B)$. So $m(\angle A) = \dfrac{m(\angle B)}{3}$ and in a triangle the sum of the interior angles is 180 degrees so using substitution:

$$\begin{aligned} m(\angle A) + m(\angle B) + m(\angle C) &= 180^\circ . \\ \frac{m(\angle B)}{3} + m(\angle B) + 2 \cdot m(\angle B) &= 180^\circ \\ m(\angle B) + 3 \cdot m(\angle B) + 6 \cdot m(\angle B) &= 540^\circ \\ 10m(\angle B) &= 540^\circ \\ m(\angle B) &= 54^\circ \end{aligned}$$

Then $54^\circ = 3 \cdot m(\angle A)$ and $m(\angle C) = 2 \cdot 54^\circ$, so $m(\angle A) = 18^\circ$ and $m(\angle C) = 108^\circ$

5. The supplement of a 150° is $180^\circ - 150^\circ = 30^\circ$, the complement of 30° is $90^\circ - 30^\circ = 60^\circ$.

6. The supplement is

$$\begin{aligned} 180^\circ - 23^\circ 17' 18'' = 179^\circ 59' 60'' \\ \underline{-23^\circ 17' 18''} \\ 156^\circ 42' 42'' \end{aligned}$$

7.
$$\begin{aligned} 9x^\circ + (5x + 62)^\circ &= 90^\circ \\ 14x^\circ = 28^\circ &\Rightarrow x = 2^\circ \end{aligned}$$
$9(2^\circ) = 18^\circ$ and $5(2^\circ) + 62^\circ = 72^\circ$

8. **(a)** **Yes**. A pair of corresponding angles are 50° each (note that the supplementary angles formed by lines n and ℓ are 130° and 50°).

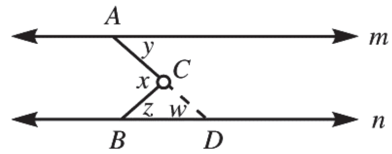
(b) **Yes**. A pair of corresponding angles are 70° each (note that the supplementary angles formed by lines n and ℓ are 110° and 70°).

(c) **Yes**. A pair of alternate interior angles are 40° each.

(d) **Yes**. A pair of corresponding angles are 90° each.

(e) The marked angles are alternate exterior angles. $k \parallel \ell$ by the alternate exterior angle theorem.

(f) Extend AC and label as follows:



$x = 180^\circ - [180^\circ - (z + m\angle xwz)]$
$= z + m\angle xwz$. It is given that $x = y + z$, so $z + m\angle xwz = y + z \Rightarrow y = m\angle xwz$.

Because $\angle xwz$ and y are measures of alternate interior angles, $k \parallel \ell$.

9. The given ratio could be written $7x : 2x$. The angles must add to 90°, thus $7x + 2x = 90$ $\Rightarrow x = 10$. The angles are $7(10) = \mathbf{70^\circ}$ and $2(10) = \mathbf{20^\circ}$.

10. Each exterior angle $= 180^\circ - 162^\circ = 18^\circ$. Since the sum of the measurements of the exterior angles $= 360^\circ$, then $360^\circ \div 18^\circ = 20$ angles, thus **20 sides**.

11. Since the sum of the angles in a triangles is 180°, $m(\angle 1) + m(\angle 3) + m(\angle 5) + m(\angle 2) + m(\angle 4) + m(\angle 6) = 180^\circ + 180^\circ = \mathbf{360^\circ}$.

12. The general form of an arithmetic sequence is $a_n = (n-1)d + a_1$. Since $a_6 = 130^\circ$, we have that $130^\circ = 5d + a_1$. Since the sum of the interior angles of a hexagon is 720°, we have that $a_1 + (d + a_1) + (2d + a_1) + (3d + a_1) + (4d + a_1) + 130^\circ = 720^\circ$. Simplifying, $5a_1 + 10d = 590^\circ$. Combining this information:

$$\begin{aligned} 5a_1 + 10d &= 590^\circ \\ \underline{-2a_1 - 10d} &= \underline{260^\circ} \\ 3a_1 \qquad &= 330^\circ \Rightarrow a_1 = 110^\circ. \end{aligned}$$

Thus, $130^\circ = 5d + 110^\circ \Rightarrow 5d = 20^\circ \Rightarrow d = 4$. Therefore, the angles are as follows: **110°; 114°; 118°; 122°; 126°; 130°**.

13. **(a)** $m(\angle 3) = 180^\circ - [m(\angle 1) + m(\angle 2)]$
$= 180^\circ - (45^\circ + 65^\circ) = \mathbf{70^\circ}$.

(b) $ABCD$ is parallelogram, thus opposite angles are equal. $m(\angle D) = m(\angle 3) = \mathbf{70^\circ}$.

(c) $AECB$ is a parallelogram, thus $m(\angle E) = m(\angle 2) = \mathbf{65°}$.

(d) $ACFB$ is a parallelogram, thus $m(\angle F) = m(\angle 1) = \mathbf{45°}$.

14. (a) $x = \mathbf{40°}$ (congruent vertical angles).

(b) $x + 4x = 90°$ (complementary angles). $5x = 90 \Rightarrow x = \mathbf{18°}$.

(c) Let $y = m(\angle ABC)$ and $z = m(\angle BAC)$. Then $180° - 3y = 3z$ (alternate interior angles) $\Rightarrow 3y + 3z = 180° \Rightarrow y + z = 60°$. Since $x + y + z = 180° \Rightarrow x + 60° = 180°$ $\Rightarrow \boldsymbol{x} = \mathbf{120°}$.

15. (a) Let the unknown angle $= x$. Then $m(\angle x) + \frac{1}{2}m(\angle x) = 90° \Rightarrow \frac{3}{2}m(\angle x) = 90°$ $\Rightarrow m(\angle x) = \frac{2}{3}(90°)$. $m(\angle x) = \mathbf{60°}$.

(b) m(third angle) $= 180° - 90°$ (the sum of the complementary angles) $= \mathbf{90°}$.

16. The six angles surrounding the center point sum to 360°. They can be compared to three pairs of vertical angles with the angles contained by triangles equal to those not contained. There are two of each angle making up the circle around the intersection point (the included angle and its vertical angle). Since the included angle and its vertical angle are always congruent, the contained angles must sum to half of $360° = 180°$. The sum of the angles in the three triangles is $3 \cdot 180° = 540°$. Thus the numbered angles sum to $540° - 180° = \mathbf{360°}$.

17. The sum of the measures of the interior angles of a hexagon $= (6 - 2) \cdot 180° = 720°$. $m(\angle x) = 720° - (110° + 105° + 142° + 122° + 130°) = \mathbf{111°}$.

18. (*i*) $m(\angle APT) = 90°$. $m(\angle 1) = 90° - 30° = \mathbf{60°}$.

(*ii*) $m(\angle 2) = \mathbf{30°}$ (alternate interior angles).

(*iii*) In ΔAPR, $m(\angle 3) = 180° - (30° + 40°) = \mathbf{110°}$.

19. (a) $m(\angle ACB) = 180° - (70° + 30°) = 80°$. $m(\angle 1) = 180° - 80° = \mathbf{100°}$ (supplementary angles).

(b) (*i*) The measure of the exterior angle equals the sum of the measures of the remote interior angles.

(*ii*) Since $\angle 1$ and $\angle C$ are supplementary, $m(\angle 1) + m(\angle C) = 180°$.
Since the sum of the measures of the interior angles of a triangle $= 180°$, $m(\angle A) + m(\angle B) + m(\angle C) = 180°$.
Then
$m(\angle 1) + m(\angle C) = m(\angle A) + m(\angle B) + m(\angle C)$, so $m(\angle 1) = m(\angle A) + m(\angle B)$.

20. If $y = 2x$, then $x = \frac{1}{2}y$. If $y = \frac{1}{2}z$, then $z = 2y$. If $y = \frac{1}{3}w$, then $w = 3y$.

Thus $\frac{1}{2}y + y + 2y + 3y = 180° \Rightarrow \frac{13}{2}y = 180° \Rightarrow$

$\boldsymbol{y} = \frac{360°}{13} = 27\frac{9°}{13} \approx \mathbf{27.69°}$;

$\boldsymbol{x} = \frac{1}{2}\left(\frac{360}{13}\right)° = \frac{360°}{26} = 13\frac{11°}{13} \approx \mathbf{13.85°}$;

$\boldsymbol{z} = 2\left(\frac{360}{13}\right)° = \frac{720°}{13} = 55\frac{5°}{13} \approx \mathbf{55.38°}$; and

$\boldsymbol{w} = 3\left(\frac{360}{13}\right)° = \frac{1080°}{13} = 83\frac{1°}{13} \approx \mathbf{83.08°}$.

21. Supplementary angles: $m(\angle 1)$: $180° - 60° = 120°$

Corresponding angles $m(\angle 1) = m(\angle 4) = 120°$.

Vertical angles: $m(\angle 5) = 60°$

Corresponding angles:

$m(\angle 5) = m(\angle 8) = m(\angle 12) = 60°$.

Corresponding angles $m(\angle 1) = m(\angle 11) + 70°$ so $m(\angle 11) = m(\angle 1) - 70° = 120° - 70° = 50°$.

Corresponding angles:

$m(\angle 11) = m(\angle 3) = m(\angle 6) = m(\angle 10) = 50°$.

Supplementary angles: $70° + m(\angle 11) + m(\angle 8) = 180°$. $m(\angle 8) = 180° - 70° - m(\angle 11)$. $m(\angle 8) = 180° - 70° - 50° = 60°$.

Supplementary angles: $m(\angle 2)$: $180^\circ - 50^\circ = 130^\circ$

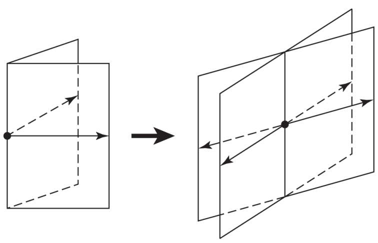
Corresponding angles: $m(\angle 2) = m(\angle 7) = 130^\circ$.

Vertical angles: $m(\angle 9) = 70^\circ$

22. $m(\angle 1) + m(\angle 2) = 180^\circ$ (straight angle)
$m(\angle 3) + m(\angle 4) = 180^\circ$ (straight angle)
Then $m(\angle 1) + m(\angle 2) = m(\angle 3) + m(\angle 4)$ and since $m(\angle 2) = m(\angle 3)$, we have $m(\angle 1) + m(\angle 3) = m(\angle 3) + m(\angle 4)$. Subtracting $m(\angle 3)$ from both sides, we have $m(\angle 1) = m(\angle 4)$.

Assessment 11-3B

1. Answers vary. Vertical angles are created by intersecting lines and are the pair of angles whose sides are two pairs of opposite rays. Dihedral angles are formed by two rays, one from each of two intersecting half-planes, perpendicular to the line of intersection. If we extend these half-planes by adjoining their "opposite" half planes and corresponding rays, we create vertical angles as shown below:



2. **(a)** $180^\circ - 130^\circ = 50^\circ$.
$180^\circ - 50^\circ - 60^\circ = \mathbf{70^\circ}$.

(b) $180^\circ - 65^\circ - 90^\circ = \mathbf{25^\circ}$.

(c) $180^\circ - 60^\circ - (180^\circ - 2x) = x \Rightarrow$
$-60^\circ + 2x = x \Rightarrow \boldsymbol{x = 60^\circ}$.

(d) $180^\circ - 90^\circ - 20^\circ = \mathbf{70^\circ}$.

3. **(a)** **(i)** **Complementary** angle:
$90^\circ - m(\angle A) = 90^\circ - 20^\circ = 70^\circ$

(ii) **Supplementary** angle:
$180^\circ - m(\angle A) = 180^\circ - 20^\circ = 160^\circ$

(iii) **Vertical angle:** 20°.

(b) **(i)** **Complementary** angle:
$$90^\circ - m(\angle B) = 90^\circ - (10^\circ + x^\circ) = 80^\circ - x^\circ$$

(ii) **Supplementary** angle:
$$180^\circ - m(\angle B) = 180^\circ - (10^\circ + x^\circ) = 170^\circ - x^\circ$$

(iii) Vertical angle: $10^\circ + x^\circ$.

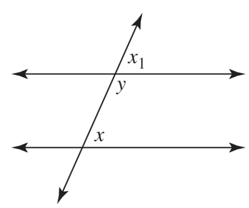
4 . $x^\circ = 15^\circ + 4 \cdot (\text{complement of } x^\circ)$
$x^\circ = 15^\circ + 4 \cdot (90^\circ - x^\circ)$
$x^\circ = 15^\circ + 360^\circ - 4x^\circ$
$5x^\circ = 375^\circ \Rightarrow x^\circ = 75^\circ$

5. The complement of a 20° is $90^\circ - 20^\circ = 70^\circ$, the supplement of 70° is $180^\circ - 70^\circ = 110^\circ$.

6. The supplement is
$$180^\circ - 128^\circ 19' 28'' = 179^\circ 59' 60'' \;-\; 128^\circ 19' 28'' = 51^\circ 40' 32''$$

7. $3x^\circ + (6x + 18)^\circ = 180^\circ$
$9x^\circ = 152^\circ \Rightarrow x = 18^\circ$
$3(18^\circ) = 54^\circ$ and $6(18^\circ) + 18^\circ = 126^\circ$

8. **(a)** Label the angle supplementary to y as x_1.



Then $x_1 + y = 180^\circ$, and since $x + y = 180^\circ$ it follows that $x = x_1$. Since x and x_1 are measures of corresponding angles, $k \parallel l$ and $k \parallel \ell$ if and only if $x = x_1$.

(b) Extend the 30° ray from line a to line b; a triangle is formed. The interior angles are 20°, 30° (alternate interior angles), and 130° (total of 180° interior angles). x and 130° are supplementary, thus $\boldsymbol{x = 50^\circ}$.

(c) Extend the 135° ray from line b to line a; a triangle is formed. The interior angles are 45° (supplement of 135°), 40° (supplement of 140°), and 95° (total of 180° interior angles). x and 95° are supplements, thus $\boldsymbol{x = 85°}$.

9. $x^\circ = \frac{2}{3}(\text{complement}) \Rightarrow x^\circ = \frac{2}{3}(90^\circ - x^\circ)$

$x^\circ = 60^\circ - \frac{2}{3}x^\circ \Rightarrow x^\circ + \frac{2}{3}x^\circ = 60^\circ$

$\frac{5}{3}x^\circ = 60^\circ \Rightarrow x^\circ = 60^\circ \cdot \frac{3}{5} = 36^\circ$

10. A dodecagon has twelve exterior angles. Each is $360^\circ \div 12^\circ = 30^\circ$; interior angles are therefore $180^\circ - 30^\circ = \mathbf{150^\circ}$.

11. $m(\angle 1) + m(\angle 3) + m(\angle 5) = 180^\circ$. Likewise, $m(\angle 2) + m(\angle 4) + m(\angle 6) = 180^\circ$. m(sum of marked angles) $= 180^\circ + 180^\circ = \mathbf{360^\circ}$.

12. The sum of the measures of the interior angles of a pentagon is $(5 - 2) \cdot 180^\circ = 540^\circ$.

The angles are $60^\circ, 60^\circ + d^\circ, 60^\circ + 2d^\circ, 60^\circ + 3d^\circ$, and $60^\circ + 4d^\circ \Rightarrow 5 \cdot 60^\circ + 10d^\circ = 540^\circ \Rightarrow d = 24^\circ$.

The angles are $\mathbf{60^\circ, 84^\circ, 108^\circ, 132^\circ,}$ and $\mathbf{156^\circ}$.

13. $\angle BCA$ and $\angle DEF$ are opposite angles of a parallelogram, so $\angle DEF$ (or $\angle E$) has measure **70°**. $\angle BAC$ and $\angle DFE$ are also opposite angles of a parallelogram, so $\angle DFE$ (or $\angle F$) has measure **60°**. Thus, $\angle EDF$ has measure **50°**.

14. **(a)** If $y = 3x$, then because it is an isosceles triangle, $x + 2(180^\circ - 3x) = 180^\circ \Rightarrow$ $^-5x = {}^-180^\circ$. $\boldsymbol{x = 36^\circ}$; $\boldsymbol{y} = 3(36^\circ) = \mathbf{108^\circ}$.

(b) One interior angle is $180^\circ - 115^\circ = 65^\circ$. x is an exterior angle to the triangle, so $\boldsymbol{x} = 55^\circ + 65^\circ = \mathbf{120^\circ}$.

(c) The total of the measures of the angles in a hexagon is $(n - 2) \cdot 180^\circ = 720^\circ$. Then starting with x, where $v = 136^\circ$, $(136^\circ - 4d) + (136^\circ - 3d) + (136^\circ - 2d) + (136^\circ - d) + 136^\circ + (136^\circ + d) = 720^\circ \Rightarrow 6(136^\circ) - 10d + d = 720^\circ \Rightarrow {}^-9d = {}^-96^\circ \Rightarrow d = 10\tfrac{2}{3}^\circ$.

So $x = 136^\circ - 4\left(10\tfrac{2}{3}^\circ\right) = 93\tfrac{1}{3}^\circ = \mathbf{93^\circ 20'}$;

$y = 136^\circ - 3\left(10\tfrac{2}{3}^\circ\right) = \mathbf{104^\circ}$;

$z = 136^\circ - 2\left(10\tfrac{2}{3}^\circ\right) = 114\tfrac{2}{3}^\circ = \mathbf{114^\circ 40'}$;

$u = 136^\circ - 10\tfrac{2}{3}^\circ = 125\tfrac{1}{3}^\circ = \mathbf{125^\circ 20'}$;

$v = \mathbf{136^\circ}$; and

$w = 136^\circ + 10\tfrac{2}{3}^\circ = 146\tfrac{2}{3}^\circ = \mathbf{146^\circ 40'}$.

15. Other than the given angles:

(a) **(*i*)** $m(\angle ABD) = 180^\circ - (50^\circ + 90^\circ) = \mathbf{40^\circ}$.

(*ii*) $m(\angle ACB) = 180^\circ - (90^\circ + 50^\circ) = \mathbf{40^\circ}$.

(*iii*) $m(\angle DBC) = 90^\circ - 40^\circ = \mathbf{50^\circ}$.

(b) **(*i*)** $m(\angle ABD) = 180^\circ - (90 + \alpha)^\circ = \mathbf{(90 - \alpha)^\circ}$.

(*ii*) $m(\angle ACB) = 180^\circ - (90 + \alpha)^\circ = \mathbf{(90 - \alpha)^\circ}$.

(*iii*) $m(\angle DBC) = 90^\circ - (90 - \alpha)^\circ = \boldsymbol{\alpha^\circ}$.

16. The measure of each interior angle of a regular octagon $= \frac{(8-2)\cdot 180^\circ}{8} = 135^\circ$. The triangle has two equal interior angles of $180^\circ - 135^\circ = 45^\circ$ (i. e., supplementary angles). Thus $m(\angle 1) = 180^\circ - (2 \cdot 45^\circ) = \mathbf{90^\circ}$.

17. Home plate is a pentagon; total interior angle measure is 540°. The measure of the sum of the two congruent angles is $540^\circ - 3 \cdot 90^\circ = 270^\circ$, or **135°** each.

18. Six interior angles add up to 720°. Two angles are right angles: 720° − 180° = 540°. The remaining four angles are the same: $4x = 540^\circ$. So $x = 135^\circ$.

19. Because the sum of the interior angles is 360°, $\angle A$ and $\angle D$ are supplementary. Also, $\angle D$ and $\angle C$ are supplementary. If we extend $\overline{DC}$ to form an alternate interior angle to $\angle C$, we see that these angles are congruent and thus $\overline{BC}$ is parallel to $\overline{AD}$. (Similarly, $\overline{AB}$ is parallel to $\overline{DC}$.) Thus $ABCD$ is a parallelogram.

20. In a pentagon the total interior angle measure is 540°. $540^\circ - (125^\circ + 120^\circ + 110^\circ + 68^\circ) = 117^\circ$. $x = 180^\circ - 117^\circ = 63^\circ$.

21. Corresponding angles: $m(\angle 1) = 70°$.
In triangle ABC,
$m\,(\angle ACB) = 180^\circ - 40^\circ - 70^\circ = 70^\circ$, then
$m\,(\angle 3) = 180^\circ - 40^\circ - 70^\circ = 70^\circ$ (straight angle).
Alternate interior angles $m\,(\angle 3) = m\,(\angle 4) = 70°$.
$m(\angle 5) = 180^\circ - 40^\circ - 70^\circ = 70^\circ$ (interior angle sum of a triangle)
$m(\angle 2) = 180^\circ - m(\angle 5) = 180^\circ - 70^\circ = 110^\circ$ (straight angle)

22. $m(\angle 1) = m(\angle 2)$ (vertical angles)
$m(\angle 2) = m(\angle 3)$ (given)
$m(\angle 3) = m(\angle 4)$ (vertical angles)
Therefore, $m(\angle 1) = m(\angle 2) = m(\angle 3) = m(\angle 4)$ and so $m(\angle 1) = m(\angle 4)$.

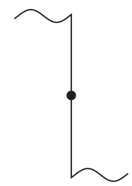
Mathematical Connections 11-3: Review Problems

20. **(a)** All angles must be right angles and all diagonals are the same length.
(b) All sides are the same length and all angles are right angles.
(c) Impossible. All squares are parallelograms.

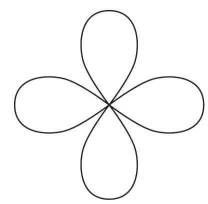
21. **(a)** (***i***) Two sets of parallel sides: *A*, *B*, *C*, *D*, *E*, *F*, *G*.
(***ii***) One set of parallel sides: *I*, *J*.
(***iii***) No parallel lines: *H*.
(b) (***i***) Four right angles: *D*, *F*, *G*.
(***ii***) Two right angles: *I*.
(***iii***) No right angles: *A*, *B*, *C*, *E*, *H*, *J*.
(c) (***i***) Four congruent sides: *B*, *C*, *F*,*G*.
(***ii***) Two pairs of congruent sides: *A*, *D*, *E*, *H*.
(***iii***) One pair of congruent sides: *J*.
(***iv***) No congruent sides: *I*.

22. The symbol reads the same forward, backwards, and upside down, i.e., reflected about a vertical line and turned 180°.

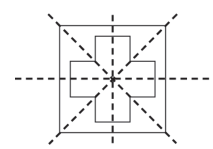
23. Answers vary.
(a) The figure has 180° turn symmetry but no line symmetry.



(b) The figure has point symmetry, 90° turn symmetry, and vertical and horizontal line symmetry.



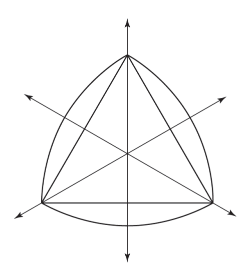
24. **(a)** (***i***) Four line symmetries, the diagonals and horizontal and vertical lines through the center.
(***ii***) No lines of symmetry.
(b) (***i***)



(***ii***) No lines of symmetry.

25. **(a)** Turn symmetries of 90°, 180° (also called point symmetry), and 270° about the center of the square.
(b) Point symmetry about the center of the square.

26. **(a)** None.
(b) 3, one through each vertex and the midpoint of its opposite side.



Assessment 11-4A: Geometry in Three Dimensions

1. **(a)** **Quadrilateral pyramid;** possibly square pyramid. This is a polyhedron determined by a simple closed polygonal region, a point not in the plane of the region, and triangular regions determined by the point and each pair of consecutive vertices of the polygonal region.
   (b) **Quadrilateral prism;** possibly trapezoidal or right trapezoidal prism. This is a polyhedron in which two congruent polygonal faces lie in parallel planes and the other faces are bounded by parallelograms.
   (c) **Pentagonal pyramid**. The closed polygonal region is a pentagon.

2. **(a)** (*i*) There are $4 + 2 = \mathbf{6}$ **cubes** in the stack.
   (*ii*) Each cube has six faces, so there are 36 total faces. From the left and right only three faces are seen; from front, back, top or bottom only four are seen. Thus there are $4 \cdot 4 + 2 \cdot 3 = 22$ visible faces. Of the 36 total faces, therefore, there must be **14** glued together.
   (b) (*i*) There are $2 \cdot 3 \cdot 4 = \mathbf{24}$ cubes in the stack.
   (*ii*) Each cube has six faces, so there are 144 total faces; $12 + 24 + 16 = 52$ are visible. $144 - 52 = \mathbf{92}$ **faces** glued together.

3. **(a)** Vertices: $\boldsymbol{A, D, R, W}$.
   (b) Edges: $\overline{\boldsymbol{AR}}, \overline{\boldsymbol{RD}}, \overline{\boldsymbol{AD}}, \overline{\boldsymbol{AW}}, \overline{\boldsymbol{WR}}, \overline{\boldsymbol{WD}}$.
   (c) Faces: $\triangle \boldsymbol{ARD}, \triangle \boldsymbol{DAW}, \triangle \boldsymbol{AWR}, \triangle \boldsymbol{DRW}$.
   (d) Intersection of $\triangle DRW$ and $\overline{RA}$: point $\boldsymbol{R}$.
   (e) Intersection of $\triangle DRW$ and $\triangle DAW$: $\overline{\boldsymbol{DW}}$.

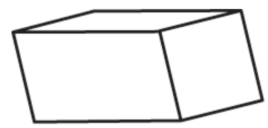
4. There is 1 possible pair of bases that lie in parallel planes.

5. **(a)** If it is a prism with n sides on the base, then it has n lateral faces. The top and bottom base are not counted in lateral faces. So the number of lateral faces is equal to the number of sides of the base.
   (b) The total number of faces is then $n + 2$ because the total number of faces includes the top and bottom base.

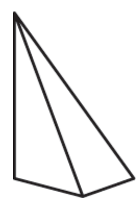
6. **(a)** **Five;** triangular prism.
   (b) **Four;** triangular pyramid.
   (c) **Four;** tetrahedron.

7. **(a)** **True**. This is the definition of a right prism.
   (b) **False**. No pyramid is a prism; i.e., a pyramid has one base and a prism two bases.
   (c) **True**. The definition of a pyramid starts with the fact that it is a polyhedron.
   (d) **False**. They lie in parallel planes.
   (e) **False.** A prism must have two bases that lie in parallel planes.
   (f) **True.** A cylinder has congruent circles for bases that lie in the parallel plane.

8. Answers may vary, but all are possible; e.g.,
   (a) An oblique square prism. Some faces are not bounded by squares.

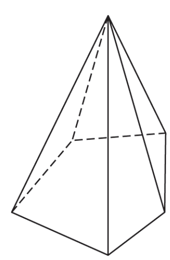


   (b) An oblique square pyramid. The base is a square but a vertical line segment from the vertex does not intersect the base at its center.



9. **(a)** Top: ○; Side △; Front △.
   (b) Top: ○; Side: ○; Front: ○.

10.



11. **(a)** **Right regular hexagonal pyramid**. Assume the base is a regular hexagon and the triangles are congruent. Points, if folded up, would meet at a vertex.
   (b) **Right square pyramid**. Assume the base is a square and triangles are congruent. Points would meet at a vertex.
   (c) **Cube**. All six faces are squares; assume they are congruent.

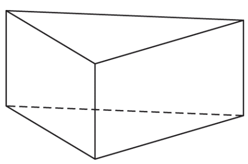
(d) **Right square prism**. Assume the lateral bases are congruent and the ends congruent. Lateral faces of the prism would all be bounded by rectangles.

(e) **Right regular hexagonal prism**. Assume both bases are congruent regular hexagons and the lateral faces are congruent.

12. **(a)** No, one face is missing. An extra triangular face would be the net of a triangular pyramid.

(b) Yes, it is the net of a rectangular prism.

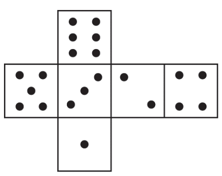
13.



14. **(a)** **(i)** The bottom has 4 dots.

(ii) The left side of the die has 5 dots.

(b)



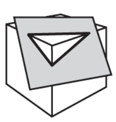
15. **(a)** ***iv***. The cross-section of a cylinder is a rectangle.

(b) ***ii***. The cross-section of a cone is a triangle.

16. **(a)** ***i*** (end view), ***ii*** (top view), ***iii*** (side view).

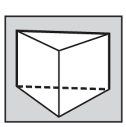
(b) ***i, ii***, ***iii***, ***iv*** (top views).

17. **(a)** The intersection is a **triangle:**



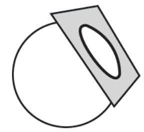
Triangle

(b) The intersection is a **rectangle:**



Rectangle

(c) The intersection is a **circle:**



Circle

18. **(a)** There will be three pairs of parallel faces, determined by any three pairs of opposite sides of the hexagonal base, e.g., $AA'B'B$ and $EE'D'D$.

(b) **120°**, e.g., $\angle FAB$ is a dihedral angle between two adjacent faces (each of the interior angles of a regular hexagon measure 120°).

19. The relationship $V + F - E$ is known as Euler's formula, and always equals 2:

(a) $V = 10, F = 7$, and $E = 15$.
$V + F - E = 10 + 7 - 15 = \mathbf{2}$.

(b) $V = 9, F = 9$, and $E = 16$.
$V + F - E = 9 + 9 - 16 = \mathbf{2}$.

20. Each triangle has 3 vertices: $8 \cdot 3 = 24$. Each octagon has 8 vertices: $6 \cdot 8 = 48$. Each vertex of a triangle is shared with a vertex of an octagon. The total number of vertices is then $48 - 24 = 24$.

Assessment 11-4B

1. **(a)** triangular prism

(b) quadrilateral pyramid

(c) cylinder

2. **(a)** **(*i*)** There are $4 + 2 + 1 = \mathbf{7}$ **cubes** in the stack.

(*ii*) Each cube has six faces, so there are 42 total faces. From the left and right only three faces are seen; from front, back, top or bottom only five are seen. Thus there are $4 \cdot 5 + 2 \cdot 3 = 26$ faces accounted for. Of the 42 total faces, therefore, there must be $42 - 26 = \mathbf{16}$ glued together.

(b) **(*i*)** There are $2 \cdot 3 \cdot 4 + 4 = \mathbf{28}$ **cubes** in the stack.

(*ii*) Each cubes has six faces, so there are 168 total faces; $4 \cdot 12 + 2 \cdot 7 = 62$ are visible. $168 - 62 = \mathbf{106}$ **faces** are glued together.

3. **(a)** A, B, C, D, E, F

(b) $\overline{AB}, \overline{BC}, \overline{CE}, \overline{BE}, \overline{EF}, \overline{CD}, \overline{AD}, \overline{DF}, \overline{AF}$

(c) $\triangle BCE, \triangle ADF,$ quadrilateral $CDFE$, quadrilateral $ABEF$, quadrilateral $ABCD$.

(d) E

(e) $\varnothing$

4. An octagonal prism has 8 lateral faces. There is one lateral face for each side of the octagonal base.

5. **(a)** If it is a pyramid with n sides on the base, then it has n lateral faces. The bottom base is not counted in the lateral faces. So the number of lateral faces is equal to the number of sides of the base.

(b) The total number of faces is then $n + 1$ because the total number of faces includes the bottom base.

6. **(a)** The bases of a prism are polygonal regions. The minimal number of sides for a polygonal region is 3, a triangle. Such a figure will have 6 edges on the bases and 3 edges on the lateral faces for a total of **9** edges.

(b) The minimal number of edges for a base is 3, a triangle, which results in 3 edges for the lateral faces for a total of **6** edges.

(c) Polyhedron is a broader term that includes pyramids. Thus, **6** edges.

7. **(a)** **False**. The base can be any simple closed curve.

(b) **False**. It has two bases.

(c) **False**. They are parallelograms; if they were rectangles it would be a right prism.

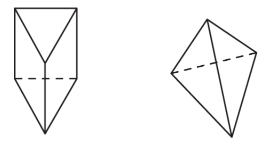
(d) **True,** by definition.

(e) **True.** A cube has all equal faces made up of squares. Each side of a square is congruent so all edges in a cube are congruent.

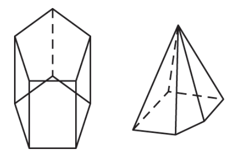
(f) **True.** Any pair of opposite faces in a rectangular prism can be the top and bottom faces. They are congruent and lie in the parallel planes.

8. Answers may vary; e.g.:

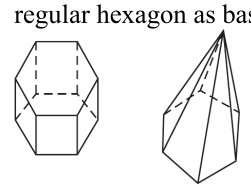
(a) Prism and pyramid (respectively) with a triangle as base:



(b) Prism and pyramid (respectively) with a pentagon as base:



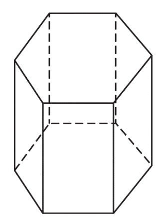
(c) Prism and pyramid (respectively) with a regular hexagon as base:



9. **(a)** Top: ○; Side □; Front □.

(b) Top, ; side, ; front, 

10.



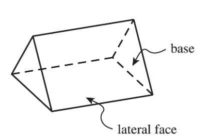
11. **(a)** **Right equiangular triangular prism**. Assuming the squares and triangles are congruent, the two congruent triangular bases lie in parallel planes.

(b) **Right square pyramid**. Four triangular faces, assuming the triangles are congruent and the rectangle is a square.

12. **(a)** **No.** The top faces do not match.

(b) **No.** There are two bottom bases but no top.

13.



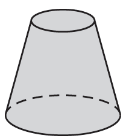
14. **(a)** $E = 3n$ **.** There are edges around each base ($2n$) and edges connecting the lateral faces (n).

(b) No, because 17 is not a multiple of 3.

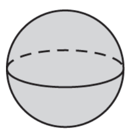
(c) $V = 2n$. There are n vertices connecting the bottom base to the lateral faces and n vertices connecting the top base to the lateral faces.

(d) No, because 23 is not a multiple of 2.

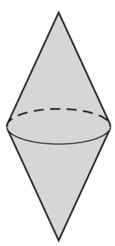
15. **(a)** Truncated right circular cone:



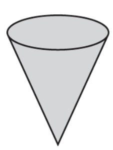
(b) Sphere:



(c) Two right circular cones joined at their bases:



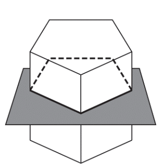
(d) Inverted right circular cone:



16. **(a)** **Object 2**. Note the relationship between numbered faces and the orientation of the numbers.

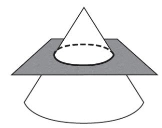
(b) **Object 4**. The two designs cannot be on adjoining faces.

17. **(a)** The intersection is a **pentagon:**



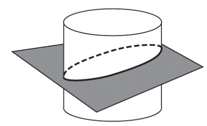
Pentagon

(b) The intersection is a **circle:**



Circle

(c) The intersection is an ellipse (or **oval**):



Ellipse or oval

18. **(a)** **None**. A pentagon has no parallel edges, thus there are no parallel lateral faces.

(b) The measure of any of the five congruent interior angles of either of the regular pentagon bases is $\frac{(5-2)\cdot 180^\circ}{5} = 108^{\text{o}}$ [per theorem 11-5, the sum of measure of the interior angles of any convex n-gon is $(n-2)\cdot 180^{\text{o}}$ and there are five interior angles in a pentagon]. Each of these angles is a dihedral angle between adjacent faces $\Rightarrow$ **108°**.

19. **(a)** (*i*) A **pyramid** has $\boldsymbol{n+1}$ **faces:** n lateral faces plus one base.

(*ii*) A **prism** has $\boldsymbol{n+2}$ **faces:** n lateral faces plus two bases.

(b) (*i*) A **pyramid** has $\boldsymbol{n+1}$ **vertices:** n vertices on the base plus the apex.

(*ii*) A **prism** has **2*n* vertices:** n vertices on each base.

(c) (*i*) A **pyramid** has **2*n* edges:** n edges on the base plus one connecting each vertex of the base to the apex.

(*ii*) A **prism** has **3*n* edges:** n on each base plus one connecting each corresponding pair of base vertices.

(d) **(*i*)** Pyramids: $(n + 1) + (n + 1) - 2n = \mathbf{2}$.

(*ii*) Prisms: $(n + 2) + 2n - 3n = \mathbf{2}$.

20.

Prism	Vertices per Base	Diagonals per Vertex	Total Number of Diagonals
Quadrilateral	4	1	4
Pentagonal	5	2	10
Hexagonal	6	3	18
Heptagonal	7	4	28
Octagonal	8	5	40
⋮	⋮	⋮	⋮
n-gon	n	$n - 3$	$n(n - 3)$

21. Each hexagon has 6 vertices. Each vertex is shared with another hexagon's vertex. If there are 20 regular hexagons in a soccer ball then there are $\frac{20 \cdot 6}{2} = 60$ vertices.

Mathematical Connections 11-4: Review Problems

20. The supplement is

$$\begin{aligned} 180^\circ - 18^\circ 13' 42'' = 179^\circ 59' 60'' \\ \underline{-18^\circ 13' 42''} \\ 161^\circ 46' 18'' \end{aligned}$$

21. Answers vary.

(a) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

(b) $\overleftrightarrow{AC} \cup \overrightarrow{BC} = \overleftrightarrow{AC}$

(c) $\overleftrightarrow{BC} \cup \overrightarrow{DC} = \overleftrightarrow{BC}$

(d) $\overrightarrow{BC} \cap \overline{CD} = \overline{CD}$

22. No, two adjacent angles cannot be vertical angles, because adjacent angles share a common side, and vertical angles do not.

23. Answers vary. For example the letter A has line symmetry but does not have turn symmetry.

24. A decagon has 35 diagonals. In a decagon (10 sides), start with a vertex and draw all possible diagonals (8) then move to the adjacent vertex and draw all possible diagonals (7), continue this until all diagonals have been drawn.
$8 + 7 + 6 + 5 + 4 + 3 + 2 = 35$.

25. In a regular 20-gon the sum of the interior angles is $180^\circ(20 - 2) = 3240^\circ$ and each interior angle measures $\frac{3240^\circ}{20} = 162^\circ$.

26. Two angles of the triangle are given so the third angle is $180^\circ - 30^\circ - 80^\circ = 70^\circ$. The angle adjacent to 70° is $180^\circ - 70^\circ = 110^\circ$. Angle $m(\angle 1) = 110^\circ$ (corresponding angles).

Chapter 11 Review

1. **(a)** $\overleftrightarrow{AB}, \overleftrightarrow{BC}, \overleftrightarrow{AC}$. One line can be drawn through any two given points.

(b) $\overrightarrow{BA}$ and $\overrightarrow{BC}$. A ray contains one endpoint and all points on the line on one side of the endpoint.

(c) $\overline{AB}$, the line segment between A and B.

(d) $\overline{AB}$, the only line segment containing all points common to both rays.

2. **(a)** Answers may vary; e.g., $\overleftrightarrow{PQ}$ **and** $\overleftrightarrow{AB}$ are skew. They do not intersect and are non-coplanar.

(b) **Any plane containing** $\overleftrightarrow{PQ}$ is perpendicular to α. Planes APQ and BPQ are two such.

(c) $\overleftrightarrow{AQ}$ is common to each.

(d) **No**. $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are skew lines, so no single plane contains them.

3. **(a)** $113°57' + 18°14' = 131°71' = \mathbf{132°11'}$.

(b) $84°13' - 27°45' = 83°73' - 27°45' = \mathbf{56°28'}$.

(c) $113°57' + 18.4° = 113°57' + 18°24' = 131°81' = \mathbf{132°21'}$.

(d) $0.75^\circ = 0^\circ + 0.75(60)' = \mathbf{0°45'0''}$.

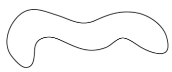
(e) $\mathbf{35°8'35''}$.

4. **(a)** The measure of one of the dihedral angles formed by planes α and γ is $m(\angle APS) = 90°$, as given. The measure of one of the dihedral angles formed by planes β and γ is $m(\angle BPQ) = 90°$, since it is given that $\overline{PQ} \perp \overline{AB}$.

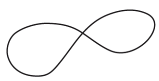
(b) $\overline{AB}$ is perpendicular to $\overline{PQ}$ and $\overline{PS}$ in plane γ.

5. Examples may vary.

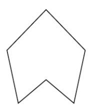
(a) A simple closed curve:



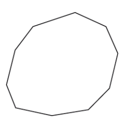
(b) A closed curve that is not simple:



(c) A concave hexagon:



(d) A convex decagon:



(e) Draw a regular pentagon *ABCDE*. Draw a line *CD* and reflect *CD* and *DE* about the line *CD*. The will create an equilateral pentagon that is not equiangular.

(f) Draw a rectangle that is not a square.

6. **(a)** **No**. The sum of the measures of two obtuse angles is greater than 180°, which is the sum of the measures of all the angles of any triangle.

(b) **No**. The sum of the measures of the four angles in a parallelogram must be 360°. If all the angles are acute the sum would be less than 360°.

7. Let α be the measure of the smallest angle. Then $\alpha + 2\alpha + 7\alpha = 180° \Rightarrow 10\alpha = 180°$. $\alpha = \mathbf{18°}$, $2\alpha = \mathbf{36°}$, $7\alpha = \mathbf{126°}$.

8. $180° - (90° + 42°) = \mathbf{48°}$.

9. $\frac{(n-2)\cdot 180}{n} = 176 \Rightarrow 180n - 360 = 176n \Rightarrow$ $4n = 360$. $n =$ **90 sides**.

10. $m(\angle 2) = m(\angle 3) = 45°$ because of alternate interior angles. Likewise, $m(\angle 1) = m(\angle 4) = 45°$. Thus $\boldsymbol{m(\angle 3) = m(\angle 4) = 45°}$.

11. **(a)** $m(\angle 3) = m(\angle 1) = \mathbf{60°}$ (vertical angles).

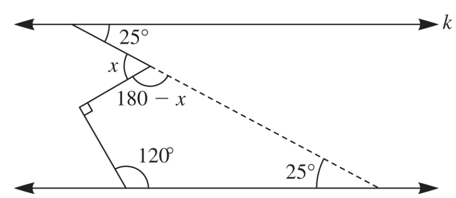
(b) $m(\angle 5) = m(\angle 3) = 60°$ (alternate interior angles) and $m(\angle 6) = 180° - m(\angle 5)$ (supplementary angles).
$m(\angle 6) = 180° - 60° = \mathbf{120°}$.

(c) $m(\angle 8) = m(\angle 6) = \mathbf{120°}$ (vertical angles).

12. $m(\angle ABC) = 180° - (90° + 55°) = 35°$.
$m(\angle x) = 180° - (90° + 35°) = \mathbf{55°}$.

13. **(a)** $x + 30° + (180° - 70°) = 180°$
$\Rightarrow x + 30° + 110° = 180°$
$\Rightarrow \boldsymbol{x = 40°}$.

(b) Extend a line segment as follows.



$90° + 120° + 180° - x + 25° = 360°$

$\Rightarrow {}^-x + 415° = 360°$

$\Rightarrow {}^-x = 360° - 415°$

$\Rightarrow {}^-x = {}^-55° \Rightarrow \boldsymbol{x = 55°}$.

14. **(a)** Consider ΔABC. This tells us that $x + y + 70° = 180°$. Also, $\boldsymbol{x = 50°}$ because $\overline{AB} \parallel \overline{CD}$. Thus, $\boldsymbol{y = 60°}$.

(b) $x + (180° - 125°) + 42° = 180°$
$\Rightarrow x + 97° = 180° \Rightarrow x = 83°$.
$y = 180° - 83° = 97°$.

15. ***(i)*** $m(\angle 1) = 180° - (70° + 45°) = \mathbf{65°}$.

(ii) $m(\angle 2) = m(\angle 1) = \mathbf{65°}$ (alt interior and corresponding angles).

(*iii*) $m(\angle 3) + m(\angle 4) = 360° - (2 \cdot 65°) = 230°$.
$m(\angle 3) = m(\angle 4)$
(opposite $\angle$s of a parallelogram)
$m(\angle 3) = \dfrac{230°}{2} = \mathbf{115°}$.

(*iv*) $m(\angle 4) = m(\angle 3) = \mathbf{115°}$.

(*v*) The third angle of $\Delta BDF = 65°$ (alternate interior angle with $\angle 2$).
$m(\angle 5) = 180° - (65° + 45°) = \mathbf{70°}$.

16. **(a)** $(5 \cdot 180°) - 360° = 3 \cdot 180° = 540°$.

(b) The measure of all the angles with vertices at $P = 360°$. If there are n triangles then there is a total of $n \cdot 180°$ minus the $360°$ surrounding P, or $n \cdot 180° - 360° = (n - 2) \cdot 180°$.

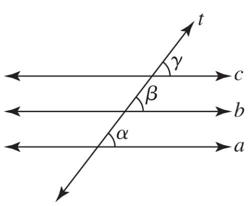
(c) Answers may vary. One way would be to connect B with E and connect A with F. There will then be two quadrilaterals and one triangle, or $2 \cdot 360° + 180° = \mathbf{900°}$.

17. **(a)** Alternate interior angles are congruent by construction. $\overline{AB} \parallel \overline{BC}$ by the alternate interior angle theorem.

(b) Corresponding angles are congruent.

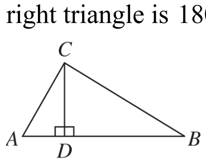
(c) $m(\angle B) + m(\angle C) + m(\angle BAC) = m(\angle BAD) + m(\angle DAE) + m(\angle BAC) = 180°$.

18. Given three parallel lines with line l intersecting them:



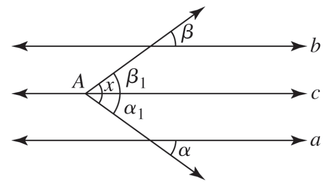
$a \parallel b \Rightarrow \alpha = \beta$. Then $b \parallel c \Rightarrow \beta = \gamma$. Thus $\alpha \parallel \gamma \Rightarrow a \parallel \mathrm{c}$ (congruent corresponding angles).

19. **Correct**. If it is assumed that a rectangle has four right angles for a total angle measurement of $360°$ and that a diagonal divides it into two congruent triangles, then the sum of the measures in each right triangle is $180°$.



Thus the sum of the measures of the angles in ΔACD and ΔBCD is $2 \cdot 180° = 360°$. In this sum all the angles of the original triangle are included as well as the two right angles at D, so the sum of the measures of the angles in the original triangle is $360° - 2 \cdot 90° = 180°$.

20. One approach would be to draw a line c through A as labeled below,



parallel to a and then prove it to be parallel to b. If $c \parallel a$, then $\alpha_1 = \alpha$ (corresponding angles). Since $x = \alpha + \beta$ and $x = \alpha_1 + \beta_1 = \alpha + \beta_1$ then $\beta_1 = \beta \Rightarrow c \parallel b$. By transitivity, $a \parallel b$.

21. **(a)** $\angle 3$ and $\angle 4$ are supplements of congruent angles.

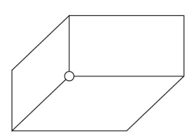
(b) $m(\angle 3) = 180° - m(\angle 1)$ and
$m(\angle 4) = 180° - m(\angle 2)$. If
$m(\angle 1) < m(\angle 2)$ then
${}^-m(\angle 1) > {}^-m(\angle 2)$.
$180° - m(\angle 1) > 180° - m(\angle 2)$
$\Rightarrow m(\angle 3) > m(\angle 4)$.

22. **(a)** $m(\angle ABO) + 60° + 90° = 180°$
$\Rightarrow m(\angle ABO) = \mathbf{30°}$.

(b) $m(\angle ABD) = m(\angle ABO) + m(\angle DBO) = \mathbf{60°}$.

(c) $m(\overset{\frown}{AXD}) = 360° - m(\overset{\frown}{AOD}) = \mathbf{240°}$.

23. Example may vary.
Three planes that intersect in a point:



24. **(a)** A triangle requires three points. The order of the points is not important, thus the number of possible triangles is the combination of ten points taken three at a time, or
${}_{10}C_3 = \frac{10!}{(10-3)!3!} = \mathbf{120}$.

(b) $nC_3 = \frac{n!}{(n-3)!3!} = \mathbf{\frac{n(n-1)(n-2)}{6}}$.

25. **(a)** $\frac{360°}{20} = 18$, so there are **18 sides**.

(b) 25 does not divide $360°$; such a regular polygon does not exist.

(c) Does not exist; the sum is always $360°$.

(d) Does not exist; the equation $\frac{n(n-3)}{2} = 4860$ has no natural number solution.

26. **(a)** 4

(b) 1

(c) 1

(d) None

(e) 2

(f) 2

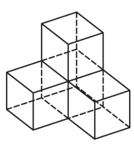
27. **(a)** Line and turn

(b) Line, turn, and point

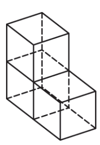
(c) Line.

28. Answers may vary.

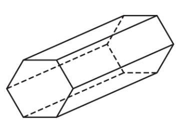
(a)



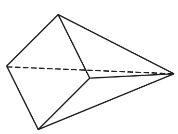
(b)



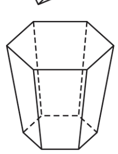
(c)



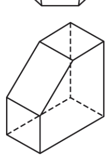
(d)



(e)



(f)



29. **(a)** $180° - 30° - 90° = \mathbf{60°}$.

(b) $\mathbf{30°}$.

(c) $180° - 120° - 30° = \mathbf{30°}$.

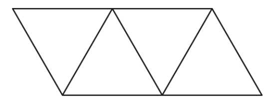
(d) $180° - 120° = \mathbf{60°}$.

(e) $m(\widehat{CD}) = m(\angle COD) = \mathbf{60°}$.

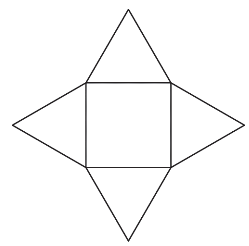
(f) $m(\widehat{BC}) = m(\angle BOC) = \mathbf{120°}$.

30. There are as many lateral faces as sides. Thus, there are **eight** lateral faces.

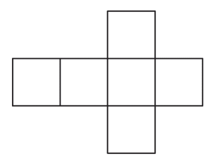
31. **(a)**



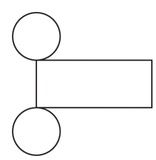
(b)



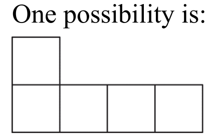
(c)



(d)



32. **(a)** One possibility is:



(b) Eight different nets are possible, i.e., the bottom may be placed in any of eight positions.

33. **(d)** $5 + 5 - 8 = 2. (V + F - E = 2.)$

(e) $12 + 8 - 18 = 2$.

(f) $10 + 7 - 15 = 2$.

CHAPTER 12

CONGRUENCE AND SIMILARITY WITH CONSTRUCTIONS

Assessment 12-1A: Congruence Through Constructions

1. **(a)** **True**; Corresponding Parts of Congruent Triangles are Congruent (CPCTC).

(b) **False**; the two angles are not corresponding angles, so there is no guarantee that they are congruent.

(c) **False**; The two line segments are not corresponding line segments, so there is no guarantee that they are congruent.

2. The two triangles are congruent by the Side, Side, Side (**SSS**) congruent condition, so therefore, $\angle R \cong \angle B$ by CPCTC.

3. It is given that $\overline{AD} \cong \overline{BC}$, and that $\angle CBD$ and $\angle ADB$ are right angles, so $\angle CBD \cong \angle ADB$. Note that $\overline{BD} \cong \overline{DB}$ since they are the same segment. Therefore $\Delta ABD \cong \Delta CDB$ by the Side, Angle, Side (**SAS**) property. That means $\angle A \cong \angle C$, $\angle ABD \cong \angle CDB$; and, $\overline{AB} \cong \overline{CD}$, by CPCTC.

4. Since both are 20 units long, $\overline{AB} \cong \overline{BC}$; since both angles are 60 degrees, $\angle ABD \cong \angle CBD$; and $\overline{BD} \cong \overline{BD}$. So by SAS, $\Delta ABD \cong \Delta CBD$.

5. They are not necessarily congruent. Angle, Angle, Angle (AAA) is **not** a congruence property. It is possible that one triangle is larger than the other triangle. Without knowing anything about the side lengths, it cannot be determined if the triangles are congruent.

6. We are given that $\overline{BC} \cong \overline{DC}$ and that $\overline{AC} \cong \overline{EC}$. Also, $\angle ACB \cong \angle ECD$, because they are vertical angles, and vertical angles are always congruent. So $\Delta ACB \cong \Delta ECD$ by **SAS**. That means $\angle CAB \cong \angle CED$ by CPCTC. Note that $\overline{AE}$ is a transversal, with $\angle CAB$ and $\angle CED$ alternate interior angles. Since the alternate interior angles are congruent, that means $\overline{AB} \parallel \overline{DE}$.

7. **(a)** Since l is the perpendicular bisector of $\overline{AC}$ we know that $\overline{AD} \cong \overline{CD}$ and $\angle ADB \simeq \angle CDB$. Since $\overline{BD} \cong \overline{BD}$, then $\Delta ADB \cong \Delta CDB$ by **SAS**. Therefore, $\overline{AB} \cong \overline{CB}$ by CPCTC. So, set their lengths equal to each other and solve for x:

$$6x + 15 = 12x - 9$$
$$24 = 6x$$
$$\mathbf{4} = x$$

(b) Since l is the perpendicular bisector of $\overline{AC}$ we know that $\overline{AB} \cong \overline{CB}$ and $\angle ABD \cong \angle CBD$. Since $\overline{BD} \cong \overline{BD}$, then $\Delta ABD \cong \Delta CBD$ by **SAS**. Therefore, $\overline{AD} \cong \overline{CD}$ by CPCTC. So, set their lengths equal to each other and solve for x:

$$6x - 3 = 3x + 15$$
$$3x = 18$$
$$x = \mathbf{6}$$

8. We are given that $\overline{AB} \cong \overline{BC}$ and $\overline{AD} \cong \overline{CD}$. Also, $\overline{BD} \cong \overline{BD}$. Therefore $\Delta ABD \cong \Delta CBD$ by **SSS**. That means $\angle ABD \simeq \angle CBD$ by CPCTC.

9. If the 3 points are collinear, then it is not possible. If the 3 points are not collinear, construct a triangle whose vertices are the 3 points. Now choose any 2 sides and construct perpendicular bisectors. The intersection of these bisectors will be a point equidistant from the 3 points.

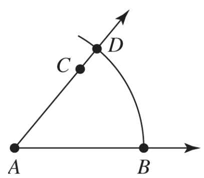
10. **Obtuse triangles**. While it will not produce a general argument, constructing an obtuse triangle and then constructing the perpendicular bisectors to the sides adjacent to the obtuse angle will demonstrate why this is so.

11. **(a)** The angle opposite $\overline{BC}$ is greater than the angle opposite $\overline{AC}$; i.e., $\boldsymbol{m(\angle A) > m(\angle B)}$.

(b) In any triangle, the side of greater length is opposite the angle of greater measure.

12. ΔABD and ΔCDB share a common side, $\overline{BD}$. Since both triangles are right triangles and $\overline{AB} \cong \overline{CD}$, $\Delta ABD \cong \Delta CDB$ by the H-L theorem. Thus, $\angle ABD \cong \angle CDB$. Since these are alternate interior angles created by transversal $\overline{BD}$, $\overline{AB} \parallel \overline{CD}$. Similarly, $\overline{BC} \parallel \overline{AD}$ by noting that $\angle ADB \cong \angle CBD$. Since opposite sides are parallel, $ABCD$ is a parallelogram.

13. **(a)** $\overline{AD}$ was created with a compass as shown below. Therefore, $\overline{AD} \cong \overline{AB}$. To construct an angle congruent $\angle CAB$, follow the instructions on page 697 in the text, Figure 11.



(b) One strategy is to construct a line segment for any of the three lengths, for example 2 cm. Then use a compass to draw circles of radii 3 cm and 4 cm from either endpoint. Where the circles intersect is a third vertex for a triangle with the desired properties. Alternatively, cut three pieces of string of the desired lengths and arrange to form a triangle.

(c) Use a strategy similar to that in part (b).

(d) Since $4+5<10$, it is **not possible**.

(e) Use a strategy similar to that in part (b).

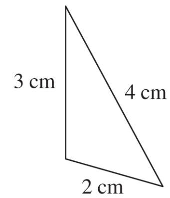
(f) Use a protractor to create an angle whose measure is $75°$. Along the either ray of this angle draw line segment from the vertex of length 6 cm and 7 cm. Connect the end points of these line segments.

(g) Use a protractor to construct a $40°$ angle, and label this angle as $\angle A$. On one of the sides, mark point B such that $m\overline{AB} = 7$ cm. Now let C be the intersection of the other side of $\angle A$ with a circle centered at B with radius 6 cm. Two triangles are possible.

(h) Use a similar strategy to that in (g). Note two triangles are possible.

(i) Use a strategy similar to that in (f), i.e., *SAS* construction.

(j) (b) **Unique** by *SSS*. If the three sides of one triangle are congruent, respectively, to the three sides of a second triangle, then the triangles are congruent, i.e., all triangles with sides 2 cm, 3 cm, and 4 cm are the same. The figure below is similar to a 2 cm, 3 cm, 4 cm triangle.

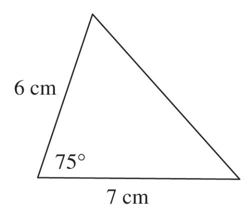


(c) **Unique** by *SSS*. A scalene right triangle.

(d) **No triangle**. The triangle inequality states that the sum of the measures of any two sides of a triangle must be greater than the measure of the third side, and $10>4+5$.

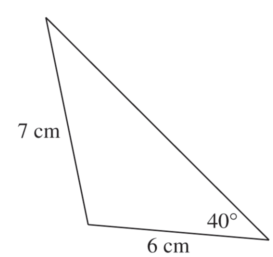
(e) **Unique** by *SSS*.

(f) **Unique** by *SAS*. If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, respectively, then the two triangles are congruent. The figure below is similar to a *SAS* 6 cm-75°-7 cm triangle.

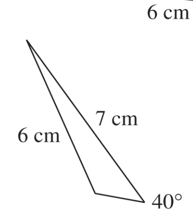


(g) **Not unique**. In the construction given, the $40°$ angle is adjacent to the side of length 7, but it could have been adjacent to the side of length 6. In either case, two triangles are possible, one acute and one obtuse. Two of the obtuse cases are shown below.

(*i*)



(*ii*)



(h) **Unique** by *SAS.* With one angle given in an isosceles triangle the others are determined.

(i) **Unique** by *SAS* with an included right angle.

14. Construct a right isosceles triangle. The angle opposite the base is the right angle, and the perpendicular bisector of the hypotenuse (the base) bisects the right angle, creating two 45 degree angles.

15. **(a)** Construct an angle congruent to $\angle A$. Construct an angle congruent to $\angle B$ such that $\angle A$ and $\angle B$ are adjacent angles (i.e., they share the same vertex, share a side, but have no intersections of their interiors).

(b) Construct an angle congruent to $\angle B$. Construct an angle congruent to $\angle A$ such that $\angle A$ and $\angle B$ share the same vertex, share a side, yet have the other side of $\angle A$ fall inside the interior of $\angle B$.

16. **No**. If all sides are congruent then all angles must be congruent; thus no right angle is possible.

17. **(a)** Construction. The triangles *ABO*, *BCO*, *CDO*, and *DAO* are congruent (*SAS*) isosceles right triangles. Therefore the congruent angles in each triangle measure $45°$. Consequently all the angles in *ABCD* measure $\overleftrightarrow{PP'}$ and all the sides are congruent.

(b) Because the arcs $\overset{\frown}{BE}$ and $\overset{\frown}{EC}$ each measure $45°$, the chords *BE* and *EC* are congruent.

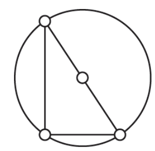
(c) Bisecting each of the right central angles we get the vertices of the regular octagon.

18. Fold the square in half so that one pair of opposite sides fall on each other. Unfold and repeat for the other pair of sides. Where the two fold lines intersect is the center of the circle and the radius is the line segment from the center to one of the square's vertices.

19. This point can be found only if *A* and *B* are not points on a line perpendicular to *m*. Assuming *A* and *B* have this property, construct the line segment $\overline{AB}$. Use the process illustrated in Figure 12-19 to construct the perpendicular bisector to $\overline{AB}$. By Theorem 12-4, the intersection of *m* and this perpendicular bisector is a point on *m* equidistance from *A* and *B*.

20. The center of the circle is the intersection of the $\perp$ bisectors of any two adjacent sides of the square. The radius of the circle is the distance from the center to any of the vertices of the square. Alternatively, the center can be found as the intersection of the diagonals.

21. **(a)** **Yes**. The center of the circle is at the intersection of any two of the three perpendicular bisectors.



(b) **Yes**. Construct a regular hexagon *ABCDEF*. Bisect adjacent angles *B* and *C*. Label the intersection of these bisectors *G*. This is the center of the circle that circumscribes the hexagon. To prove that the circle circumscribes the regular hexagon, construct $\overline{AG}$, $\overline{DG}$, $\overline{EG}$ and $\overline{FG}$ noting that the triangles adjacent to ΔBCG are congruent to ΔBCG by SAS. Use this same approach to move around the figure noting that adjacent triangles are congruent. This shows that $\overline{AG}$, for example, is congruent to $\overline{BG}$.

(c) **No.** Construct a quadrilateral inside a circle and note that the center is an equal distance from each of the vertices of the quadrilateral. By Theorem 12-3, the center is on the perpendicular bisectors of the sides of the quadrilateral. So the four perpendicular bisectors are concurrent. But, parallelograms that are not rectangles do not have this property since the perpendicular bisectors of opposite sides are parallel.

22. **(a)** Construct point C as the intersection of two circles; one circle is centered at point A, the other at point B, both with diameter (length) $\overline{AB}$. Each angle will measure $60°$.

(b) The construction in part (a) produces three $60°$ angles.

(c) Extend one of the sides of the triangle in (a). The exterior angle to the equilateral triangle formed will be supplementary to a $60°$ angle and thus $120°$.

23. **(a)** Answers vary. One possible solution: $\Delta CAB \cong \Delta CBA$

(b) From part (a) $\angle A \cong \angle B$, by corresponding parts of congruent triangles.

24. **No.** If the three points are collinear, then a point the same distance from two of the points will not be the same distance from the third. The answer is yes if the points are not collinear. To see this, construct line segments connecting the three points forming a triangle. The center of the circumscribing circle will be the same distance from each of the three points.

Assessment 12-1B

1. **(a)** **True**; Corresponding Parts of Congruent Triangles are Congruent (CPCTC).

(b) **True**; Corresponding Parts of Congruent Triangles are Congruent (CPCTC).

(c) **False**; The two line segments are not corresponding line segments, so there is no guarantee that they are congruent.

2. The two triangles are congruent by the Side, Angle, Side (SAS) congruence property, so therefore, $\overline{RI} \cong \overline{BC}$ by CPCTC.

3. We are given that $\overline{BC} \cong \overline{CE}$ and that $\overline{AC} \cong \overline{DC}$. Also $\angle ACB \simeq \angle DCE$ because they are vertical angles, and vertical angles are always congruent. Therefore $\Delta ABC \cong \Delta DEC$ by the Side, Angle, Side (SAS) property. That means $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle ACB \cong \angle DCE$; also, $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DC}$ and $\overline{BC} \cong \overline{EC}$ by CPCTC.

4. We are given that $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$. Also, $\overline{BD} \cong \overline{BD}$. Therefore, $\Delta ABD \cong \Delta CBD$ by **SSS**.

5. We are given convex quadrilateral *ABCD*, with $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$. Draw diagonal segment $\overline{BD}$. Two triangles are created, ΔABD and ΔCDB. Since $\overline{BD}$ is congruent to itself, by **SSS**, $\Delta ABD \cong \Delta CDB$. This means that $\angle ABD \cong \angle CDB$ by CPCTC. Note that $\overline{BD}$ is a transversal, with $\angle ABD$ and $\angle CDB$ alternate interior angles. Since the alternate interior angles are congruent, that means $\overline{AB} \parallel \overline{CD}$. Similarly, $\angle ADB \cong \angle CBD$ by CPCTC, which means $\overline{AD} \parallel \overline{BC}$ since they are also congruent alternate interior angles.

6. Since $\overline{CA} \cong \overline{CB}$, by Theorem 12.5, part a (page 700), the angles opposite those sides are congruent; i.e., $\angle A \cong \angle B$. So, $x - 40 = \frac{x}{2}$.

$$\begin{aligned} x - 40 &= \tfrac{x}{2} \\ 2x - 80 &= x \\ x &= 80 \end{aligned}$$

Substituting 80 in for *x*, both $\angle A$ and $\angle B$ measure $40°$ each $(m(\angle A) = 40°,\ m(\angle B) = 40°)$. Since the interior angles of a triangle (in Euclidean Geometry) must sum to 180 degrees, $m(\angle C) = 100°$

7. **(a)** **CDAB**, since $\angle A$ corresponds to $\angle E$, $\angle B$ corresponds to $\angle F$, $\angle C$ corresponds to $\angle G$, and $\angle D$ corresponds to $\angle H$.

(b) **Solving for *x***: Since $\angle C$ corresponds to $\angle G$, and $m(\angle G) = 40°$, then $m(\angle C) = 40°$ as well. Since $m(\angle C) = (2x)°$ as well, then $x = 20$.

Solving for *y*: Since $\overline{CD}$ corresponds to $\overline{GH}$ and the length of $\overline{CD}$ = 18, then the length of $\overline{GH}$ must also be 18. Since $GH = 7y - 10$, then

$$\begin{aligned} 7y - 10 &= 18 \\ 7y &= 28 \\ y &= 4 \end{aligned}$$

Solving for *z*: Side $\overline{AD}$ corresponds to $\overline{EH}$, and the length of $\overline{EH}$ = 12, so the length of $\overline{AD}$ is also 12. Since $AD = z - 10$, then $z - 10 = 12; \Rightarrow z = 22$

8. Since in ΔABD, $\overline{AB} \cong \overline{BD} \cong \overline{AD}$, all the angles must be congruent as well: $\angle A \cong \angle B \cong \angle C$. So, $x° = m(\angle A) = 60°$, which means $\boldsymbol{x = 60}$. Now, in ΔCBD, $m(\angle CBD) = 120°$, since $m(\angle A) = m(\angle B) = 60°$, and $\angle ABD$ and $\angle CDB$ are supplementary angles. Note also that $\overline{BD} \cong \overline{BC}$, meaning $y° = m\angle(BCD)$ $= m\angle(BDC)$ So, since $m\angle(CBD) = 120°$, $m\angle(BCD) + m\angle(BDC) = 60°$. This means that $y° = m\angle(BCD) = m\angle(BDC) = 30°$. So, $\boldsymbol{y = 30}$.

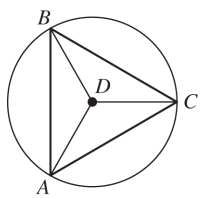
9. **(a)** **Yes**, *SAS*.

(b) **Yes**, *SSS*.

(c) **No**. Since the angle is not between the sides (adjacent to only one side) more than one triangle can be formed.

10. **No**. Construct a parallelogram that is not a rectangle. Draw the $\perp$ bisectors of three adjacent sides. Points equidistant from the vertices must lie on all three of the $\perp$ bisectors. But the $\perp$ bisectors of the opposite sides do not intersect.

11. **No**. If 4 points A, B, C, D existed in a plane and were equidistant, three would determine the vertices of an equilateral triangle, ΔABC. Thus, D would have to be at the center of the circle that circumscribes ΔABC. This would create three congruent triangles: ΔADB, ΔBDC, and ΔCDA. Thus, ΔADB for example would have angle measures $120°, 30°$, and $30°$. Thus, $\overline{AD}$ would not be congruent to $\overline{AB}$.



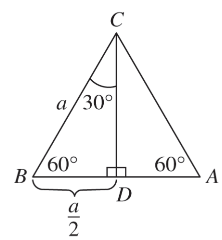
12. (a) Construct an angle and mark off a line segment of any length on one of the rays. Use this line segment to construct a larger angle at the opposite endpoint to the first angle. The third vertex will be the intersection of the two remaining rays.
(b) The side opposite the angle of greatest measure appears to be the one of greatest length. For all triangles, the side opposite the angle of greatest measure has greater length than a side opposite an angle of lesser measure.
(c) The hypotenuse is longer than any of the legs. This is true because the hypotenuse is opposite the angle of measure $90°$, which is the angle of greatest measure in a right triangle.

13. (a) The lengths must be the same. They are corresponding parts of congruent triangles.
(b) **No**. If the ground is not level, or perpendicular to the antenna, at least one of the guy wires no longer forms a right triangle. Thus the lengths are no longer the same.

14. Draw a line segment with the straightedge. From each endpoint, sweep out arcs whose radii are the length of the line segment. The intersection of the arcs provides the third vertex for an equilateral triangle. Then construct the perpendicular bisector of the base; it will form the $30°$ angle bisector of the $60°$ included angle.

15. (a) Use the procedure of Figure 12-11 in the text, to create two adjacent angles congruent to $\angle A$ and then an angle congruent to $\angle B$ adjacent to one of the outer most rays of one of the angles congruent to $\angle A$.
(b) Create two angles B adjacent to each other. Create angle A inside this newly formed angle, $2B$. The difference is $m(\angle C) = 2m(\angle B) - m(\angle A)$.

16. (a) By Hypotenuse-Leg (HL).
(b) Let the length of each of the sides of the equilateral triangle ABC be a. Because $BD = DA$, $BD = \frac{1}{2}a$. Thus, in the $30°$, $60°$ triangle BCD, BD is opposite the $30°$ angle and $BD = \frac{1}{2}a = \frac{1}{2}BC$.



17. (a) Because $\angle BOC$ measures $60°$ and $\overline{OG}$ bisects $\overline{BC}$, $\angle BOG$ and $\angle COG$ measure $30°$. Since $\Delta BOG \cong \Delta COG$ by SAS, $\overline{BG} \cong \overline{CG}$. This same argument can be made for each of the adjacent vertices of the hexagon. A similar argument shows that all the constructed triangles are congruent. Since we have constructed a 12-gon with congruent sides and interior angles, we have constructed a regular 12-gon.
(b) Because the hexagon is regular, each interior angle measures $\frac{360°}{60°} = 60°$. Because ΔBOC is isosceles, the equal base angles measure $\frac{180°-60°}{2} = 60°$. Thus, ΔBOC is equilateral.

(c) Using (b) offers an approach to constructing a regular hexagon by adjoining six congruent equilateral triangles. After this step, part (a) can be used to create the 12-gon.

18. **(a)** F is at the midpoint of both diagonals. $\Delta ABC \cong \Delta ADC$ by SSS; $\angle CAB \cong \angle CAD$ by CPCTC; and $\Delta BAF \cong \Delta DAF$ by SAS. Thus $\overline{BF} \cong \overline{FD}$ and F is a midpoint. A similar argument will show $\overline{AF} \cong \overline{FC}$.

(b) **90°**. They are congruent because $\Delta BAF \cong \Delta DAF$ and they are supplementary.

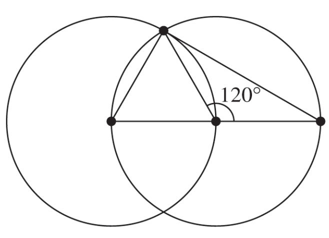
19. Place three points at different locations on the circle. Construct line segments connecting the points to form a triangle. Use the process described in Figure 12-19 to construct the perpendicular bisectors for two of the sides of the triangle. The intersection of these bisectors is the center of the circle.

20. Answers vary on ways to construct the rectangle. The construction used in Figure 12-19 can be used to create a perpendicular to a line segment. Draw a line; now, create two lines perpendicular to the original line. On one of these new lines, create a line perpendicular to it; this last line should be parallel to the first line. The center of the circle is the intersection of the perpendicular bisectors of any two adjacent sides. The radius of the circle is the distance from the center to any of the vertices.

21. **(a)** **Not possible** unless the rhombus is a square.

(b) **Possible**. The center O of the circle is the intersection of any two perpendicular bisectors of two adjacent sides. The radius is the segment from O to any of the vertices.

22. **(a)** Construct an equilaterial triangle and use an exterior angle as shown below.



(b) Bisect a $60°$ angle and use the supplement of a $30°$ angle.

23. **(a)** The three sides are congruent in any order;

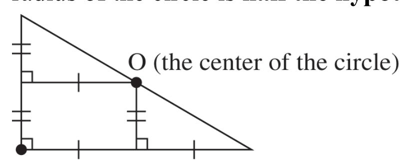
$$\begin{aligned} \Delta ABC &\cong \Delta ABC \\ &\cong \Delta BAC \\ &\cong \Delta CAB \\ &\cong \Delta ACB \\ &\cong \Delta BCA \\ &\cong \Delta CBA. \end{aligned}$$

(b) The four sides are congruent as follows;

Quadrilateral $ABCD$ ≅ Quadrilateral $ABCD$
≅ Quadrilateral $DCBA$
≅ Quadrilateral $CDAB$
≅ Quadrilateral $BADC$.

24. **(a)** Answers will vary. One way to construct the right triangle is using the method in 12-19 to construct a right angle.

(b) Use the construction described in 12-20. The center of the circumscribing circle lies on the hypotenuse. The figure below offers an intuitive argument for why this is so. Thus, the radius of the circle is **half the hypotenuse**.

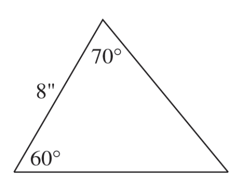


(c) Since O is the center of the circumscribing circle and the vertex associated with the right angle lies on the circle, the radius of the circumscribing circle is half the length of the hypotenuse.

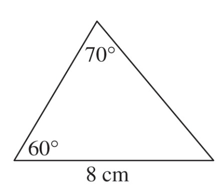
Assessment 12-2A: Additional Congruence Theorems

1. Constructions may vary; those below are representative.

(a) *ASA*:



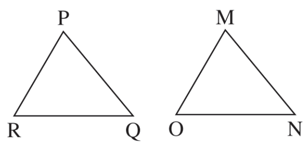
(b) *AAS*:



(c) **Infinitely many** are possible. All would be similar; *AAA* determines a unique shape, but not size.

2. **(a)** **No**. The triangle is unique by *ASA*.
(b) **No**. The triangle is unique by *AAS*.
(c) **Yes**. *AAA* determines a unique shape, but not size.

3. Compare the following triangles:



(a) **Yes**. Congruent by *ASA*.
(b) **Yes**. Congruent by *AAS*.

4. At any setting of the ruler, it is given that $\overline{AB} \cong \overline{DC}$ and $\overline{AC} \cong \overline{BD}$. Note that $\overline{BC} \cong \overline{CB}$. Then $\Delta ABC \cong \Delta DCB$ by *SSS*. $\angle ABC \cong \angle DCB$ by CPCTC. Because these are alternate interior angles formed by lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ with transversal line $\overleftrightarrow{BC}$, $\overline{AB} \parallel \overline{DC}$.

5. **(a)** $\angle ABD \cong \angle CBD$ is given. Note that D is a point on $\overline{AC}$ such that $A-D-C$. Since $\angle CDB$ is a right angle, then $\angle ADB$ is also a right angle (the two angles are supplementary); therefore they are congruent. Finally, $\overline{BD} \cong \overline{BD}$. So, by **ASA**, $\Delta ABD \cong \Delta CBD$

(b) There are no congruent triangles in this figure, based on the information that is given. AAA is not a congruence property.

6. Since the student has $\angle B \cong \angle E$ and $\overline{BC} \cong \overline{EF}$, congruence can be established in two different ways. If the student can show that $\angle C \cong \angle F$ then the triangles will be congruent by the **ASA** congruence property. If the student can show that $\overline{AB} \cong \overline{DE}$, then the triangles will be congruent by the **SAS** congruence property.

7. **(a)** Since $\Delta ABC \cong \Delta DEF$, $\angle E$, corresponds to $\angle B$, meaning they must have the same measure. Therefore $m(\angle E) = 85°$.

(b) By Theorem 11.3, it is known that the sum of the interior angles of a triangle is $180°$. Since $m(\angle A) = 40°$, and $m(\angle B) = 85°$, then $m(\angle C) = 55°$. And, since $\Delta ABC \cong \Delta DEF$, $\angle F$ corresponds to $\angle C$, meaning they must have the same measure. Therefore, $m(\angle F) = 55°$.

(c) Since $\Delta ABC \cong \Delta DEF$, $\overline{DE}$ corresponds to $\overline{AB}$, meaning they must have the same measure. Therefore $m(\overline{DE}) = 6$.

(d) Since $\Delta ABC \cong \Delta DEF$, $\overline{EF}$ corresponds to $\overline{BC}$, meaning they must have the same measure. Therefore $m(\overline{EF}) = 5$.

8. We are given that $\angle ABC \cong \angle DEF$, $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. So $\Delta ABC \cong \Delta DEF$ by **SAS**. That means $\angle BAC \cong \angle EDF$ by CPCTC.

9. We are given $\overline{AD} \parallel \overline{EC}$ and $\overline{BC} \cong \overline{BD}$. Since $\overline{AD} \parallel \overline{EC}$, alternate interior angles $\angle DAB$ and $\angle CEB$ are congruent. Also $\angle ABD \cong \angle EBC$ since they are vertical angles. Therefore, $\Delta ABD \cong \Delta EBC$ by **AAS**.

10. Since $ABCD \cong EFGH$, $\overline{CD}$ corresponds to $\overline{GH}$, meaning they must have the same length. So

$$2x - 4 = 10$$
$$2x = 14$$
$$x = 7.$$

Also, Since $ABCD \cong EFGH$, $\angle A$ corresponds to $\angle E$, so their measures are equal. This means

$$7y° + 9° = 72°$$
$$7y° = 63°$$
$$y = 9.$$

11. See Table 12-1 for properties.
(a) **Parallelogram**, by properties *d* and *e*.
(b) **None**. It must be known that the quadrilateral is a parallelogram before it can be known that it is a rectangle. Otherwise it could be an isosceles trapezoid.
(c) **None**. The quadrilateral could be a parallelogram, rectangle, or trapezoid even if the diagonals are not perpendicular.

12. An isosceles trapezoid is formed. The angles formed with one base as a side are congruent, as are the angles formed on the other base.

The congruent angles are sometimes referred to as the base angles of the isosceles trapezoid.

13. See Table 12-1.

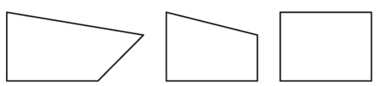
(a) **True**, by property.

(b) **True**, by definition.

(c) **True**, by definition.

(d) **False**. A trapezoid may have only one pair of parallel sides, while in a parallelogram each pair of opposite sides must be parallel. A trapezoid may also have two consecutive right angles with the other two not right angles.

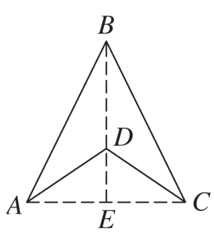
14. (a) Constructions may vary; possibilities include:



(b) **No**. The sum of the measures of the angles in a quadrilateral must be 360°. If the quadrilateral has three right angles, then the fourth must also be a right angle.

(c) **No**. Any parallelogram with a pair of right angles must have right angles as its other pair. Thus it must also be a rectangle (a rectangle is a parallelogram with right angles).

15. *ABCD* is a kite.



Because $AB = BC$, B is equidistant from A and C. By Theorem 12-3, point B is on the perpendicular bisector of $\overline{AC}$. Likewise, D is on the perpendicular bisector of $\overline{AC}$. Because two points determine a unique line $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC} \Rightarrow \overleftrightarrow{BD} \perp \overleftrightarrow{AC}$.

16. (a) **Rectangle**. $\overline{AB} \parallel \overline{ED}$ and $\overline{AE} \parallel \overline{BD}$. Because $\Delta AFE \cong \Delta BCD$ adjacent angles are congruent.

(b) **Isosceles trapezoid**. $\overline{BE} \parallel \overline{AF}$ but $\overline{AB} \nparallel \overline{EF}$; $EF = AB$.

17. A **rhombus**. All sides are congruent.

18. Either the arcs or the central angles must have the same measure. Radii are already the same, since the sectors are part of the same circle.

19. (a) A **kite**.

$\overline{DX} \cong \overline{AX}$ (given). $\angle D$ and $\angle A$ are right angles of the rectangle. $\overline{DP} \cong \overline{AQ}$ (given). Thus $\Delta PDX \cong \Delta QAX$ by SAS and $\overline{PX} \cong \overline{QX}$ by CPCTC.

$\overline{DC} \cong \overline{BA}$ as opposite sides of the rectangle. Since $\overline{DP} \cong \overline{AQ}$ then $\overline{CP} \cong \overline{BQ}$.

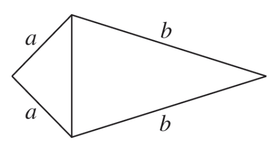
$\overline{CY} \cong \overline{BY}$ (given). $\angle C$ and $\angle B$ are right angles of the rectangle. Therefore $\Delta CYP \cong \Delta BYQ$ by SAS so that $\overline{QY} \cong \overline{PY}$.

By definition, then, *PXQY* is a kite.

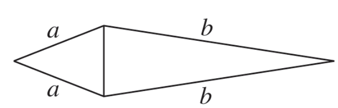
(b) The quadrilateral is **still a kite**. When P and Q, however, are midpoints of $\overline{DC}$ and $\overline{AB}$ respectively, *PXQY* is a rhombus.

20. Make one of the quadrilaterals a square and the other a rectangle that is not a square.

21. Construct the first kite by constructing a segment to become a diagonal and then construct two isosceles triangles with the segment as a common base (see below).



Construct the second kite starting with a segment not congruent to the first. Construct two isosceles triangles with that segment as their common base but the sides congruent to the corresponding sides of the isosceles triangles in the first construction (see below).



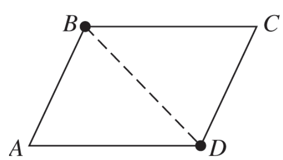
22. **Rhombus**. Use *SAS* to prove that $\Delta ECF \cong \Delta GBF \cong \Delta EDH \cong \Delta GAH$. Then $\overline{EF} \cong \overline{GF} \cong \overline{EH} \cong \overline{GH}$.

23. **(a)** The lengths of one side of each square must be congruent.

(b) Two adjacent sides of one must be congruent to the corresponding sides of the other.

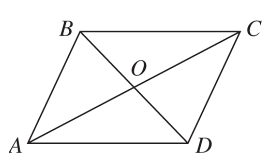
(c) Answers may vary. E.g., one solution is that two adjacent sides and the included angle of one must be congruent to the corresponding parts in the other.

24. **(a)** In the figure, $\overline{BC} \parallel \overline{AD}$ and $\overline{BC} \cong \overline{AD}$.



Since $\overline{BC} \parallel \overline{AD}$, $\angle ADB \cong \angle CBD$. Also, $\overline{BD} = \overline{DB}$. Thus, by SAS, $\Delta ADB \cong \Delta CBD$. Thus, corresponding parts are congruent, and $\overline{AB} \cong \overline{CD}$ and $\angle ABD \cong \angle CDB$. Since these last angles are alternate interior angles for transversal $\overline{BD}$, $\overline{AB} \parallel \overline{CD}$. Since opposite sides are parallel and congruent, the quadrilateral is a parallelogram.

(b) If the diagonals bisect each other, then $AO = OC$ and $BO = OD$ in the figure below.

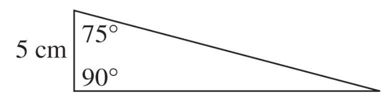


In addition, vertical angles $\angle BOC$ and $\angle DOA$ are congruent. Thus, by *SAS*, $\Delta BOC \cong \Delta DOA$. $\angle OBC \cong \angle ODA$ because they are corresponding parts of $\cong$ Δ's, and $\overline{BC} \parallel \overline{AD}$, because the angles are alternate interior angles formed by transversal $\overline{BD}$. Since quadrilateral *ABCD* has a pair of opposite sides parallel
and congruent, it must be a parallelogram by part (a).

25. Answers may vary, e.g., the polygons must have the same number of sides with one pair congruent. All regular polygons with the same number of sides are similar, so if they have the same number of sides with one pair congruent they are congruent.

Assessment 12-2B

1. **(a)** *ASA*:

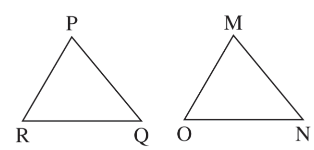


(b) **Infinitely many** are possible. All would be similar; *AAA* determines a unique shape, but not size.

2. **(a)** **No**. Because it is specified that the side given is a leg (not the hypotenuse) on a side of the given angle, the triangle is unique by *ASA*.

(b) **Yes**. *AAA* determines a unique shape, but not size.

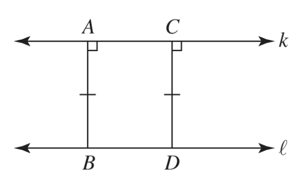
3. Compare the following triangles:



(a) **No**. *SSA* does not assure congruence.

(b) **No**. *AAA* does not assure congruence.

4. Because the angles at *A* and *C* are congruent corresponding angles, $\overline{AB} \parallel \overline{CD}$.



Since $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$ we can conclude that *ABDC* is a parallelogram, so $k \parallel l$.

5. **(a)** In the figure, it is given that $\angle DAC \cong \angle BAC$ and $\angle DCA \cong \angle BCA$. Also $\overline{AC} \cong \overline{AC}$. Therefore by **ASA**, $\Delta ACD \cong \Delta ACB$.

(b) There is not enough information given to determine if the triangles are congruent. It is known that $\angle ACB \cong \angle DCE$, since they are vertical angles; and we are given $\overline{AB} \cong \overline{ED}$ and $\overline{AC} \cong \overline{DC}$, but SSA is not a congruence property.

6. To show $\Delta ABC \cong \Delta DEC$, it needs to be shown that $\overline{AB} \cong \overline{DE}$; then the triangles would be congruent by **ASA**.

7. Note that $\overline{AD}$ and $\overline{BD}$ are congruent, since they are both radii of the same circle. Since they are congruent, ΔABD is an isosceles triangle, and $\angle A \cong \angle 5$. So, $m(\angle 5) = 30°$. This means $m(\angle 1) = 120°$, since $\angle 1$, $\angle 5$, and $\angle A$ are all interior angles of the same triangle. From knowing $m(\angle 1)$, we can calculate $m(\angle 2) = 60°$, since $\angle 1$ and $\angle 2$ are supplementary angles. Now, note that $\overline{BD}$ and $\overline{CD}$ are both radii of the same circle, so they must be congruent. That means $\angle 4 \cong \angle 3$. So, using that, combined with $m(\angle 2) = 60°$, it can be found that $m(\angle 3) = 60°$ and $m(\angle 4) = 60°$.

8. It is given that $\overline{PQ} \parallel \overline{RS}$ and $\angle P \cong \angle R$. Draw in diagonal $\overline{SQ}$. Since alternate interior angles are congruent, $\angle PSQ \cong \angle RQS$. Also, $\overline{SQ} \cong \overline{SQ}$. So, $\Delta PSQ \cong \Delta RQS$. by **AAS**. That means $\overline{PQ} \cong \overline{RS}$ by CPCTC.

9. It is given that ΔISO is an isosceles triangle, with $\overline{SI} \cong \overline{SO}$. Now, ΔANG is also an isosceles triangle, with $\overline{NA} \cong \overline{NG}$. Since it is given that $m(\angle A) = 35°$, we can quickly deduce $m(\angle G) = 35°$ as well, since the angles opposite congruent sides of isosceles triangles are congruent. So, $m(\angle N) = 110°$. That means by **SAS**, $\Delta ISO \cong \Delta ANG$.

10. Since $\Delta ABC \cong \Delta DEF$, $\angle B$ corresponds to $\angle E$, meaning they have the same measure. So.
$4x° - 4° = 48°$
$4x° = 52°$
$x = 13$

Also, since $\Delta ABC \cong \Delta DEF$, $\angle C$ corresponds to $\angle F$, meaning they have the same measure. So
$5y° - 3° = 62°$
$5y° = 65°$
$y = 13$

11. See Table 12-1 for properties.

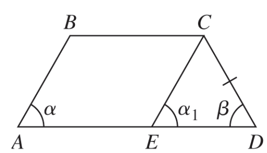
(a) **Rectangle**, by property *d* and *e*.

(b) **Rhombus**, by property *c* and *d*.

(c) **Square**. By parallelogram property *d*, rectangle property *d and e*, and rhombus property *c* and *d*.

(d) **Parallelogram**, by definition. See exercise 24 in section 12.2A

12. Construct $\overline{CE}$ such that $\overline{CE} \parallel \overline{AB} \Rightarrow ABCE$ is a parallelogram:



Then $AD = CE$, which makes ΔBEC isosceles $\Rightarrow \alpha_1 = \beta$. Because $\alpha = \alpha_1$ (corresponding angles created by the parallels $\overline{AB}$ and $\overline{CE}$) $\alpha = \beta$, or $\angle A \cong \angle D$.

13. See Table 12-1.

(a) **False**. A square is both a rectangle and a rhombus.

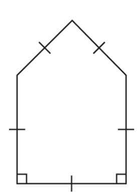
(b) **False**. A square satisfies all conditions of a trapezoid, thus some trapezoids must be squares.

(c) **True**. All squares, by definition, are trapezoids.

(d) **True**. A rhombus is a parallelogram with all sides congruent.

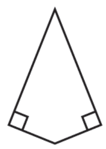
(e) **True**. A square is a rhombus with all angles congruent.

14. **(a)** Construction may vary:



(b) **No**. The sum of the interior angles of a quadrilateral must be $360°$, but the sum of four acute angles is less then $360°$.

(c) **Yes**, To construct a kite with exactly two right angles, construct a right triangle in which no angle measures $45°$ and then reflect it across its hypotenuse, as shown:



15. **Yes**. $\Delta ABD \cong \Delta CBD$ by $SSS \Rightarrow \angle BAD \cong \angle BCD$ by CPCTC.

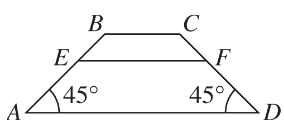
16. (a) **Isosceles trapezoid**. $\overline{AD} \parallel \overline{BC}$ and $\angle BAD \cong \angle ADC$.

(b) **Kite**. $\overline{AE}$ and $\overline{CE}$ are adjacent and congruent; likewise $\overline{AB}$ and $\overline{BC}$ are adjacent and congruent.

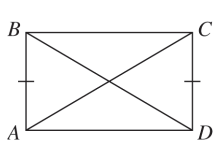
17. (a) Draw a line segment. Use the construction method from 12-1 Figure 19 to construct a perpendicular line segment (Note the method can be modified so that the perpendicular segment is not necessarily a bisector.). Using these two segments as diagonals, construct the sides of the kite.

(b) At the vertices construct lines perpendicular to the diagonals. The intersections of the lines are the vertices of the rectangle.

18. Answers vary. For example, two isosceles trapezoids $AEFD$ and $ABCD$ as shown where $\overline{EF} \parallel \overline{AD}$.



19. (a) Because the diagonals bisect each other from 24 (b) in Assessment 12–2A, we know that the quadrilateral is a parallelogram. We only need to prove that one of the angles of the quadrilateral is a right angle. Because we know that the diagonals are congruent, we know that $\Delta ABD \cong \Delta DCA$ in the figure below.



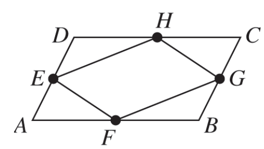
The fact $\Delta ABC \cong \Delta DCA$ follows from SSS by observing that $\overline{AB} \cong \overline{DC}$ (opposite sides of a parallelogram), $\overline{AD} = \overline{DA}$, and $\overline{BD} = \overline{CA}$ (given). Thus, $\angle BAD \cong \angle CDA$. These angles are also supplementary since $\overline{AB} \parallel \overline{CD}$. Thus, each must be a right angle.

(b) Construct two congruent line segments that bisect each other and the endpoints of these line segments will be the vertices of a rectangle.

(c) If the angles created by the diagonals are not congruent, the corresponding sides of the two rectangles will not be $\cong$.

20. (a) **Parallelogram**.

(b) Suppose $ADCB$ in part (a) is a parallelogram. By SAS, $\Delta EDH \cong \Delta GBF$, thus $\overline{EH} \cong \overline{GF}$. Similarly, $\Delta EAF \cong \Delta GCH$ so $\overline{EF} \cong \overline{GH}$. If opposite sides of a quadrilateral are congruent, it is a parallelogram.



21. (a) Two adjacent sides of one must be congruent to the corresponding adjacent sides of the other.

(b) Possibilities include:

(*i*) The sides and one angle are congruent to the corresponding parts in the other kite, or

(*ii*) Corresponding diagonals are congruent.

(c) The vertex (where the congruent sides intersect) angles are congruent, and one of the two congruent sides in one triangle is congruent to the corresponding side in the other.

22. (a) $\angle ABD \cong \angle CDB$ and $\angle CBD \cong \angle ADB$ (alternate interior angles with respect to parallel lines). Then $\Delta ABD \cong \Delta CDB$ (by ASA). Thus $\angle BAD \cong \angle DCB$ (CPCTC); similarly, $\angle ABC \cong \angle CDA$.

(b) $\Delta ABD \cong \Delta CDB$, thus $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$ (CPCTC).

(c) $\angle BAC \cong \angle DCA$ and $\angle ABD \cong \angle CDB$ (alternate interior angles), and $\overline{AB} \cong \overline{DC}$ [from part (b)]. Then $\Delta BAF \cong \Delta DCF$, so $\overline{AF} \cong \overline{CF}$ and $\overline{BF} \cong \overline{DF}$ (CPCTC).

(d) In ΔABD, $m(\angle BAD) + m(\angle ABD) + m(\angle ADB) = 180°$.
From part (a), $\angle CBD \cong \angle ADB$. Substitution yields
$m(\angle BAD) + m(\angle ABD) + m(\angle CBD) = 180°$.
Since $m(\angle ABC) = m(\angle ABD) + m(\angle CBD)$, then $m(\angle BAD) + m(\angle ABC) = 180°$. $\angle ABC$ and $\angle BAD$, therefore, are supplementary.

23. Two rhombi are congruent if a side of one is congruent to a side of the other and an angle of one is congruent to an angle of the other. To see this, note that in a rhombus all four sides are congruent. Thus, establishing that the two rhombi have one pair of congruent sides is sufficient to conclude all the sides are congruent. To see that a pair of congruent angles is sufficient, use the properties of a parallelogram established in 19(d).

Mathematical Connections 12-2: Review Problems

21. Triangles ***BCD*, *CDE*, *DEA*,** and ***EAB***. By definition, all five sides and angles are congruent. Therefore the triangles are congruent by *SAS*.

22. Use the procedure shown in textbook Figure 12-10.

23. Follow the procedure of Figure 12-10, using the given segment for all three sides.

24. **(a)** **Yes**, by *SAS*.
(b) **Yes**, by *SSS*.
(c) **Yes**, by Hypotenuse-Leg (HL).

Assessment 12-3A: Additional Constructions

1. **(a)** Refer to textbook Figure 37 in Chapter 12. Draw a line through *P* that intersects ℓ, but copy α as an alternate interior angle (i.e., vertical to α as shown in the Figure).
(b) Refer to Figure 12-36. Draw the line through *P* that intersects ℓ, and use the construction shown to create a rhombus.

2. Draw a line l and a point P not on l. Let B be any point on line l and construct $\overline{PB}$. Let C be any other point on line l. Copy $\angle PBC$ to create new angle $\angle BPD$ (see page 697, figure 11 on how to copy an angle) so that the vertex of the new angle is point P and the side $\overline{PD}$ of the new angle $\angle BPD$ is such that points D and C are on opposite sides of $\overline{PB}$. Now extend $\overline{PD}$ to create line m. If this is done correctly, you should have $l \parallel m$, since $\angle PBC$ and $\angle BPD$ are congruent alternate interior angles.

3. Copy $\angle A$ using the directions for copying an angle on page 697 (figure 11). Now, place the compass point on vertex A and draw an arc that intersects both sides of the angle. Label the points of intersection B and C. Keeping the compass opening the same as before, place the point of the compass on point B. Construct an arc in the interior of $\angle BAC$. Place the compass point on point C, and construct another arc in the interior of $\angle BAC$, so that the two arcs intersect. Label the point of intersection D. Construct ray $\overrightarrow{AD}$ using a straightedge. This ray should be the bisector of $\angle BAC$, and $\angle BAD \cong \angle CAD$.

4. Since $\overrightarrow{AC}$ bisects the angles, they are congruent. That means they have the same measure. So, $x - 1 = 24 \Rightarrow x = 25$

5. **(a)** Since $\overrightarrow{BC}$ bisects $\angle ABD$, $\angle ABC \simeq \angle DBC$. That means they have the same measure. So, $4x - 5 = 3x + 4 \Rightarrow x = 9$

(b) To find the measure of $\angle ABD$, put the answer from part (a) into either of the expressions. That will give you the angle measure for one-half of $\angle ABD$. Then, double it to obtain the final answer. For example, using the expression $4x - 5$,
$4(9) - 5 = 31;$
$31 \times 2 = 62.$
So, $m(\angle ABD) = 62°$

6. **(a)** A **right triangle**. Assuming the ground is level, the cable will hang perpendicularly to the ground.
(b) The altitude is the extension of the cable from vertex *A* to the ground. It will lie outside the triangle.

7. **(a)** The perpendicular bisectors of the sides of an acute triangle meet inside the triangle.

(b) The perpendicular bisectors of the sides of a right triangle meet at the midpoint of the hypotenuse.

(c) The perpendicular bisectors of the sides of an obtuse triangle meet outside the triangle.

(d) Use the point of intersection of the perpendicular bisectors as the center of the circle, then use the distance from the intersection to any of the vertices as the radius of the circle.

8. (a) Construct the perpendicular bisectors of any two sides of the triangle. The intersection of the two is a point *P*, equidistant to *A*, *B*, and *C*.

Any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment. Point *P* is equidistant from all vertices because it is on two perpendicular bisectors. Being at the intersection of two of the perpendicular bisectors forces the point to be equidistant from all three vertices.

(b) Same as part (a), except that point *P* will be outside the obtuse triangle. In part (a) it was inside.

9. (a) If the rectangle is not a square it is **impossible** to construct an inscribed circle. The angle bisectors of a rectangle do not intersect in a single point.

(b) **Possible**. The center of the circle is the intersection of the diagonals (which are also the angle bisectors of the vertices). The radius of the circle is the distance from the center to any of the sides.

(c) **Possible**. The intersection of the three longest diagonals is the center of the circle.

10. Extend $\overline{AB}$ and construct a right angle at *A*, using the method of Figure 12-45. Measure $\overline{AB}$ with a compass and use it to mark off side $\overline{AC}$ along the perpendicular.

From *C* and *B*, mark off the length of $\overline{AB}$ by drawing compass arcs at the approximate location of the final vertex. Call the intersection of the two arcs point *D*.

Construct $\overline{AD}$ and $\overline{CD}$ to form square *ABCD*.

11. (a) Make an arc of radius *BC* with center *A* and one with radius *AB* and center *C* so that the two intersect. This intersection is the location of the fourth vertex.

Another technique would be to construct a parallel to $\overline{BC}$ through *A*, then mark off the distance *BC* from *A*.

(b) Since one use of a compass or straight edge is not sufficient to find the fourth vertex, we must use the tools twice to locate the fourth vertex. Then, we must use the straight edge twice to construct the remaining sides of the parallelogram after the fourth vertex is located. The cheapest way is **40¢**.

12. See pages 701–702 (figure 19) on how to construct the perpendicular bisector of a segment. Following the direction on those pages as given, and drawing segments $\overline{AC}$, $\overline{CB}$, $\overline{BD}$ and $\overline{DA}$, the quadrilateral that will be created should be a rhombus, since the four segments just drawn should be congruent. Further, since the diagonals of the rhombus bisect each other, the figure is also a parallelogram.

13. Construct line m perpendicular to $\overline{AB}$ at point A. With the compass point on B and the compass opening wider than $\overline{AB}$, draw an arc that intersects m in two points. Call these points C and D.

Draw segments $\overline{BC}$, $\overline{BD}$ and $\overline{CD}$; ΔBCD should be an isosceles triangle.

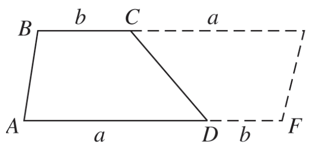
14. Because *O* is the incenter, the three angle bisectors of triangle *ABC* intersect at *O*. By Theorem 12-10, *OD* is equal to the distance from *O* to $\overline{AC}$. Label the point of intersection of $\overline{AC}$ and a line perpendicular to $\overline{AC}$ through *O* as *P*. *OPCD* forms a rectangle. Since opposites sides of rectangles are congruent, $\boldsymbol{OP = CD = 2''}$. Since $m(\overline{OD})$ is the distance from the incenter to $\overline{CB}$, $m(\overline{OD}) = 2''$.

15. The incenter is located where the angle bisectors of the interior angles meet. The radius is the distance from the incenter to any side.

16. The side opposite the $7''$ side is also $7''$ by parallelogram properties. Let $x = EF$. Then $5 + 5 - x = 7 \Rightarrow x = 3''$.

17. (a) If the parallelogram is not a rectangle, cut along any altitude. If the parallelogram is a rectangle, cut along any line through the point where the diagonals meet. The line must not be a diagonal nor parallel to any side.

(b) Make a copy of the given trapezoid and put it upside down next to $\overline{CD}$; i.e.,



Extend $\overline{BC}$ so that $CE = a$ and extend $\overline{AD}$ so that $DF = b$. Since $\overline{BE} \parallel \overline{AF}$ and $\overline{BE} \cong \overline{AF}$ (the length of each is $a + b$), $ABEF$ is a parallelogram.

18. As **close to 26 inches** as the jack can close; i.e., twice the length of a side.

19. **(a)** **Possible**. Draw line $\overline{AB}$ the length of a side of the square, then construct a right angle at A using the method of Figure 12-45. Measure $\overline{AB}$ with a compass and use it to mark off side $\overline{AC}$ along the perpendicular. From C and B, mark off the length of $\overline{AB}$ by drawing compass arcs at the approximate location of the final vertex. Call the intersection of the two arcs point D. Construct $\overline{CD}$ and $\overline{BD}$ to from square $ABDC$.

(b) **No unique rectangle**. The endpoints of two segments bisecting each other and congruent to the given diagonal determine a rectangle, but since the segments may intersect at any angle there are infinitely many such rectangles.

(c) **Not possible**. The sum of the measures of the angles would be greater than $180°$.

(d) **No unique parallelogram**. Given three right angles, the fourth angle must also be a right angle. The parallelogram would be a rectangle or square, an infinite number of which could be constructed.

20. **(a)** Construct an equilateral triangle and bisect one of its angles.

(b) Bisect a right angle.

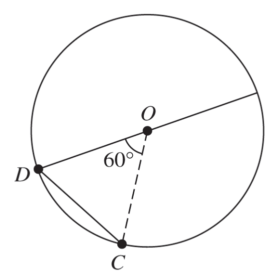
(c) Add $45°$ and $30°$ angles.

21. Make arcs of the same radius from A and B above $\overline{AB}$ and label their intersection C. Repeat the process with a new radius, labeling this intersection D. $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$.

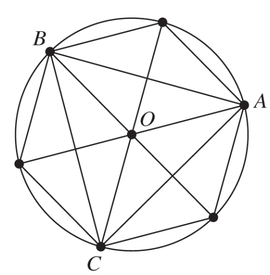
22. **(a)** **Possible**. The point is determined by the intersection of the angle bisector of $\angle A$ and the perpendicular bisector of $\overline{BC}$. Since the point is on the angle bisector of $\angle BAD$, it is equidistant from its sides. It is on the perpendicular bisector of $\overline{BC}$, thus it is equidistant from B and C.

(b) **Possible**. The point is determined by the intersection of the angle bisectors of $\angle A$ and $\angle B$.

23. Answers vary. One method is to construct a circle with center O and radius r. Then draw a diameter and label one of the intersections of the diameter and the circle as D. Construct an arc with radius r and center O. This will determine an equilateral triangle whose vertex O determines a central angle of $60°$.



This method can be continued to construct six congruent triangles as shown below. Two adjacent triangles determine a central angle of $120°$ with vertex O. Because ΔAOB, ΔBOC, and ΔAOC are congruent by SAS, $\overline{BC}$, $\overline{BA}$, and $\overline{AC}$ are congruent and hence we have constructed an equilateral inscribed triangle.



Assessment 12-3B

1. **(a)** Construct line m through P perpendicular to ℓ; then a line through P perpendicular to m.

   (b) Draw any line through P that intersects ℓ. Let Q be the point where the line intersects line ℓ. Find the midpoint M of $\overline{PQ}$. Draw any line through M intersecting ℓ at Q'. Find P' so M is the midpoint of $\overline{P'Q'}$. Then the line $\overleftrightarrow{PP'}$ is the desired figure.

2. Draw a line ℓ and a point P not on ℓ. Let B be any point on line ℓ and construct line $\overleftrightarrow{PB}$. Let C be any other point on line l, and let D be a point on line $\overleftrightarrow{PB}$ such that B is between D and P. Copy $\angle CBD$ to create new angle $\angle C'PD'$ (see page 697, figure 11 on how to copy an angle) so that the vertex of the new angle is point P, the side $\overline{PC'}$ of the new angle $\angle C'PD'$ is such that points C' and C are on opposite sides of $\overline{PB}$, and side $\overline{PD'}$ of the new angle $\angle C'PD'$ is on line $\overleftrightarrow{PB}$. Now extend $\overline{PC'}$ to create line m. If this is done correctly, you should have $\ell \parallel m$, since $\angle CBD$ and $\angle C'BD'$ are congruent alternate exterior angles.

3. To copy line segment $\overline{AB}$, first draw a point on the paper where you wish to copy the segment: call it point C. Now, put one point of the compass on point A the other on point B. Then, put the point of the compass on your drawn point C, and mark off a small arc. Select any point on that arc (label it point D) and construct segment $\overline{CD}$. So, $\overline{AB} \cong \overline{CD}$. To bisect $\overline{CD}$, place the compass point on point C and open the compass to a length such that it is more than 1/2 the length of $\overline{CD}$. Now, construct two arcs, one on each side of $\overline{CD}$. With the exact same compass opening, move the point of the compass over to point D. Construct two arcs, one on each side of $\overline{CD}$ such that the arcs intersect. Let points X and Y be the points of intersection on the arcs. Draw $\overline{XY}$ using a straightedge. The point where $\overline{XY}$ intersects $\overline{CD}$ will bisect $\overline{CD}$. In fact, $\overleftrightarrow{XY}$ is the perpendicular bisector of $\overline{CD}$.

4. **(a)** Since $\overrightarrow{OP}$ bisects $\overline{MN}$, then $\overline{MP}$ and $\overline{NP}$ have the same measure, so they are equal.

   $$2x + 5 = 15 - 3x$$
   Then, $$5x = 10$$
   $$x = 2$$

   (b) Since $\overrightarrow{OP}$ is the perpendicular bisector of $\overline{MN}$, then $\overline{MP} \cong \overline{NP}$ and $\angle MPO \cong \angle NPO$. Also, $\overline{OP} \cong \overline{OP}$. Therefore, $\Delta MPO \cong \Delta NPO$ by **SAS**. So, $\overline{MO} \cong \overline{NO}$ by CPCTC. Then

   $$5x + 1 = m(\overline{MO})$$
   $$5(2) + 1 = m(\overline{MO})$$
   $$11 = m(\overline{MO})$$

5. **(a)** It is given that $\overrightarrow{AC}$ is the angle bisector of $\angle BAD$. That means $\angle BAC \cong \angle DAC$. Then

   $$(8x - 34)° = (5x + 2)°$$
   $$3x = 36$$
   $$x = 12$$

   (b) To find $m(\angle BAD)$, put the answer from part (a) into either of the equations. That will give you the angle measure for one-half of $\angle BAD$. Then, double it to obtain the final answer. For example, using the expression $(8x - 34)°$ we obtain

   $$2(8x° - 34°) = m(\angle BAD)$$
   $$2[8(12°) - 34°] = m(\angle BAD)$$
   $$2(96° - 34°) = m(\angle BAD)$$
   $$2(62°) = m(\angle BAD)$$
   $$124° = m(\angle BAD)$$

6. Use the construction methods in figures 12-48 and 12-49 to construct the three altitudes. Extend these altitudes until all three intersect.

7. The intersection of the two perpendicular bisectors will be the center of the circumscribing circle. The center will be outside the triangle.

8. Draw the obtuse triangle; the center of the inscribed circle will be the intersection of the angle bisectors. Since the center is on each angle bisector it is equidistant from the sides of each angle; therefore from the sides of the triangle.

9. Yes, because the angle bisectors are concurrent. One of the diagonals of the kite is on the angle bisectors of opposite angles. For the other opposite angles, the angle bisectors intersect on that diagonal. The distances from this intersection point to the sides are all equal. Thus, the distance from the intersection of the angle bisectors to any one of the sides defines the radius of an inscribing circle.

10. Answers may vary. One way would be to find the midpoint of $\overline{AB}$ and label it M. Construct the perpendicular to $\overline{AB}$ at M. Set the compass to radius $\overline{AM}$ and sweep out a circle with center M. Where the circle intersects the perpendicular line will provide the two other vertices of the square.

11. **(*i*)** With a compass construct a circle with center A and radius r (10¢).

(*ii*) Choose any point B on the circle and construct a circle with radius r and center B. Choose one of the two points where the two circles intersect and name it point C (10¢).

(*iii*) Use a straight edge to construct line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ (30¢).

Total: **50¢**.

12. Copy $\overline{EF}$ to create new segment $\overline{AC}$, then bisect segment $\overline{AC}$ to find its midpoint; label the midpoint M (see problem 3 for instructions on how to do this). Now, construct a circle with M as the center with points A and C as points on the circle; in other words, $\overline{AC}$ is the diameter of the circle. Now, draw another diameter; label the endpoints of that diameter B and D. Now draw segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. If done correctly, rectangle $ABCD$ should have been created.

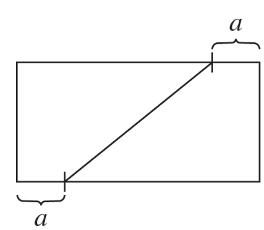
13. Given points B and E, place the compass point on E, and the pencil part of the compass on B. Draw a circle so that E is the center of the circle, and B is a point on the circle. Now, draw $\overline{BE}$, extending the segment until it intersects with the circle. Mark this point of intersection as point D. Note that $\overline{BD}$ is a diagonal of the square $ABCD$. Now, construct the perpendicular bisector of $\overline{BD}$ through point E, ensuring that the perpendicular bisector intersects the circle in two places (see problem 3 in this section to construct a perpendicular bisector of a segment). Label the points where the perpendicular bisector intersects the circle as points A and C. Now draw segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. If done correctly, square $ABCD$ should have been created.

14. Since O is the incenter, $\overline{XY}$ is parallel to $\overline{CB}$ and XY (its length) is equal to r. $\angle AYX \cong \angle ABC$ and so ΔAXY is isosceles and $\overline{AX} \cong \overline{XY}$. Thus $AX = XY = r$.

15. The center is the intersection of any two angle bisectors. The radius of the incircle is the distance from the incenter to any side, measured along a perpendicular to that side.

16. The side opposite side b is also b by parallelogram properties. Therefore $a + a - x = b$ and $x = 2a - b$.

17. The following figure shows two such congruent trapezoids. For different values of a we get different trapezoids.



18. **(a)** $\overrightarrow{PQ}$ is the perpendicular bisector of $\overline{AB}$.

(b) Q is on the perpendicular bisector of $\overline{AB}$ because $\overline{AQ} \cong \overline{QB}$. Similarly, P is on the perpendicular bisector of $\overline{AB}$.
Two points determine a unique line, thus the perpendicular bisector contains $\overrightarrow{PQ}$.

(c) $\overrightarrow{PQ}$ is the angle bisector of $\angle APB$. $\overrightarrow{QC}$ is the angle bisector of $\angle AQB$.

(d) $\Delta APQ \cong \Delta BPQ$ by *SSS* so $\angle APQ \cong \angle BPQ$ by CPCTC. Thus, $\overrightarrow{PQ}$ is the angle bisector of $\angle APB$. Using the now established fact that $\angle APQ \cong \angle BPQ$, $\Delta APC \cong \Delta BPC$ by *SAS*. Thus, $\Delta AQC \cong \Delta BQC$ by *SSS*, and $\angle AQC \cong \angle BQC$ by CPCTC.

19. **(a)** **No unique parallelogram**. Without the angle between the sides, there are infinitely many parallelograms which could be constructed from the two given sides.

(c) **Possible**. Construct two perpendicular segments bisecting each other and congruent to the given diagonals. Then connect the endpoints.

(d) **No unique parallelogram**. Given a side and all the angles, the other pair of congruent sides can be of any length.

20. **(a)** Bisect a 60° angle twice.

(b) Add 60°, 30°, and 15° angles, or a right angle plus 15°.

(c) 120° is the supplement of a 60° angle.

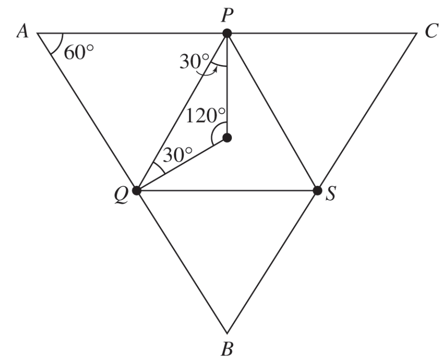
21. Make the edge of the ruler coincide with ℓ. Let one of the legs of the right triangle slide along the edge of the ruler until the other leg goes through *P*. The line along the edge containing *P* is the required perpendicular.

22. **(a)** **Possible**. The point is determined by the intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$.

(b) **Not possible**. If such a point existed it would have to be on the perpendicular bisectors of all the sides. Because the perpendicular bisectors of all the four sides of a quadrilateral do not necessarily intersect in a single point, as is the case for the given quadrilateral, such a point does not exist.

(c) In general, **not possible**. It is possible only if the angle bisectors are concurrent—not the case for a general quadrilateral.

23. Construct an equilateral triangle *PQS* inscribed in the circle (see Assessment 12-3A problem 23) with center *O*. Next construct perpendiculars to $\overline{OP}$, $\overline{OQ}$ and $\overline{OS}$ at *P*, *Q*, and *S* respectively. The points of intersection of the perpendiculars are the vertices of a required triangle. To prove that the triangle is equilateral, let *A* be the intersection of the perpendiculars at *P* and *Q*. Because $m(\angle QOP) = 120°$, $m(\angle OPA) = m(\angle OQA)$ $= 90°$, and *APOQ* forms a quadrilateral, $m(\angle A) = 60°$. Similarly, the other angles of ΔABC measure 60° and hence the triangle is equilateral.



Mathematical Connections 12-3: Review Problems

23. $\Delta ABC \cong \Delta DEC$ by *ASA* ($\overline{BC} \cong \overline{CE}$; $\angle ACB \cong \angle ECD$ as vertical angles, and $\angle B \cong \angle E$ as alternate interior angles formed by the parallels $\overline{AB}$ and $\overline{ED}$ and the transversal $\overleftrightarrow{EB}$). Thus $\overline{AB} \cong \overline{DE}$ by CPCTC.

24. **(a)** Copy the angle, then measure off each side along a side of the angle. Then connect the end points.

(b) Copy $\overline{AB}$. Make arcs from *A* and *B*, one with radius *AC* and the other with radius *BC*. Their intersection is *C*.

(c) Copy the side. Copy the angles at opposite ends, extending their sides until they meet to from the triangle.

25. Since $\angle B \cong \angle E$ and $\angle A \cong \angle D$, then $\angle C \cong \angle F$. Since it is given that $\overline{AC} \cong \overline{DF}$, $\Delta ABC \cong \Delta DEF$ by **ASA**.

26. Since the triangle is isosceles, the angles opposite the congruent sides are congruent. So, the unmarked angle has a measure of $65°$. Since all interior angles of a triangle must sum to $180°$, we have $65° + 65° + x° = 180°$. Solving for *x*, $130° + x° = 180° \Rightarrow x = 50°$

Assessment 12-4A: Similar Triangles and Other Similar Figures

1. A scale of 1 cm: 110 km, means that every 1 cm of shoelace represents 110 km of river. Since the shoelace is 61 cm long, to find the length of the Nile River, multiply $61 \times 110 = 6710$ km. Another solution method would be to create a proportion, and solve for the unknown variable x:

$$\frac{1\,\text{cm}}{110\,\text{km}} = \frac{61\,\text{cm}}{x\,\text{km}}$$

2. A scale of 1 in: 100 ft means every inch of the drawing should represent 100 feet of the Washington Monument. Since the Washington Monument is 555 feet tall, to find the height of the drawing divide $555 \div 100 = 5.5$ inches. Another solution method would be to create a proportion, and solve for the unknown variable x:

$$\frac{1''}{100'} = \frac{x''}{555'}$$

3. **Yes.** The left/right sides of the small rectangle are of length 4, while the left/right side of the large rectangle is of length $4 + 4 = 8$; so the length of the left/right sides of the large rectangle is twice that of the small rectangle. Similarly, the top/bottom sides of the small rectangle are of length 8.5, while the top/bottom sides of the large rectangle are of length $17 = 8.5 + 8.5$; so once again those lengths are double that of the small rectangle. Since corresponding sides are in proportion, the two rectangles are similar.

4. In a rhombus, adjacent angles are supplementary. Since $\angle BAD \cong \angle B_1A_1D_1$ we know that $\angle ABC \cong \angle A_1B_1C_1$. Thus, the corresponding angles are congruent. Since, in a rhombus, adjacent sides are congruent, $\frac{AD}{AB} = 1 = \frac{A_1D_1}{A_1B_1}$. This implies that $\frac{AD}{A_1D_1} = \frac{AB}{A_1B_1}$, in other words, corresponding sides are proportional. Therefore, the rhombuses are similar.

5. **(a)** **Always similar**, by AA. All angles measure $60°$.

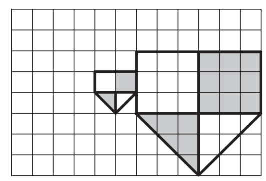
(b) **Always similar**. Sides are proportional and angles congruent.

(c) **Not** always similar. Consider a square and a nonsquare rectangle.

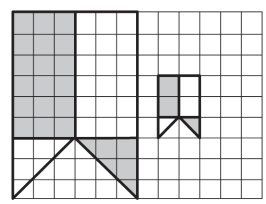
(d) **Always similar**. The ratio of corresponding sides longer sides will be equal to the ratio of corresponding shorter sides. Suppose the first rectangle has side lengths of a and $2a$, and the second rectangle has side lengths of b and $2b$. Then the shorter sides are in the ratio of $\frac{a}{b}$ and the longer sides are in the ratio of $\frac{2a}{2b} = \frac{a}{b}$.

6. Make all dimensions three times as long. For example, in part (c) each side would be three diagonal units long. One possible solution set is below.

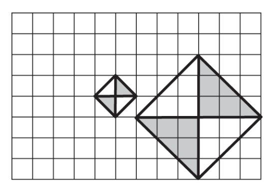
(a)



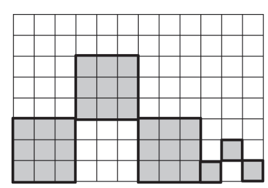
(b)



(c)



(d)



7. **(a)** **(*i*)** $\triangle ABC \sim \triangle DEF$ by AA.

(*ii*) $\triangle ABC \sim \triangle EDA$ by AA.

(*iii*) $\Delta ACD \sim \Delta ABE$ by AA. $\angle A$ is in both triangles.

(*iv*) $\Delta ABE \sim \Delta DBC$ by SAS, since $\frac{2}{3} = \frac{3}{4.5}$ and the vertical angles at B are congruent.

(b) **(*i*)** $\frac{2}{3}$. Sides of length 2 and length 3 are corresponding.

(*ii*) $\frac{1}{2}$. Corresponding sides of ΔAED are twice the length of ΔACB.

(***iii***) $\frac{3}{4}$. Corresponding sides are of length 6 and $6+2=8$; $\frac{6}{8}=\frac{3}{4}$.

(***iv***) $\frac{2}{3}$. Corresponding sides are AB and BD. The lengths of their sides are in the ratio $2:3$.

8. **(a)** $\frac{\text{Short side}}{\text{Long side}}=\frac{5}{10}=\frac{x}{x+7} \Rightarrow 5(x+7)=10x \Rightarrow$ $5x+35=10x \Rightarrow x=7$.

(b) $\frac{3}{x}=\frac{7}{8} \Rightarrow 7x=3\cdot 8 \Rightarrow x=\frac{24}{7}$.

9. The two triangles are similar by **AA** similarity; it is given that one pair of angles is congruent, and the two vertical angles are congruent. Therefore, the corresponding sides will be in proportion. Note that the side of length 10 on the left triangle corresponds with the side of length 15 of the large triangle. Using that information, these proportions can be set up to solve for sides x and y. (Note that it is possible to set up other correct proportions):

$$\frac{10}{15}=\frac{x}{21} \Rightarrow 210=15x \Rightarrow 14=x \text{ and}$$

$$\frac{10}{15}=\frac{12}{y} \Rightarrow 10y=180 \Rightarrow y=18.$$

10. We know ΔAEB is similar to ΔCED by **AA** similarity; the two triangles share $\angle E$, and corresponding angles (such as $\angle B$ and $\angle EDC$) will be congruent because $\overline{AB} \parallel \overline{CD}$. So the corresponding sides of the two triangles will be in proportion.

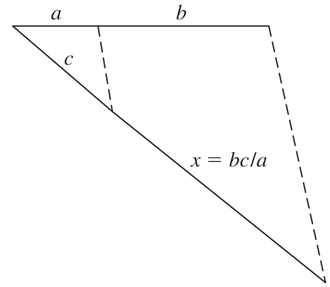
(a) $\frac{y}{x}=\frac{9}{6}$, simplifying, $\frac{y}{x}=\frac{3}{2}$.

(b) $\frac{x}{x+y}=\frac{6}{6+9}$; so $\frac{x}{x+y}=\frac{6}{15}=\frac{2}{5}$.

(c) $\frac{6}{x}=\frac{9}{y}$

11. Follow the procedure illustrated by Figure 12-59 in the textbook.

12. If $\frac{a}{b}=\frac{c}{x}$, then $x=\frac{bc}{a}$. Construct as below, using the technique of Figure 12-56.

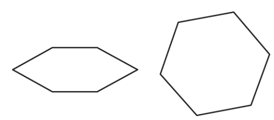


Note that the two dashed lines are parallel.

13. **(a)** Yes, because the angles stay the same and the lengths of corresponding sides are proportional.

(b) Let S_1 and S_2 represent the lengths of a side of the original polygon and the reduced polygon respectively. Then, $S_2=.8\,(.8\,S_1) \Rightarrow \frac{S_2}{S_1}=0.64$.

14. Sketches may vary. Make the sides proportional but interior angles not congruent; e. g.:



15. The new page will be $\frac{75}{100}$ the size of the original. The ratio to return to the original size would thus be $\frac{100}{75}=133\frac{1}{3}\%$. (Most copy machines do not include this setting.)

16. Let the height above ground be h and convert all measurements to inches (3 feet = 36 inches: 7 feet = 84 inches) and use similar triangles. $\frac{36}{13}=\frac{h}{84} \Rightarrow 13h=36\cdot 84 \Rightarrow h \approx 232.6$ inches or **about 19.38 feet**.

17. $\Delta DCP \sim \Delta BAP$. Thus $\frac{6}{AB}=\frac{4}{10} \Rightarrow 4AB=6\cdot 10 \Rightarrow$ $AB=$ **15 m**.

18. We know ΔACB ΔDCG and ΔFCG are all similar by **AA** similarity; the three triangles share $\angle C$, and corresponding angles (such as $\angle B$, $\angle FGC$, and $\angle DEC$) will be congruent because $\overline{AB} \parallel \overline{DE} \parallel \overline{FG}$. So the corresponding sides of the three triangles will be in proportion.

(a) It is given that F is one-fourth of the distance from C to A. From that, it can be deduced that $\overline{CF}$ is one-fourth the length of $\overline{AC}$. It is given that $AC = 4$; therefore, $CF = 1$.

(b) It is given that G is one-fourth of the distance from C to B. From that, it can be deduced that $\overline{CG}$ is one-fourth the length of $\overline{BC}$. It is given that $BC = 4.5$; therefore, $CG = 1.125$.

(c) From parts (a) and (b), we know that the sides of ΔFCG are one-fourth the length of the sides of ΔACB; so $\overline{FG}$ will be one-fourth the length of $\overline{AB}$. It is given that $AB = 3$; therefore $FG = 0.75$.

19. It is given that $\overline{BC} \parallel \overline{ED}$ and $\overline{AB} \parallel \overline{DC}$. Since $\overline{BC} \parallel \overline{ED}$, it is true that $\angle BCA \cong \angle DEC$, since they are alternate interior angles with $\overline{AC}$ as the transversal. Also, since $\overline{AB} \parallel \overline{DC}$, it is true that $\angle BAC \cong \angle DCA$, since they are alternate interior angles, with $\overline{AC}$ as the transversal. So, $\Delta ABC \sim \Delta CDE$ by **AA** similarity.

20. **(a)** A perimeter is the sum of the lengths of the sides, so the ratio of the perimeters is the same as the ratio of the sides.

(b) If $a, b, c,$ and d are the sides of one quadrilateral and $a_1, b_1, c_1,$ and d_1 are the corresponding sides of a similar quadrilateral, then $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \frac{d}{d_1} = r$ the scale factor.

Then,

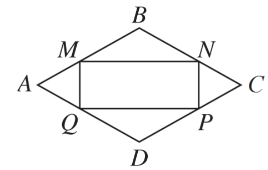
$$\frac{a+b+c+d}{a_1+b_1+c_1+d_1} = \frac{a_1r+b_1r+c_1r+d_1r}{a_1+b_1+c_1+d_1} = \frac{(a_1+b_1+c_1+d_1)r}{a_1+b_1+c_1+d_1} = r$$

So the ratio of the perimeters is r. An analogous proof works for any two similar n-gons.

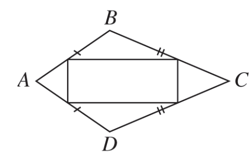
21. **(a)** From the Midsegment Theorem, we know that each midsegment is half the measure of the opposite side of ΔABC. Thus, the sides of each of the smaller triangles are half as long as the corresponding sides of ΔABC. Hence by SSS they are congruent to each other.

(b) **Yes**. The ratio of the corresponding sides is $\frac{1}{2}$ and therefore by SSS the triangles are similar.

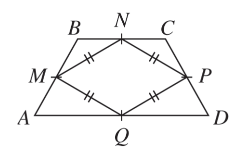
22. **(a)** $ABCD$ is a rhombus $\Rightarrow \overline{AB} = \overline{BC} = \overline{CD} = \overline{DA}$, thus all angles of $MNPQ$ are right and it is a **rectangle**.



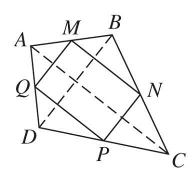
(b) If $ABCD$ is a kite, $MNPQ$ is a **rectangle**.



(c) If $ABCD$ is an isosceles trapezoid, $MNPQ$ is a **rhombus**.



(d) If $ABCD$ is quadrilateral which is not a rhombus nor a kite but whose diagonals are perpendicular to each other, $MNPQ$ is a **rectangle**.



23. Yes**.** All such parallel cross-sections are circular. They have exactly the same shape but are not necessarily the same diameter; i.e., they are similar.

24. 4.2 cm represents the distance from city *A* to city *B* on the map: 512 miles represents the distance from city *A* to city *B* in reality. This means the distance from city *B* to city *C* can be found using the ratio $\frac{4.2 \text{ cm}}{512 \text{ miles}}$ to solve for the unknown miles:

$$\frac{4.2 \text{ cm}}{512 \text{ miles}} = \frac{10.1 \text{ cm}}{\text{BC miles}}; \quad BC \approx 1231 \text{ miles}.$$

Similarly, the distance from city A to city C can be found using the same ratio $\frac{4.2 \text{ cm}}{512 \text{ miles}}$ as follows:

$$\frac{4.2 \text{ cm}}{512 \text{ miles}} = \frac{12.2 \text{ cm}}{AC \text{ miles}}; \quad AC \approx 1487 \text{ miles}.$$

Assessment 12-4B

1. **No.** When comparing the 8 inch by 10 inch picture to the 5 inch by 7 inch picture, the short sides have a ratio of $8:5 = 1.6:1$, but the long sides have a ratio of $10:7 = 1.4:1$, so they are not in proportion. Similarly, it can be found that the wallet sized pictures, which are $2\frac{1}{4}$ by $3\frac{1}{4}$ are not in proportion to either the 8 by 10 picture or the 5 by 7 picture.

2. The scale is 16 feet (for the model) to 563 feet (for the actual object). to find the scale, solve $\frac{16 \text{ ft}}{563 \text{ ft}} = \frac{1 \text{ ft}}{x \text{ feet}};\ x = 35.1875 \text{ feet}$
   The scale is 1:35.1875.

3. **No**. Two corresponding sides of the two quadrilaterals are in the ratio of $\frac{4}{6}$; a third corresponding side on each quadrilateral is in the ratio $\frac{6}{10}$; and $\frac{4}{6} \neq \frac{6}{10}$.

4. **Not necessarily** similar. Consider a parallelogram $ABCD$ in which $BC = 2AB$, and the rhombus $ABEF$ where E and F are the midpoints of $\overline{BC}$ and $\overline{AD}$ respectively.

5. **(a)** **Not** always similar. We can construct a counterexample as follows. Consider a rectangle with sides 3 and 4 units. Its diagonal is 5 units. Now construct a rectangle with one side 3 units and a diagonal 10 units. The other side will be $\sqrt{10^2 - 3^2} = \sqrt{91}$.
   (b) **Not** always similar. Angles may be different.
   (c) **Always similar**. Radii are proportional.
   (d) **Not** always similar, unless they have the same number of sides.
   (e) **Always similar**. Sides are proportional and angles congruent.

6. Answers vary. For example, choose any natural number a and draw $a, a, a\sqrt{2}$ by starting at a corner of one grid and constructing vertical and horizontal line segments. Note that all right isosceles triangles are similar.

7. **(a)** **Similar** by SSS since $\frac{4}{6} = \frac{4}{6} = \frac{2}{3}$. The correspondence is $\Delta ABD \sim \Delta DEF$ and the scale factor is $\frac{2}{3}$.
   (b) **Similar** by SAS since $\frac{3}{6} = \frac{4}{8}$ and each has a right angle. The correspondence is $\Delta ABD \sim \Delta EDA$ and the scale factor is $\frac{1}{2}$.
   (c) **Similar** by AA. The correspondence is $\Delta ABE \sim \Delta ACD$ and the scale factor is $\frac{6}{6+2} = \mathbf{\frac{3}{4}}$.
   (d) **Similar** by AA. The correspondence is $\Delta ABE \sim \Delta DBC$ and the scale factor is $\frac{6}{8} = \mathbf{\frac{3}{4}}$.

8. **(a)** Similar by SAS because $\frac{5}{5+5} = \frac{7}{7+7} = \frac{1}{2}$ $\Rightarrow x = 9\left(\frac{1}{2}\right) = 4.5$.
   (b) Similar by AA. Scale factor is $\frac{8}{6} \Rightarrow$ $x = \left(\frac{8}{6}\right)5 = \mathbf{\frac{20}{3}}$.
   (c) Similar by SAS because $\frac{4+3.2}{3.2} = \frac{4+5}{4} = \frac{9}{4}$. $x = \left(\frac{9}{4}\right)3 = \mathbf{\frac{27}{4}}$.
   (d) Similar by AA because each has a right angle and $m(\angle CBA) = 180° - 90° - m(\angle DBE)$ and in $\Delta BDE, m(\angle E) = 90° - m(\angle DBE)$ $\Rightarrow m(\angle CBA) = m(\angle E)$. Scale factor is $\frac{2}{3} \Rightarrow x = \left(\frac{2}{3}\right)4 = \mathbf{\frac{8}{3}}$ and $y = \left(\frac{2}{3}\right)5 = \mathbf{\frac{10}{3}}$.

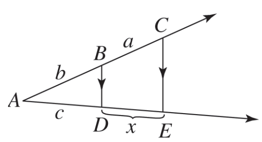
9. We know ΔAGD, ΔBFD, and ΔCED are all similar by **AA** similarity; the three triangles share $\angle D$, and corresponding angles (such as $\angle A$, $\angle DBF$, and $\angle DCE$) will be congruent because $\overline{AG} \parallel \overline{BF} \parallel \overline{CE}$. So the corresponding sides of the three triangles will be in proportion. The lengths of two sides ΔCED of are given; we can use those lengths to set up a ratio of $\frac{ED}{CD} = \frac{4.5}{6}$ to solve for $x = FE$. Since $\frac{ED}{CD} = \frac{FE}{BC}$, after substituting in the proper numbers and variable, the proportion is $\frac{4.5}{6} = \frac{x}{6};\ x = 4.5$. To solve for $y = AG$, we can use $\frac{FE}{BF} = \frac{GF}{AG}$ as the proportion; after substituting in the proper numbers and variable, the proportion is $\frac{4.5}{5} = \frac{9}{y};\ y = 10$

10. **(a)** We know ΔABC and ΔDBE are similar by **AA** similarity; the two triangles share $\angle B$, and corresponding angles (such as $\angle C$, and $\angle BED$) will be congruent because $\overline{AC} \parallel \overline{DE}$. So the corresponding sides of the three triangles will be in proportion.

(b) The proportion $\frac{EB}{DB} = \frac{CE}{AD}$ can be used to find the length of $\overline{CE}$. Substituting in the proper numbers, the proportion is $\frac{10}{18} = \frac{CE}{4}$; therefore, $CE = 2\frac{2}{9}$ or . $CE = 2.\overline{2}$

11. Follow the procedure illustrated by Figure 12-59 in the textbook.

12. $\frac{a}{b} = \frac{x}{c} \Rightarrow \frac{b}{a} = \frac{c}{x}$. Let b and a be end-to-end on a side of an arbitrary angle. On the other side of the angle, mark D such that $DA = c$. Connect B with D, and draw a line through C that is parallel to $\overline{BD}$, labeling its intersection with $\overrightarrow{AD}$ as C. Then $DE = x$.



13. **(a)** **Yes**, by the transitive property. If $\Delta ABC \sim \Delta DEF$ and $\Delta DEF \sim \Delta GHI$, then $\Delta ABC \sim \Delta GHI$. The proof of this property can be constructed by considering the AA similarity theorem.

(b) Given that a side of ΔABC is 75% of the corresponding side of ΔDEF, and that a side of ΔGHI is 32% of the corresponding side of ΔDEF, then the ratio of a side of ΔABC to the corresponding side of ΔGHI is 75 to 32. Conversely, the ratio of a side of ΔGHI to the corresponding side of ΔABC is 32 to 75.

14. Sketches may vary. Make the sides proportional but interior angles not congruent. One could be concave and one could be convex to make sure the shapes are not congruent.

15. Let s represent the length of one side of the paper. Our goal is to reduce to $\frac{1}{6}s$. We first reduce by 25%, so the side now has length $0.75s$. Let x be the next percent reduction: $0.75s - x(0.75s) = 0.75(1-x)s$.

Thus, $\frac{1}{6}s = 0.75(1-x)s \Rightarrow \frac{1}{6} = 0.75 - 0.75x \Rightarrow$

$x = \frac{.75 - \frac{1}{6}}{.75} \approx$ **77.8%**.

16. The figure shows similar triangles with proportional sides. Convert the girl's height to 1.5 m and let h be the height of the tree. Then $\frac{1.5}{3} = \frac{h}{15+3} \Rightarrow$ $3h = 1.5(15+3)$. $h =$ **9 m**.

17. The proportion can be set up using two ratios comparing distances from the mirror (d) over heights (h); or $\frac{d(\text{girl})}{h(\text{girl})} = \frac{d(\text{pole})}{h(\text{pole})}$. Substituting in the proper numbers, the proportion is $\frac{5.5}{5} = \frac{20}{h(\text{pole})}$.

Therefore, the height of the pole $= 18.\overline{18}$, or about 18 feet.

18. **(a)** Since D is the midpoint of $\overline{AC}$, and $AC = 4$, then $DC = 2$.

(b) Since E is the midpoint of $\overline{BC}$, and $BC = 4.5$, then $CE = 2.25$.

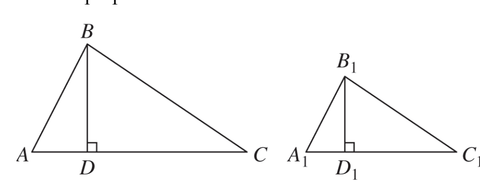
(c) From parts (a) and (b), we know that the sides of ΔFCG are one-half the length of the sides of ΔACB; so $\overline{DE}$ will be one-half the length of $\overline{AB}$. It is given that $AB = 3$; therefore $DE = 1.5$.

19. $\Delta ABC \sim \Delta ADE$, by the AA Similarity Theorem. Therefore, $\frac{DE}{BC} = \frac{AD}{AB} = \frac{EA}{AC} \Rightarrow \frac{DE}{6} = \frac{4}{10} = \frac{EA}{8}$.

Thus, $DE = \frac{12}{5}$ and $EA = \frac{16}{5}$.

20. **(a)** The ratio of corresponding heights is the same as the ratio of corresponding sides.

(b) Let $ABC \sim A_1 B_1 C_1$. Construct altitudes BD and $B_1 D_1$.



By the AA Similarity Theorem, $\Delta ABD \sim \Delta A_1 B_1 D_1$. Thus, $\frac{AB}{A_1 B_1} = \frac{BD}{B_1 D_1}$.

21. (a) (*i*) Connect B with D and then apply Theorem 12-13 to triangles ABD and BCD in turn.

(*ii*) Theorems 12-13 and 12-14 imply, where P is the midpoint of $\overline{MN}$, that $MP = \frac{1}{2}a$ and $PN = \frac{1}{2}b.$ Thus $MN = MP + PN = \frac{1}{2}a + \frac{1}{2}b = \frac{1}{2}(a+b).$

(b) From part (a), $c = \frac{1}{2}(a+b)$ or $2c = a + b \Rightarrow$

$$c + c = a + b$$
$$c + c - b = a$$
$$c - b = a - c.$$

which shows that b, c, and a form an arithmetic sequence having difference $c - b$, which is equal to $a - c$. One way to express this sequence is $a_n = (c-b)n + (2b - c)$, where $n = 1, 2,$ and 3.

(c) Part (b) proved that the length of a bisecting segment parallel to the bases forms an arithmetic sequence with the lengths of the bases. By viewing the new figure as the compilation of four such trapezoids, a similar argument shows the lengths of all ten segments form an arithmetic sequence. Denote the lengths of the consecutive parallel segments by $b_1, b_2, b_3, \ldots, b_{10}$, where $b_1 = b$ and $b_{10} = a.$ Because b_1, b_2, b_3 from an arithmetic sequence, $b_2 - b_1 = b_3 - b_2 = d.$ (where d is the common difference). But b_2, b_3, b_4 are also lengths of three consecutive segments, and $b_3 - b_2 = b_4 - b_3 = d.$ Proceed in the same way to show that $b_5 - b_4 = d,$ and so on. In the formula for the sum of an arithmetic sequence, $S_{10} = \frac{10(b_1 + b_{10})}{2} = 5(a+b).$

22. (a) $\overline{NP}$ and $\overline{MN}$ are midsegments, so $\overline{NP} \parallel \overline{BD}$ and $\overline{MN} \parallel \overline{AC} \Rightarrow NSTV$ is a parallelogram. $\angle VNS \cong \angle VTS$ since these are opposite angles in parallelogram.

(b) (*i*) **Perpendicular**. Then the parallelogram will have a right angle and therefore be a rectangle.

(*ii*) **Perpendicular and congruent**. Then and only then $MN = NP$ (each is half as long as a diagonal) and MN and NP will be perpendicular.

23. **Yes**, because like circles, all semicircles have the same shape.

24. A proportion can be set up, with the two distances between E 24th and E 33rd as one of the ratios, and the two distances between E 30th and E 33rd (with a variable for the missing distance) as the other ratio. The 5th Avenue distances will be in the numerators, while the Broadway Avenue distances will be in the denominator. Therefore, we have $\frac{2600\text{ ft}}{2800\text{ ft}} = \frac{x\text{ ft}}{1120\text{ ft}};\ x = 1040\text{ feet}.$

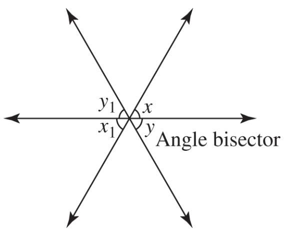
Mathematical Connections 12-4: Review Problems

25. Start with the given base and construct its perpendicular bisector. The vertex of the required triangle must be on this perpendicular bisector. Starting at the point where the perpendicular bisector intersects the base, mark on the perpendicular bisector a segment congruent to the given altitude. The endpoint of the segment not on the base is the vertex of the required isosceles triangle.

26. Construct an equilateral triangle with the given side. Then construct the angle bisector from any vertex to the opposite side; this will form the altitude.

27. Answers may vary. E.g.: construct a perpendicular line at one of the endpoints of the hypotenuse, giving a $90°$ angle. Bisect this angle to yield a $45°$ angle. Copy the $45°$ angle at the other endpoint of the hypotenuse, extending until it meets the bisector which formed the first $45°$ angle. These will meet at a $90°$ angle across from the given hypotenuse.

28. **Yes**. The bisector of one of a pair of vertical angles bisects the other if extended, since vertical angles are congruent. The extended bisector forms two new pairs of vertical angles.



$x = x_1$ and $y = y_1$; but $x = y$ so $x_1 = y_1$.

29. Two quadrilaterals may be congruent by *SASAS* or by *ASASA*.

Chapter 12 Review

1. **(a)** $\Delta ABD \cong \Delta CBD$ by **SAS**. $\overline{BD}$ is common to both; right angles are congruent; $\overline{AD} \cong \overline{DC}$.

(b) $\Delta GAC \cong \Delta EDB$ by **SAS.** $\overline{AC} \cong \overline{DB}$; right angles are congruent; $\overline{AG} \cong \overline{DE}$.

(c) $\Delta ABC \cong \Delta EDC$ by **AAS.** Right angles are congruent; vertical angles; $\angle ACB \cong \angle DCE$; $\overline{AC} \cong \overline{CE}$.

(d) $\Delta BAD \cong \Delta EAC$ by **ASA.** $\angle A$ is common to both; $\overline{AB} \cong \overline{AE}$; $\angle B \cong \angle E$.

(e) $\Delta ABD \cong \Delta CBD$ by **ASA.** $\overline{AD} \cong \overline{DC}$; right angles are congruent; $\angle A \cong \angle C$.

(f) $\Delta ABD \cong \Delta CBD$ by **SAS.** $\overline{AB} \cong \overline{BC}$; $\angle ABD \cong \angle DBC$; $\overline{BD}$ is common to both.

(g) Given $\overline{BD} \cong \overline{BE}$; $\overline{AB} \cong \overline{BC}$; $\overline{AD} \cong \overline{EC}$. So, $\Delta ABD \cong \Delta CBE$ by **SSS.** Also, $\Delta ABE \cong \Delta CBD$ by **SSS**, since $\overline{AD} \cong \overline{EC}$ means $\overline{AE} \cong \overline{CD}$.

(h) $\Delta ABC \cong \Delta ADC$ by **SSS**; $\Delta ABE \cong \Delta ADE$ by **SSS** or **SAS**; $\Delta EBC \cong \Delta EDC$ by **SSS** or **SAS**. $\overline{BC} \cong \overline{CD}$; $\overline{AB} \cong \overline{AD}$; properties of kites contribute to most of congruency.

2. **Parallelogram**. $\Delta ADE \cong \Delta CBF$ by *SAS*, so $\angle DEA \cong \angle CFB$ by CPCTC; $\angle DEA \cong \angle EAF$ (congruent alternate interior angles between the parallels $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$ and the transversal $\overleftrightarrow{AE}$). That makes $\angle EAF \cong \angle CFB$. Since $\angle EAF$ and $\angle CFB$ are congruent corresponding angles, $\overline{AE} \parallel \overline{FC}$. It is already known that $\overline{EC} \parallel \overline{AF}$, as they are on the parallel sides of the square. Two pairs of parallel opposite sides implies a parallelogram.

3. **(a)** **(*i*)** Use the method illustrated by Figure 12-40 in the text.

(*ii*) Fold the angle down the middle so the sides match and the crease passes through *A*.

(b) **(*i*)** See text Figure 12-45.

(*ii*) Fold the line on top of itself so that the crease passes through *B*.

(c) **(*i*)** See text Figure 12-43.

(*ii*) Same as in part (b)(*ii*).

(d) **(*i*)** See text Figures 12-36 or 12-37.

(*ii*) Make line $k \perp \ell$ through P as in part (b) and (c), then make $m \perp k$ through P as in part (b) $\Rightarrow m \parallel \ell$.

4. **(a)** **(*i*)** $\frac{x}{4} = \frac{6}{3} \Rightarrow 3x = 4 \cdot 6$. $\boldsymbol{x = 8}$.

(*ii*) $\frac{y}{10} = \frac{3}{6} \Rightarrow 6y = 3 \cdot 10$. $\boldsymbol{y = 5}$.

(b) $\frac{x}{13} = \frac{2.5}{5} \Rightarrow 5x = 2.5 \cdot 13$. $\boldsymbol{x = 6.5}$.

5. Use the method illustrated by Figure 12-59 in the text.

6. $\frac{a}{b} = \frac{c}{d}$. $\frac{a}{b} = \frac{x}{y}$ in ΔABC; $\frac{x}{y} = \frac{c}{d}$ in ΔACD, so $\frac{a}{b} = \frac{c}{d}$ by the transitive property.

7. $\overline{AB}$ must be a chord of the circle, and the perpendicular bisector of a chord passes through the center. Construct this line to locate the center on ℓ, measure the radius to *A* or *B*, and draw the circle with your compass.

8. **(a)** **(*i*)** $\Delta ACB \sim \Delta DEB$ by *AA*.

(*ii*) $\frac{x}{3} = \frac{8}{5} \Rightarrow 5x = 8 \cdot 3$. $\boldsymbol{x = \frac{24}{5}}$.

(b) **(*i*)** $\Delta AED \sim \Delta ACB$ by *AA*.

(*ii*) $\frac{4}{6} = \frac{y+6}{11} \Rightarrow 6(y+6) = 4 \cdot 11$. $\boldsymbol{y = \frac{4}{3}}$.

(iii) $\frac{6}{5}=\frac{11}{x}\Rightarrow 6x=11\cdot 5.\ \boldsymbol{x=\frac{55}{6}}$.

9. We know ΔADE is similar to ΔACB by AA similarity; the two triangles share $\angle A$, and corresponding angles (such as $\angle B$ and $\angle AED$) will be congruent because $\overline{DE}\parallel\overline{CB}$. So the corresponding sides of the two triangles will be in proportion. To find AD, the proportion $\frac{AE}{EB}=\frac{AD}{DC}$ can be used. Substituting in the proper numbers, the proportion is $\frac{20}{12}=\frac{AD}{9}$; therefore the solution is $AD=15$.

10. It is given that $l\parallel m$; place a point on l called A. Construct a line (call it n) perpendicular to line l through point A. See page 726, Figure 45 on how to do this. Since $l\parallel m$, line n is also perpendicular to line m Now, plot the point of intersection that line n has with line m; label that point B Next, find the perpendicular bisector of $\overline{AB}$. See page 726, Figure 44 for an explanation on how this is done. Extend the perpendicular bisector of $\overline{AB}$ to create line p If all is done correctly, line p should be equidistant from, and parallel to, lines l and m.

11. (a) The width (left/right sides) of the pool is 16 feet, while the width of the patio is 24 feet. The widths are in the ratio of $\dfrac{16}{24}=\dfrac{2}{3}$. The length of the pool is 32 feet, while the length of the patio is 48 feet. They are in the ratio of $\dfrac{32}{48}=\dfrac{2}{3}$. Since the ratios are the same, the rectangular pool is similar to the rectangular patio.

(b) Comparing the patio to the pool is comparing the larger object to the smaller object. Therefore the scale factor of the patio to the pool is $\dfrac{3}{2}$.

(c) The total perimeter of the patio is $48+24+48+24=144$ feet. The total perimeter of the pool is $32+16+32+16=96$ feet. So the ratio of the patio to the pool would be $\dfrac{144}{96}=\dfrac{3}{2}$.

12. The two triangles are not congruent. For two triangles to be congruent, their corresponding angles must be congruent (which is true in this situation) and their corresponding sides must be congruent. The corresponding sides cannot be congruent since one triangle is larger than the other. However, we can determine that the two triangles are similar by **AA** similarity.

13. (a) Given $\Delta ABC\cong\Delta TRI$, $\angle T$ must correspond with $\angle A$. Since $m\angle A=70°$, the $m\angle T=70°$.

(b) Given $\Delta ABC\cong\Delta TRI$, $\angle C$ must correspond with $\angle I$. Since $m\angle I=62°$, the $m\angle C=62°$.

(c) Given $\Delta ABC\cong\Delta TRI$, $\angle B$ must correspond with $\angle R$. Since $m\angle R=48°$, the $m\angle B=48°$.

14. ΔABC is an isosceles triangle, since $\angle A\cong\angle B$. That means the sides opposite those angles are congruent. Setting them equal to each other,

$3x-4=2x+1$
$x=5$

Now substitute 5 in for x in for all three sides:

$3(5)-4=11$
$2(5)+1=11$
$(5)+1=6$

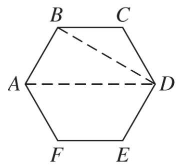
15. If h is the height of the building: $\frac{2\text{ m tall}}{1\text{ m shadow}}=\frac{h\text{ m high}}{6\text{ m shadow}}\Rightarrow 1h=2\cdot 6.\ \boldsymbol{h=12}$ **m**.

16. Polygons (ii) and (iii). Any convex regular polygon (i.e., a polygon in which all the angles are congruent and all the sides are congruent) can be inscribed in a circle. The angles in polygon (i) are not necessarily congruent.

17. $\frac{h}{8}=\frac{1.5}{2}\Rightarrow 2h=1.5\cdot 8.\ \boldsymbol{h=6}$ **m**.

18. $\frac{d}{64}=\frac{16}{20}\Rightarrow 20d=16\cdot 64.\ \boldsymbol{d=\frac{256}{5}}$ **m**.

19. In the figure below:



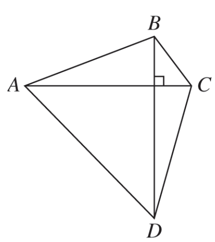
$\angle CBD = 30°$, $\angle C = 120°$, and $\angle CDB = 30°$.

(a) $\overline{AD}$ is a diameter, since it passes through the center of the circumcircle and the midpoint of $\overline{AD}$ is the circumcenter.

(b) $m(\angle CBD) = 30°$, since base angles of an isosceles triangle are congruent. Since $m(\angle ABC) = 120°$, then $m(\angle ABD) = m(\angle ABC) - m(\angle CBD) = 120° - 30° = 90°$. Thus $\angle ABD$ is a right angle.

20. The statement is false since in (i) we have a counterexample.

(i) Quadrilateral *ABCD* below is not a square, even through its diagonals are perpendicular and congruent (i.e., $\overline{AC} \cong \overline{BD}$).



(ii) If the diagonals bisect each other, the quadrilateral is a square.

21. Congruent triangles have congruent corresponding angles. Thus they are similar by AA.

22. **(a)** Since $\overline{BC} \parallel \overline{AD}$, alternate interior angles $\angle CBN$ and $\angle EDN$ are congruent. Since N is the midpoint of diagonal $\overline{BD}$, $\overline{BN} \cong \overline{DN}$. $\angle BNC \cong \angle DNE$, since these angles are vertical. Therefore, $\Delta BCN \cong \Delta DEN$.

(b) From (a), we know that $\overline{CN} \cong \overline{EN}$ by corresponding parts of congruent triangles. Since M is the midpoint of diagonal AC, we know that $\overline{AM} \cong \overline{CM}$. Thus, $\overline{MN}$ is the mid-segment of ΔACE.

(c) Because $\overline{MN}$ is a midsegment in ΔACE, $MN = \frac{1}{2} AE$. Now $AE = AD - ED = AD - BC$ because $ED = BC$ (from congruence in part (a)). Thus, $AE = a - b$ and therefore $MN = \frac{1}{2} AE = \frac{1}{2}(a-b)$.

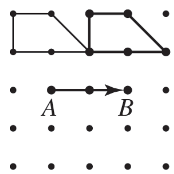
23. It is given that $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$. Since $\overline{AC} \cong \overline{AC}$, by **ASA**, $\Delta ACD \cong \Delta ACB$. That means $\overline{BC} \cong \overline{DC}$ by CPCTC. Also, $\overline{CE} \cong \overline{CE}$. Therefore, by **SAS** congruence, $\Delta BCE \cong \Delta DCE$.

CHAPTER 13

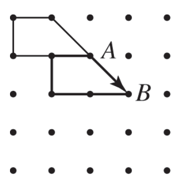
CONGRUENCE AND SIMILARITY WITH TRANSFORMATIONS

Assessment 13-1A: Translations and Rotations

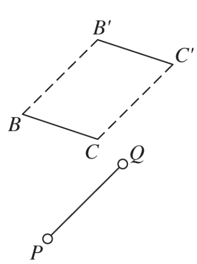
1. **(a)** Each corner of the trapezoid moves two dots to the right.



(b) Each corner of the trapezoid moves one dot down and one dot to the right.



2. **(a)** Trace $\overline{BC}$ and the line containing the slide arrow on the tracing paper and label the trace of B as B' and the trace of C as C'. Mark on the original paper and on the tracing paper the initial point of the arrow by P and the head of the arrow by Q. Slide the tracing paper along the line $\overleftrightarrow{PQ}$ so that P will fall on Q; trace $\overline{BC}$. The segment $\overline{B'C'}$ is the image of $\overline{BC}$ under the translation.



(b) Construct a parallelogram with $\overline{BC}$ and $\overline{B'C'}$ as opposite sides.

3. **(a)** $(0, 0) \to (0 + 3, 0 - 4) = \mathbf{(3, {}^-4)}$.

(b) $({}^-3, 4) \to ({}^-3 + 3, 4 - 4) = \mathbf{(0, 0)}$.

(c) $({}^-6, {}^-9) \to ({}^-6 + 3, {}^-9 - 4) = \mathbf{({}^-3, {}^-13)}$.

4. To go from an image to the coordinates of the original points, reverse the signs of the translations; i.e., $(x', y') \to (x' + 3, y' - 4) = (x, y)$.

(a) $(0, 0) \to (0 + 3, 0 - 4) = \mathbf{(3, {}^-4)}$.

(b) $({}^-3, 4) \to ({}^-3 + 3, 4 - 4) = \mathbf{(0, 0)}$.

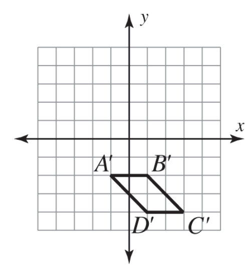
(c) $({}^-6, {}^-9) \to ({}^-6 + 3, {}^-9 - 4) = \mathbf{({}^-3, {}^-13)}$.

5. **(a)** $A({}^-4, 2) \to ({}^-4 + 3, 2 - 4) = A'({}^-1, {}^-2)$;

$B({}^-2, 2) \to ({}^-2 + 3, 2 - 4) = B'(1, {}^-2)$;

$C(0, 0) \to (0 + 3, 0 - 4) = C'(3, {}^-4)$;

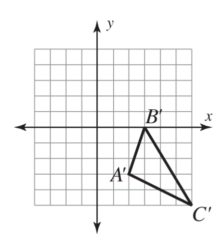
$D({}^-2, 0) \to ({}^-2 + 3, 0 - 4) = D'(1, {}^-4)$.



(b) $A({}^-1, 1) \to ({}^-1 + 3, 1 - 4) = A'(2, {}^-3)$;

$B(1, 4) \to (1 + 3, 4 - 4) = B'(4, 0)$;

$C(3, {}^-1) \to (3 + 3, {}^-1 - 4) = C'(6, {}^-5)$.



6. $(x', y') \to (x' - 3, y' + 4)$:

Label points $A'({}^-1, 0), B'(2, 0), C'(3, {}^-3)$, and $D'(0, {}^-3)$.

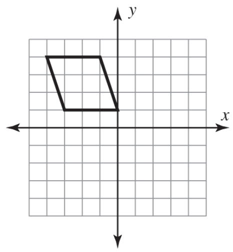
$A'(^-1, 0) \rightarrow (^-1 - 3, 0 + 4) = A(^-4, 4);$

$B'(2, 0) \rightarrow (2 - 3, 0 + 4) = B(^-1, 4);$

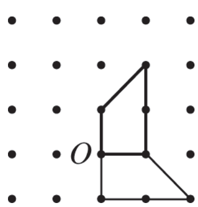
$C'(3, ^-3) \rightarrow (3 - 3, ^-3 + 4) = C(0, 1);$

$D'(0, ^-3) \rightarrow (0 - 3, ^-3 + 4) = D(^-3, 1).$

The figure whose image was shown is:



7. Rotate each corner 90° counterclockwise around O to obtain:



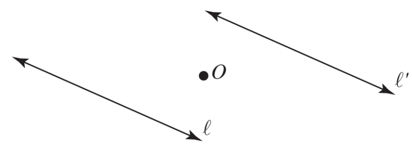
8. If $y = 2x - 1$, two sample points on line ℓ are $A(0, ^-1)$ and $B(1, 1)$. Note that in the following, the slope of the image line remains the same as the slope of line ℓ.

(a) $(x, y - 2) \rightarrow A' = (0, ^-1 - 2) = (0, ^-3)$ and $B' = (1, 1 - 2) = (1, ^-1)$. Then slope $(m) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{^-1 - ^-3}{1 - 0} = 2$. Using x and y values from A' in the general form $y = mx + b, ^-3 = 2(0) + b \Rightarrow b = ^-3$. The image is $\mathbf{y = 2x - 3}$.

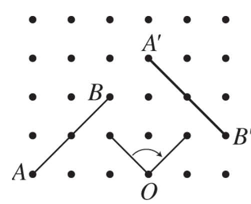
(b) $(x + 3, y) \rightarrow A' = (0 + 3, ^-1) = (3, ^-1)$ and $B' = (1 + 3, 1) = (4, 1)$ Then $m = \frac{1 - ^-1}{4 - 3} = 2$. Using x and y values from A' in the general form, $^-1 = 2(3) + b$ $\Rightarrow b = ^-7$. The image is $\mathbf{y = 2x - 7}$.

9. If y is the image of k under $(x + 3, y - 2)$, k is the translation of the image under $(x - 3, y + 2)$. Two sample points on the image are $A'(0,3)$ and $B'(1, 1)$. $A = (0 - 3, 3 + 2) = (^-3, 5)$ and $B = (1 - 3, 1 + 2) = (^-2, 3)$. Then $m = \frac{3-5}{^-2 - ^-3} = ^-2$. Using x and y values from A in the general form, $5 = ^-2(^-3) + b \Rightarrow b = ^-1$. Line k is $\mathbf{y = ^-2x - 1}$.

10. Construct a line perpendicular to ℓ passing through O. Denote the intersection of this perpendicular line with ℓ as Q. Draw a circle with radius $\overline{OQ}$ and center O; label point P where the perpendicular line intersects the circle on the opposite side of P. Then construct a perpendicular to $\overleftrightarrow{OQ}$ at $P \Rightarrow \ell'$.



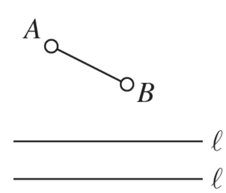
11. Reverse the rotation (i.e., counterclockwise) to locate $\overline{AB}$, which is the pre-image.



12. Answers may vary; e.g., SOS. H, I, N, O, S, X, or Z could appear in such rotational words; variations in drawing could use M and W in rotational images such as MOW.

13. **(a)** Since $\overline{AB}$ is parallel to line ℓ, the image is the line ℓ itself.

(b) Sketches may vary.



(c) ℓ and ℓ' are parallel. If P and Q are any two points on ℓ and P' and Q' their respective images, then from the definition of a translation $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent to

$\overline{AB}$. Therefore $\overline{PP'}$ and $\overline{QQ'}$ are parallel and congruent. Thus $PP'Q'Q$ is a parallelogram and by definition $\ell \parallel \ell'$.

(d) Ten successive rotations of $36° = 360°$. Thus the image of $\angle ABC$ is an identity transformation and is $\angle ABC$ itself.

14. **(a)** Connect O to P with a line using a straight edge. Construct a circle with center O through point P. Construct a circle with center P through point O. Label the circle intersections Q (left intersection) and R (right intersection). Connect Q to Q with a line using a straight edge. By construction $\overleftrightarrow{OP} \perp \overleftrightarrow{QR}$. Bisect $\angle POQ$ using the steps from "Constructing an Angle Bisector" Figure 40 in Chapter 12. The bisected angle measures 45° because it is half of the right angle created by the perpendicular lines. Put the legs of your compass on O and on P; using the same compass opening put the legs on O and on the line segment that bisected $\angle POQ$ and mark P' to rotate P 45° counterclockwise.

(b) Construct a circle with center O through point P. Construct a circle with center P through point O. Label the circle intersections Q (left intersection) and R (right intersection). Connect O to P and Q to O and Q to P with a segment using a straight edge. The triangle formed is an equilateral triangle and the interior angles measure 60°. We can replace R with P' to rotate P 60° clockwise.

(c) $105^\circ = 45^\circ + 60^\circ$. Construct the 45° angle using the steps from part (a). Construct the 60° using the steps from part (b) but start using P' created in part (a) instead of P. Label P' from part (b) with P''. The angle $\angle POP''$ is a 105° angle.

15. Signs of the x and y coordinates will be reversed under a half-turn about the origin.

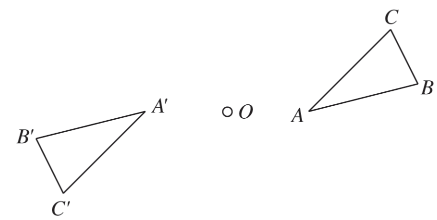
(a) $(4, 0) \rightarrow$ $\mathbf{(^-4, 0)}$.

(b) $(2, 4) \rightarrow$ $\mathbf{(^-2, ^-4)}$.

(c) $(^-2, ^-4) \rightarrow$ $\mathbf{(2, 4)}$.

(d) $(a, b) \rightarrow$ $\boldsymbol{(^-a, ^-b)}$.

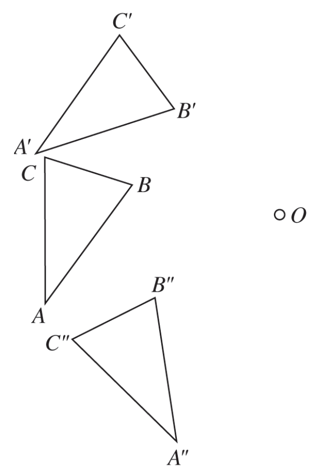
16. Label vertices of the triangles A, B, and C. Find the image of A by drawing $\overrightarrow{AO}$ and marking off $OA' = OA$ so that O is the midpoint of $\overline{AA'}$. Similarly find B' and C'. A', B', and C' are the vertices of the figures under a half-turn about O.



17. **(a)** $\ell' = \ell$.

(b) $\ell' \perp \ell$.

18. **(a)** First rotate $\triangle ABC$ by angle α to $\triangle A'B'C'$. Then rotate $\triangle A'B'C'$ by angle β to obtain $\triangle A''B''C''$.

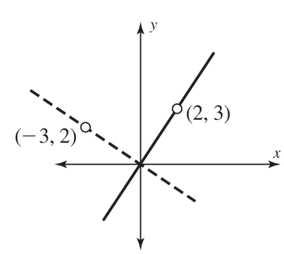


(b) **No**. Either rotation first would produce the same image.

(c) **Yes**. Rotate around O by angle $|\alpha - \beta|$ in the direction of the larger of α and β.

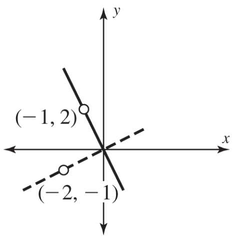
19. **(a)** The following figures show 90° counterclockwise rotation about the origin.

(i) $m_\ell = \frac{3}{2} \Rightarrow m_\perp = \frac{2}{^-3}$ because $(2, 3) \rightarrow$ $\mathbf{(^-3, 2)}$.



(***ii***) $m_\ell = \frac{2}{^{-}1} \Rightarrow m_\perp = \frac{^{-}1}{^{-}2}$ because

$(^{-}1,2) \rightarrow \mathbf{(^{-}2,^{-}1)}$.

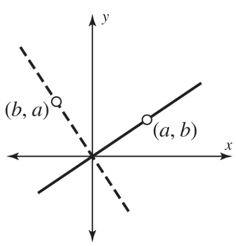


(***iii***) Under 90° counterclockwise rotation about the origin, $(m,n) \rightarrow \boldsymbol{(^{-}n,m)}$ or $m_\perp = \frac{m}{^{-}n}$.

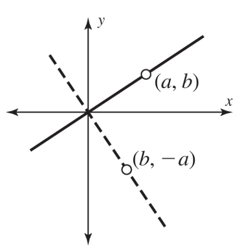
(b) In problem 19(a)(*iii*) the point (a, b) moves to $(^{-}b, a)$. Repeating this 90° counterclockwise rotation would produce a full half-turn and the point would end at $(^{-}a, ^{-}b)$.

(c) $m_\ell = \frac{b}{a} \Rightarrow m_\perp = \frac{^{-}a}{b}$ because $(a,b) \rightarrow$ $\boldsymbol{(b, ^{-}a)}$.

(***i***) (a, b) rotated 90° counterclockwise:



(***ii***) $(^{-}b, a)$ under a half-turn about the origin:



Or $(a,b) \rightarrow \boldsymbol{(b, ^{-}a)}$.

20. **(a)** Because $y = x + 2$ has slope 1, the slope of a perpendicular line is $^{-}1$. Thus, $y = {^{-}1}x + b$. To pass through $(1, 0)$, $0 = y = {^{-}1}(1) + b$. Thus, $b = 1$. Therefore, $\boldsymbol{y = {^{-}x} + 1}$.

(b) Because $y = -2x + 3$ has slope $^{-}2$, we have $^{-}2m = {^{-}1} \Rightarrow m = \frac{1}{2}$. Thus, $y = \frac{1}{2}x + b$, and $0 = \frac{1}{2}(1) + b \Rightarrow b = {^{-}\frac{1}{2}}$. Thus, $\boldsymbol{y = \frac{1}{2}x - \frac{1}{2}}$.

(c) Because, $x = -4$ is a vertical line, any horizontal line is perpendicular, and $\boldsymbol{y = 0}$ passes through $(1, 0)$.

(d) Because, $y = -3$ is a horizontal line any vertical line is perpendicular. $\boldsymbol{x = 1}$ passes through $(1, 0)$.

21. **(a)** The image of $A(h, k)$ under the translation from O to C is $B(h + a, k + 0) = B(h + a, k)$.

(b) $m_{\overline{OB}} = \frac{k-0}{(h+a)-0} = \frac{k}{h+a}$. $m_{\overline{AC}} = \frac{k-0}{h-a} = \frac{k}{h-a}$. $m_{\overline{OB}} \cdot m_{\overline{AC}} = \frac{k}{h+a} \cdot \frac{k}{h-a} = \frac{k^2}{h^2-a^2} = \frac{k^2}{^{-}k^2} = {^{-}1}$. Thus the diagonals are perpendicular.

(c) There is a turn symmetry of 180 degrees about the point of intersection of the two diagonals.

22. **(a)** Connect A to P, Q and R. Connect B to P, Q and R. Construct a line parallel to AB through point P. Find point P' such that $AB = PP'$ and $\overrightarrow{PP'}$ is in the same direction as $\overrightarrow{AB}$. Repeat for points Q' and R'. Repeat this procedure to construct $\Delta P''Q''R''$.

(b) Through B construct $\overrightarrow{BE}$ so that $\overline{BE} \parallel \overline{CD}$, $BE = CD$, and $\overrightarrow{BE}$ points in the same direction as CD. Then $\overrightarrow{AE}$ is a slide arrow that corresponds to the translation that takes $\triangle PQR$ to $\Delta P''Q''R''$.

23. **(a)** Rotation by 90° or a multiple of 90° about the point of intersection of the diagonals.

(b) Rotation by 72° or a multiple of 72° about O, where O is the point of intersection of the perpendicular bisectors of the sides.

(c) Rotation by $\frac{360^\circ}{n}$ or a multiple of $\frac{360^\circ}{n}$ about O, where O is the point of intersection about the center by any turn angle.

(d) Infinite turn symmetries.

(e) No rotational symmetries.

24. (a) Half-turn symmetry about the point of intersection of the diagonals. A rectangle has half-turn symmetry because the diagonals bisect each other.

(b) Half-turn symmetry about the point of intersection of the diagonals. A parallelogram has half-turn symmetry because the diagonals bisect each other

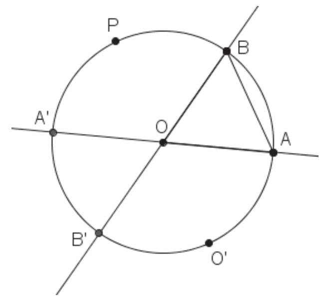
(c) No half-turn symmetry.

(d) No half-turn symmetry.

25. (a) Find the image of the center O by connecting point O and connecting P with a line using a straight edge. Put your compass legs on O and P and using the same compass opening leave one leg on point P and mark point O'. Construct the circle with point O' as the center and the same radius as $\overline{OP}$.

(b) The circles have a single point P in common and they have the same radius $\overline{OP} = \overline{PO'}$.

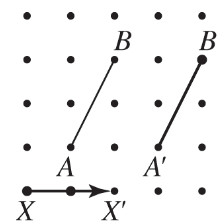
26. Construct a point O and with any compass opening construct a circle using the point O as its center. Put the legs of the compass on the center O and on any point A on the circle. Using the same compass opening leave one of the legs on A and create another point B counterclockwise from point A on the circle. This is the chord $\overline{AB}$. Then construct A' by using a straight edge and including the center O and point A. The intersection of the straight line and the circle is a half turn of point A with turn center O. Then construct B' by using a straight edge and including the center O and point B. The intersection of the straight line and the circle is a half turn of point B with turn center O. To construct O' put your compass legs on point A and point B. Using the same compass opening leave one leg on point A and mark point O' on the circle clockwise from point A. Using the same compass opening leave one leg on point B and mark point P on the circle counterclockwise from point B. $ABPA'B'O$ is a regular hexagon.



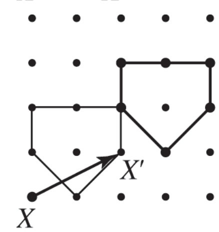
Assessment 13-1B

1. Reverse the translation so that the image completes a slide from X' to X (i.e., to its pre-image); then check by carrying out the given motion in the "forward" direction (i.e., to see if $\overline{AB}$ maps to $\overline{A'B'}$).

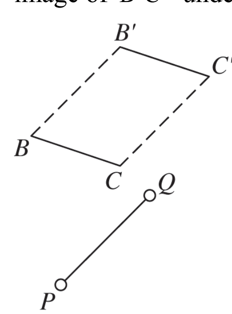
(a)



(b)



2. (a) Trace $\overline{B'C'}$ and the line containing the slide arrow on the tracing paper and label the trace of B as B' and the trace of C as C'. Mark on the original paper and on the tracing paper the initial point of the arrow by P and the head of the arrow by Q. Slide the tracing paper along the line $\overleftrightarrow{PQ}$ so that Q will fall on P; trace $\overline{B'C'}$. The segment $\overline{BC}$ is the pre-image of $\overline{B'C'}$ under the translation.



(b) Construct a parallelogram with $\overline{BC}$ and $\overline{B'C'}$ as opposite sides, similar to that of Figure 13-5 in the textbook.

3. **(a)** $(7, 14) \to (7+3, 14-4) = \mathbf{(10, 10)}$.

(b) $(^-3, {}^-5) \to (^-3+3, {}^-5-4) = \mathbf{(0, {}^-9)}$.

(c) $(h, k) \to \boldsymbol{(h+3, k-4)}$.

4. To go from an image to the coordinates of the original points, reverse the signs of the translations; i.e., $(x', y') \to (x'+3, y'-4) = (x, y)$.

(a) $(7, 14) \to (7+3, 14-4) = (10, 10)$.

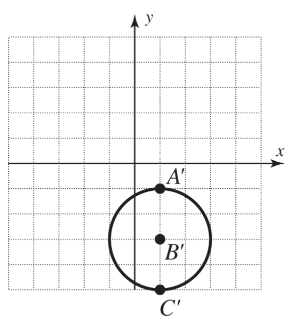
(b) $(^-7, {}^-10) \to (^-7+3, {}^-10-4) = (^-4, {}^-14)$.

(c) $(h, k) \to (h+3, k-4)$.

5. **(a)** $A(^-2, 3) \to (^-2+3, 3-4) = A'(1, {}^-1)$;

$B(^-2, 1) \to (^-2+3, 1-4) = B'(1, {}^-3)$;

$C(^-2, {}^-1) \to (^-2+3, {}^-1-4) = C'(1, {}^-5)$.

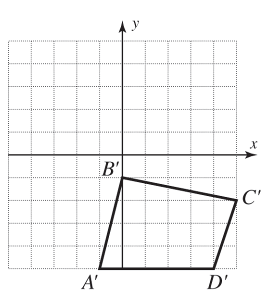


(b) $A(^-4, {}^-1) \to (^-4+3, {}^-1-4) = A'(^-1, {}^-5)$;

$B(^-3, 3) \to (^-3+3, 3-4) = B'(0, {}^-1)$;

$C(2, 2) \to (2+3, 2-4) = C'(5, {}^-2)$;

$D(1, {}^-1) \to (^-1+3, {}^-1-4) = D'(2, {}^-5)$.



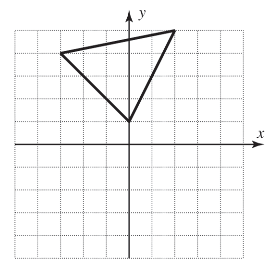
6. Label points $A'(0, 0)$, $B'(5, 1)$ and $C'(3, {}^-3)$.

$A'(0,0) \to (0-3, 0+4) = A(^-3, 4)$;

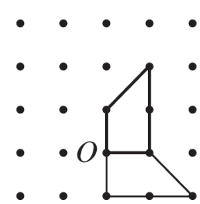
$B'(5,1) \to (5-3, 1+4) = B(2, 5)$;

$C'(3, {}^-3) \to (3-3, {}^-3+4) = C(0, 1)$.

The figure whose image was shown is:



7. Use the technique demonstrated in Example 13-3 of the text to obtain:



8. If $y = 2x - 1$, two sample points on line ℓ are $A(0, {}^-1)$ and $B(1, 1)$.

Note that in the following, the slope of the image line remains the same as the slope of line ℓ.

(a) $(x-3, y+2) \to A' = (0-3, {}^-1+2) = (^-3, 1)$ and $B' = (1-3, 1+2) = (^-2, 3)$

Then $m = \frac{3-1}{^-2-{}^-3} = 2$. Using x and y values from A' in the general form, $1 = 2(^-3) + b \Rightarrow b = 7$. The image is $\boldsymbol{y = 2x + 7}$.

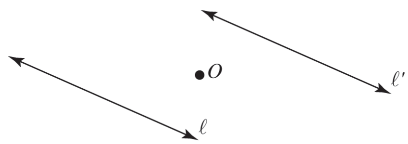
(b) $(x-5, y-4) \to A' = (0-5, {}^-1-4) = (^-5, {}^-5)$ and $B' = (1-5, 1-4) = (^-4, {}^-3)$

Then $m = \frac{^-3-{}^-5}{^-4-{}^-5} = 2$. Using x and y values from A' in the general form, $^-5 = 2(^-5) + b \Rightarrow b = 5$.

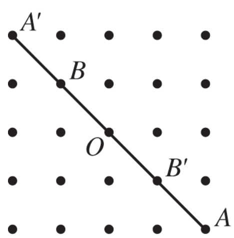
The image is $\boldsymbol{y = 2x + 5}$.

9. **P', P, and O are collinear** because the measure of $\angle POP'$ under a half-turn must be 180°.

10. Construct a line perpendicular to ℓ passing through O. Denote the intersection the perpendicular line with ℓ as Q. Draw a circle with radius $\overline{OQ}$ and center O; label point P where the perpendicular line intersects the circle opposite of Q. Then construct a perpendicular to $\overline{OQ}$ at $P \Rightarrow \ell'$.



11. Reverse the rotation in either direction to locate $\overline{AB}$; i.e., the pre-image.



12. **No**. There is no half-turn that is the image of MOM. A half-turn would yield WOW.

13. **(a)** When the figure is creased and folded along the perpendicular $\overline{PP'}$, the point P falls on $\overline{P'}$, which shows that the perpendicular also bisects $\overline{PP'}$. Alternatively, by the definition of rotation, $PO = PO'$, implying that O is equidistant from P and P' and so is on the perpendicular bisector of $\overline{PP'}$.

(b) From part (a), O is on the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$, as well as $\overline{CC'}$. Then O may be found by determining the point at which any two of the perpendicular bisectors intersect. $\angle AOA'$ is the angle of rotation.

(c) The perpendicular bisectors of the corresp-onding sides are not concurrent; i.e., they do not contain the same point; although the perpendicular bisectors of two sides may intersect.

14. An angle whose measure is 60° can be constructed by first constructing an equilateral triangle. Then an angle whose measure is 30° can be constructed by bisecting the 60° angle.

(a) Make arcs of radius OP from both O and P so that they intersect counterclockwise from P (this is the construction for an equilateral triangle). Bisect the angle and mark off OP along the bisector to locate P'.

(b) Use the procedure in (a), except to intersect the arcs clockwise from P.

(c) Construct a circle with center O through point P. Construct a circle with center P through point O. Label the circle intersections Q (left intersection) and R (right intersection). Connect O to P and Q to O and Q to P with a segment using a straight edge. The triangle formed is an equilateral triangle and the interior angles measure 60°. We can replace R with P' to rotate P 60° clockwise.

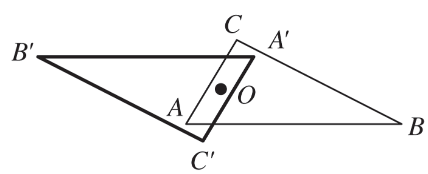
15. Signs of the x and y coordinates will be reversed under a half-turn about the origin.

(a) $(0, 3) \to$ $\mathbf{(0, {}^-3)}$.

(b) $({}^-2, 5) \to$ $\mathbf{(2, {}^-5)}$.

(c) $({}^-a, {}^-b) \to$ $\boldsymbol{(a, b)}$.

16. Label vertices of the triangles A, B, and C. Find the image of A by drawing $\overrightarrow{AO}$ and marking off $OA' = OA$ so that O is the midpoint of $\overline{AA'}$. Similarly find B' and C'. A', B', and C' are the vertices of the figure under a half-turn about O.



17. **(a)** $\ell' \parallel \ell$.

(b) ℓ' and ℓ intersect at a 60° angle.

18. **(a)** A **circle**, with radii $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$.

(b) All points O for which two vertices trace an identical path are the points on the perpendicular bisectors of their sides of the triangle.

(c) **Yes**. The intersection of the perpendicular bisectors forms the center of the circumscribed circle, which is equidistant from all vertices.

19. **(a)** Pick an arbitrary point A on the line $y = 3x - 1$; e.g., $(1, 2)$. The image of A under a half-turn about the origin is $(^{-}1, {}^{-}2)$. The line passing through $(^{-}1, {}^{-}2)$ with $m = 3$ is $\boldsymbol{y = 3x + 1}$.

(b) The image of the arbitrary point A under a 90° counterclockwise turn about the origin is $(^{-}2, 1)$. The line passing through $(^{-}2, 1)$ with $m_{\perp} = \frac{^{-}1}{3}$ is $\boldsymbol{y = \frac{^{-}1}{3}x + \frac{1}{3}}$.

20. Let each of the lines be labeled line ℓ.

(a) $m_{\ell} = 2$, so $m_{\perp} = \frac{^{-}1}{2}$. Given the point $(^{-}1, {}^{-}3)$, $^{-}3 = \frac{^{-}1}{2}(^{-}1) + b \Rightarrow b = \frac{^{-}7}{2}$. The perpendicular line has equation $\boldsymbol{y = \frac{^{-}1}{2}x - \frac{7}{2}}$.

(b) $m_{\ell} = {}^{-}2$, so $m_{\perp} = \frac{1}{2}$. Given the point $(^{-}1, {}^{-}3)$, $^{-}3 = \frac{1}{2}(^{-}1) + b \Rightarrow b = \frac{^{-}5}{2}$. The perpendicular line has equation $\boldsymbol{y = \frac{1}{2}x - \frac{5}{2}}$.

(c) The perpendicular through $(^{-}1, {}^{-}3)$ is $\boldsymbol{y = {}^{-}3}$.

(d) The perpendicular through $(^{-}1, {}^{-}3)$ is $\boldsymbol{x = {}^{-}1}$.

21. O must be the midpoint of segments $\overline{AA'}$ and $\overline{BB'}$. So A' must lie on the x-axis and B' must lie on the y-axis. Since A' lies on the x-axis, A' has coordinates $(x, 0)$. But $AB = \sqrt{a^2 + b^2} = BA' = \sqrt{x^2 + b^2} \Rightarrow a^2 + b^2 = x^2 + b^2 \Rightarrow x^2 = a^2$. Since x is positive $x = a$. So, A' has coordinates $(a, 0)$. Similarly, B' has coordinates $(0, -b)$.

22. **(a)** Yes, the same result can be accomplished by a single translation. A translation **from A to C**.

(b) A translation **from A to C**. The translation from D to E is the same as the translation from B to C.

(c) Yes. We do not land at C. This is because our starting point was different. However, both translations would have the same result if the same starting point is used.

23. **(a)** No turn symmetries.

(b) Turn symmetry about the intersection of perpendicular bisectors by 60° or a multiple of 60° in either direction.

(c) Turn symmetry by 90° or a multiple of 90° about the intersection of the diagonals in either direction.

(d) No turn symmetries.

(e) Half-turn symmetry about the center of the circle.

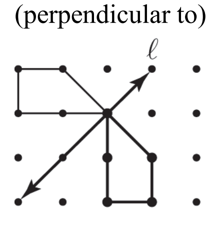
24. Answers vary. For example the number eight: 8.

25. Construct a point O and with any compass opening construct a circle using the point O as its center. Put the legs of the compass on the center O and on any point P on the circle. Connect point O and point P with a line using a straightedge. Find point O' such that $\overrightarrow{OP} = \overrightarrow{PO'}$ and $\overrightarrow{PO'}$ is in the same direction as $\overrightarrow{OP}$. Construct the circle with point O' as the center and the same radius as $\overline{OP}$. The circles have a single point P in common and they have the same radius $\overline{OP} = \overline{PO'}$.

26. Translating the figure by $\overrightarrow{AB}$ or by $\overrightarrow{BA}$ results in the same figure as the drawing extends indefinitely in both directions horizontally.

Assessment 13-2A: Reflections and Glide Reflections

1. Locate the image of vertices directly across (perpendicular to) ℓ on the geoboard.



2. Reflecting lines are described for each.

(a) **All diameters**. There are infinitely many.

(b) **Perpendicular bisector** of the segment. The line the segment is contained is a trivial line of reflection

(c) Any **line perpendicular** to the given line or the line itself.

(d) **Perpendicular bisectors** of the sides and lines containing the diagonals.

(e) **None**.

(f) **Perpendicular bisectors** of each of the sides.

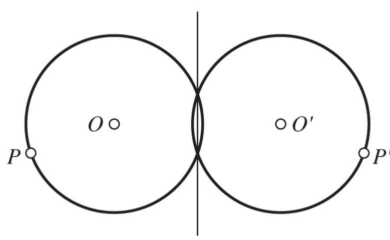
(g) **None**.

(h) If the kite is not a rhombus, the line containing the diagonal determined by the vertices of the noncongruent angles is a line of symmetry. If a rhombus then each line containing a diagonal is a line of symmetry.

(i) **Perpendicular bisectors** of parallel sides and three diagonals determined by vertices on the circumscribed circles; or the **angle bisectors** of the vertices.

3. The original figure is reflected back upon itself, i.e., if the first reflection yields $\Delta A'B'C'$, the second reflection brings $\Delta A'B'C'$ back to ΔABC.

4. Find the image of the center of the circle and one point on the circumference of the circle to determine the image of the circle. One possibility is:



5. **(a)** **No**. The final images are congruent but in different locations, thus they are not the same.

(b) A translation determined by a slide arrow from P to R. Let P be any point on ℓ and Q on m such that $\overrightarrow{PQ} \perp \ell$. Point R is on $\overrightarrow{PQ}$ such that $PQ = QR$.

6. Construct as suggested. The final image is a translation of the original.

7. The line of reflection is the **perpendicular bisector** of $\overline{AA'}$, $\overline{BB'}$, or $\overline{CC'}$.

8. **(a)** Examples may vary, but include MOM, WOW, TOOT, HAH, etc..

(b) **(*i*)** Examples include BOX, HIKE, CODE, OBOE, etc..

(*ii*) The letters B, C, D, E, H, I, K (depending on construction), O, and X may be used.

(c) Others of these numbers include combinations of 1, 8, and 0. Examples are 1, 8, 11, 88, 101, 111, 181, 808, 818, 888, 1001, or 1111.

9. Given the line $y = 2x + 1$, two sample points are $A(0, 1)$ and $B(1, 3)$.

(a) $P(x, y) \to P(x, {}^-y)$ when reflected in the x-axis $\Rightarrow A(0,1) \to A'(0, {}^-1)$ and $B(1,3) \to B'(1, {}^-3)$. Then $m = \frac{{}^-3 - {}^-1}{1-0} = {}^-2$. Using A', ${}^-1 = {}^-2(0) + b \Rightarrow b = {}^-1$. So the equation of the line is $\boldsymbol{y = {}^-2x - 1}$.

(b) $P(x, y) \to P({}^-x, y)$ when reflected in the y-axis $\Rightarrow A(0,1) \to A'(0,1)$ and $B(1,3) \to B'({}^-1, 3)$. Then $m = \frac{3-1}{{}^-1-0} = {}^-2$. Using A', $1 = {}^-2(0) + b \Rightarrow b = 1$. So the equation of the line is $\boldsymbol{y = {}^-2x + 1}$.

(c) $P(x, y) \to P(y, x)$ when reflected in the line $y = x \Rightarrow A(0,1) \to A'(1,0)$ and $B(1,3) \to B'(3,1)$. Then $m = \frac{1-0}{3-1} = \frac{1}{2}$. Using A', $0 = \frac{1}{2}(1) + b \Rightarrow b = \frac{{}^-1}{2}$. So the equation of the line is $\boldsymbol{y = \frac{1}{2}x - \frac{1}{2}}$.

10. **(a)** For glide reflections with the translation parallel to the reflection line, the **images are the same** regardless of order of operation.

(b) **Commutative**, for glide reflections with the translation parallel to the reflection line.

11. None of the images has a reverse orientation, so there are no reflections or glide reflections involved. Thus,

1 to 2 can be viewed as a counterclockwise rotation;

1 to 3 can be viewed as a clockwise rotation;

1 to 4 is a translation down;

1 to 5 is a clockwise rotation followed by a translation down or a rotation about an exterior point;

1 to 6 is a translation or a clockwise rotation; and

1 to 7 is a translation or a clockwise rotation.

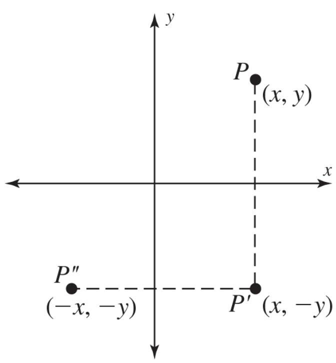
12. **(a)** When reflecting in the x-axis, $P(x, y) \to P'(x, {}^{-}y)$. Thus $A(3, 4) \to \boldsymbol{A'(3, {}^{-}4)}$**;** $B(2, {}^{-}6) \to \boldsymbol{B'(2, 6)}$**;** and $C({}^{-}2, 5), \to \boldsymbol{C'({}^{-}2, {}^{-}5)}$.

(b) When reflecting in the line $y = x$, $P(x, y) \to P'(y, x)$. Thus $A(3, 4) \to \boldsymbol{A'(4, 3)}$**;** $B(2, {}^{-}6) \to \boldsymbol{B'({}^{-}6, 2)}$**;** and $C({}^{-}2, 5), \to \boldsymbol{C'(5, {}^{-}2)}$.

13. **(a)** **(i)** $P(x, y) \to \boldsymbol{P'(x, {}^{-}y)}$ under reflection in the x-axis.

(ii) $P(x, y) \to \boldsymbol{P'(y, x)}$ under reflection about $y = x$.

(b) $\boldsymbol{({}^{-}x, {}^{-}y)}$. From part (a) the reason can be seen as below:



14. $P(x, y) \to P(x, {}^{-}y)$ when reflected in the x-axis. Thus:

(a) $y = {}^{-}x + 3 \to {}^{-}y = {}^{-}x + 3 \Rightarrow$ $\boldsymbol{y = x - 3}$.

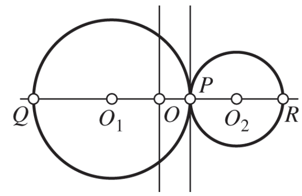
(b) $y = 0 \to {}^{-}y = 0 \to \boldsymbol{y = 0}$.

15. $P(x, y) \to P({}^{-}x, y)$ when reflected in the y-axis. Thus:

(a) $y = {}^{-}x + 3 \to y = {}^{-}({}^{-}x) + 3 \Rightarrow$ $\boldsymbol{y = x + 3}$.

(b) $y = 0 \to \boldsymbol{y = 0}$.

16. **(a)** Given the circles below:

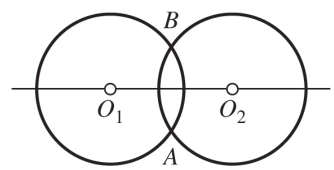


(***i***) The line through P perpendicular to $\overleftrightarrow{O_1O_2}$ will cause the image to be an interior circle. The smaller circle will be tangent to point P.

(***ii***) The line through O, the perpendicular bisector of $\overline{QR}$, will cause the image to be an interior circle. The smaller circle will be tangent to point Q.

(b) $\overleftrightarrow{O_1O_2}$

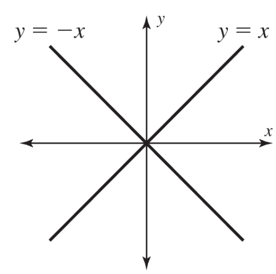
17. Given the circles below:



(a) The line $\overleftrightarrow{AB}$, since the circles are congruent.

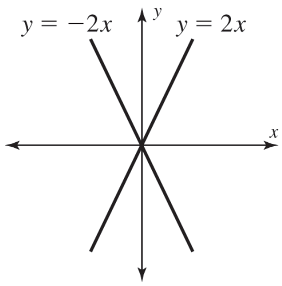
(b) **Yes**, a translation taking O_1 to O_2.

18. **(a)**



Any point (a, b) on the line $y = x$ will reflect across the vertical axis to the point $({}^{-}a, b)$ on the line $y = {}^{-}x$. Any point (a, b) on the line $y = x$ will reflect across the horizontal axis to the point $(a, {}^{-}b)$ on the line $y = {}^{-}x$. Thus the lines of reflection are the axes, $\boldsymbol{x = 0}$ and $\boldsymbol{y = 0}$.

(b)



Any point (a, b) on the line $y = 2x$ will reflect across the vertical axis to the point $(^-a, b)$ on the line $y = {}^-2x$. Any point (a, b) on the line $y = 2x$ will reflect across the horizontal axis to the point $(a, {}^-b)$ on the line $y = {}^-2x$. Thus the lines of reflection are the axes, $\boldsymbol{x = 0}$ and $\boldsymbol{y = 0}$.

19. (a) If point P was determined by a glide reflection by the slide arrow $\overrightarrow{AB}$ and line l parallel to $\overleftrightarrow{AB}$, then we can start with P' and work backwards: reflect P' across line l and translate by the slide arrow $\overrightarrow{BA}$ to P.

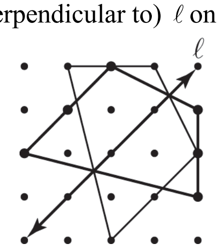
(b) Construct $\triangle DEF$ Connect A to D, E and F. Connect B to D, E and F. Construct a line parallel to AB through point D. Find point D' such that $AB = DD'$ and $\overrightarrow{DD'}$ is in the same direction as $\overrightarrow{AB}$. Repeat for points E' and F'. Connect D', E', and F' to create $\Delta D'E'F'$. Repeat this procedure to construct $\Delta D"E"F"$ starting from $\Delta D'E'F'$. A single translation by slide arrow $\overrightarrow{AC}$, where A, B, and C are collinear and B is the midpoint of $\overline{AC}$ takes $\triangle PQR$ to $\Delta P"Q"R"$.

20. (a) Any point moves right 2 and reflects across the x-axis. Point (x, y) moves to $(x+2, {}^-y)$.

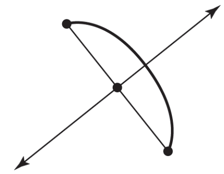
(b) If $(x+2, {}^-y) = (3,5)$ then $x+2 = 3 \Rightarrow x = 1$ and ${}^-y = 5 \Rightarrow y = {}^-5$ so the coordinates of P are $\left(1, {}^-5\right)$.

Assessment 13-2B

1. Locate the image of vertices directly across (perpendicular to) ℓ on the geoboard.

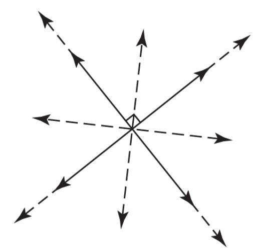


2. (a) The perpendicular bisector of the chord connecting the endpoints of the arc.

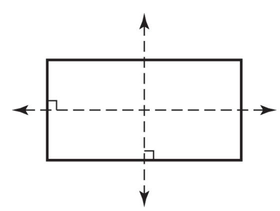


(b) The line containing the ray.

(c) The line bisecting the vertical angles formed by the perpendicular lines, and the lines themselves.

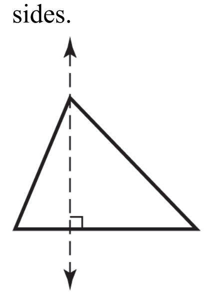


(d) Perpendicular bisectors of pairs of parallel sides.

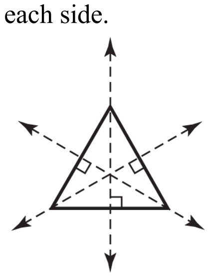


If a square, the perpendicular bisectors of each side.

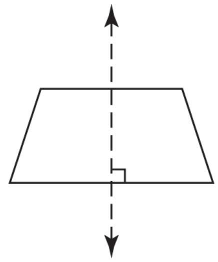
(e) If not equilateral, the perpendicular bisector of the side that is not congruent to the other two sides.



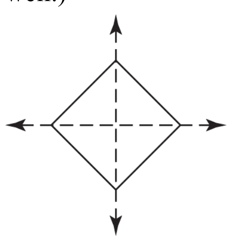
If equilateral, the perpendicular bisectors of each side.



(f) Perpendicular bisector of parallel sides if there is exactly one pair of parallel sides and they are not congruent.

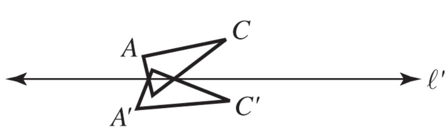


(g) The lines containing the diagonals. (If the rhombus is a square, then include the perpendicular bisectors of parallel sides as well.)

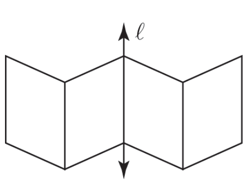


(h) There will be n reflecting lines in all. If n is even, the lines are the perpendicular bisectors of the parallel sides $\left(\frac{n}{2}\right)$ and the diameters through the vertices determined by the circumscribed circle $\left(\frac{n}{2}\right)$. If n is odd, the lines are the perpendicular bisectors of the sides.

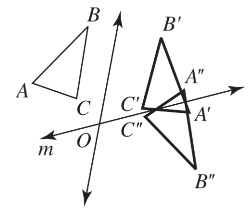
3. Methods will vary; the image is reflected in line ℓ as shown below, the reflected again to obtain the original image.



4. Locate the images of vertices on perpendiculars to line ℓ.

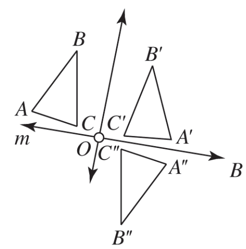


5. (a) The following image results:



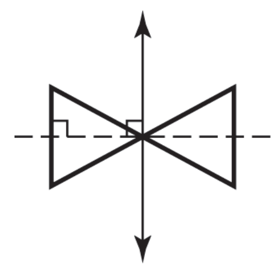
(b) A rotation about O through 2α, where α is the measure of the angle between ℓ and m in the direction from ℓ to m as shown.

(c) A half-turn about O:



6. Construction. The results should be unchanged from what is already stated in problems 4 and 5.

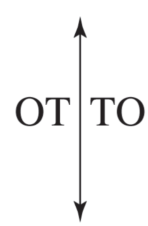
7. (a)



(b) Use the line perpendicular to an altitude of the original triangle. The bow-tie may be skewed, depending on the type of the triangle, but the two bases will be parallel.

(c) Three reflecting lines are possible. Two other than the one used in part (a) could be found in the same way at the other vertices.

8.

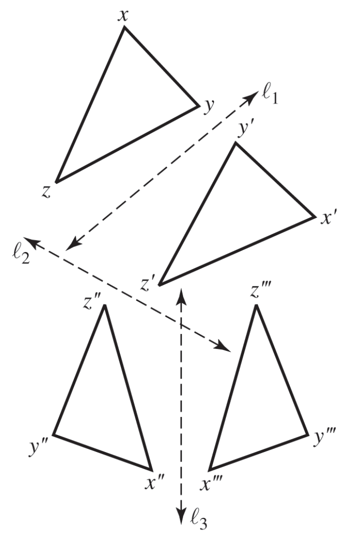


9. **(a)** The x-axis and all lines perpendicular to the x-axis.

(b) The y-axis and all lines perpendicular to the y-axis.

(c) The line $y = x$ and all lines perpendicular to the line $y = x$.

10. **(a)** Answers may vary, but no more than 3 are needed.



(b) The image cannot be obtained by a translation or rotation alone because the orientation is reversed. It cannot be a reflection because there is no single reflecting line.

11. **(a)** Answers may vary. Part (a) could be moved to part (b) by:

(***i***) A reflection in the line through the common point of the parts and containing a diagonal of the large square.

(***ii***) A 90° counterclockwise rotation about the common point of the parts.

(b) Answers may vary. A 90° clockwise rotation about the center of the large square would take part (a) to part (b).

12. **(a)** When reflecting in the y-axis, $P(x, y) \to P'(^-x, y)$. Thus $A(3, 4) \to \boldsymbol{A'(^-3, 4)}$; $B(2, ^-6) \to \boldsymbol{B'(^-2, ^-6)}$; and $C(^-2, 5) \to \boldsymbol{C'(2, 5)}$.

(b) When reflecting in the line $y = {^-x}$, $P(x, y) \to P'(^-y, ^-x)$. Thus $A(3, 4) \to \boldsymbol{A'(^-4, ^-3)}$; $B(2, ^-6) \to \boldsymbol{B'(6, ^-2)}$; and $C(^-2, 5) \to \boldsymbol{C'(^-5, 2)}$.

13. $P(x, y) \to \boldsymbol{P'(^-x, y)}$ under reflection in the y-axis; $P(x, y) \to \boldsymbol{P'(^-y, ^-x)}$ under reflection about the line $y = {^-x}$.

14. $P(x, y) \to P(x, ^-y)$ when reflected in the x-axis. Thus:

(a) $y = 3x \to {^-y} = 3x \Rightarrow \boldsymbol{y = {^-3x}}$.

(b) $y = {^-x} \to {^-y} = {^-x} \Rightarrow \boldsymbol{y = x}$.

(c) $x = 0 \to \boldsymbol{x = 0}$.

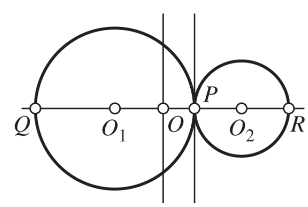
15. $P(x, y) \to P(^-x, y)$ when reflected in the y-axis. Thus:

(a) $y = 3x \to y = 3(^-x) \Rightarrow \boldsymbol{y = {^-3x}}$.

(b) $y = {^-x} \to y = {^-(^-x)} \Rightarrow \boldsymbol{y = x}$.

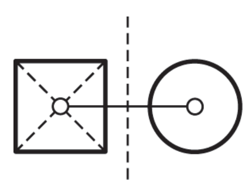
(c) $x = 0 \to {^-x} = 0 \to \boldsymbol{x = 0}$.

16. Given the intersecting circles below:



The line $\overleftrightarrow{O_1O_2}$ will reflect the circles onto themselves.

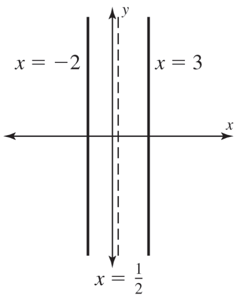
17. Given:



The diameter of the circle must be congruent to a side of the square. Then the line of reflection is the perpendicular bisector of the segment connecting the center of the circle with the point of intersection of the diagonals of the square.

18. **(a)** The lines $\boldsymbol{y = x}$ or $\boldsymbol{y = {}^{-}x}$ will cause $x = 0$ and $y = 0$ to reflect on each other.

(b)



The vertical line where $x = \frac{{}^{-}2+3}{2}$, or $\boldsymbol{x = \frac{1}{2}}$, will cause $x = {}^{-}2$ and $x = 3$ to reflect on each other.

19. **(a)** If point $P'Q'$ was determined by a glide reflection by the slide arrow $\overrightarrow{AB}$ and line $\overleftrightarrow{AB}$, then we can start with $P'Q'$ and work backwards: reflect $P'Q'$ across line $\overleftrightarrow{AB}$ and translate by the slide arrow $\overrightarrow{BA}$ to *PQ*.

(b) Construct quadrilateral *DEFG* Connect *A* to *D, E, F,* and *G.* Connect *B* to *D, E,F,* and *G.* Construct a line parallel to *AB* through point *D.* Find point D' such that $AB = DD'$ and $\overrightarrow{DD'}$ is in the same direction as $\overrightarrow{AB}$. Repeat for points E', F', and G'. Connect D', E', F', and G' to create quadrilateral $D'E'F'G'$. Repeat this procedure to construct quadrilateral $D''E''F''G''$ starting from quadrilateral $D'E'F'G'$. A single translation by slide arrow $\overrightarrow{AC}$, where *A*, *B*, and *C* are collinear and *B* is the midpoint of $\overline{AC}$ takes quadrilateral *DEFG* to quadrilateral $D''E''F''G''$.

20. **(a)** Any point reflects across the *y*-axis and moves up 2. Point (x, y) moves to $({}^{-}x, y+2)$.

(b) If $({}^{-}x, y+2) = (3,5)$ then ${}^{-}x = 3 \Rightarrow x = {}^{-}3$ and $y + 2 = 5 \Rightarrow y = 3$ so the coordinates of *P* are $\left({}^{-}3, 3\right)$.

21. **(a)** All points on line *l* are fixed points.

(b) The point *O* is a fixed point.

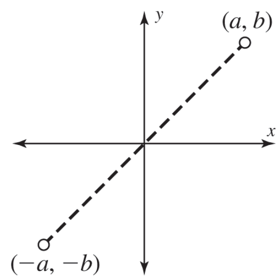
(c) No fixed points.

(d) No fixed points.

Mathematical Connections 13-2: Review Problems

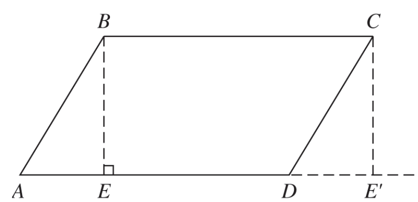
19. **0, 1, and 8,** depending upon how they are drawn. **5 and 2,** if drawn as a digital clock would show them.

20. $\boldsymbol{({}^{-}a, {}^{-}b)}$. See below, where, for example, (a, b) is in the first quadrant. The relationship is the same regardless of the quadrant in which (a, b) is located.



21. A **rotation** of any angle about the center of the circle will transform the circle into itself.

22. Construct $\overline{BE}$ perpendicular to $\overline{AD}$ as shown below:



Translate $\triangle ABE$ by the slide arrow from *B* to *C*. The image of $\triangle ABE$ is $\triangle DCE'$, thus the rectangle $BCE'E$ is the required rectangle.

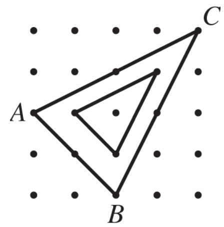
Assessment 13-3A: Dilations

1. Answers may vary.

(a) Slide the small triangle down three units (translation). Then complete the dilation with scale factor 2 (i.e., the larger triangle has sides twice as long) using the top right vertex as center.

(b) Slide the smaller triangle right 5 and up 1. Then complete the dilation with scale factor 2 using the top right vertex as center.

2. Vertices A', B', and C' will be half the distance from O as vertices A, B, and C, respectively.



3. Answers may vary.
   - **(a)** Translate B to B' followed by a size transformation with center B' and scale factor 2.
   - **(b)** Rotate a half-turn with the midpoint of $\overline{AA'}$ as the center followed by a size transformation with center A' and scale factor $\frac{1}{2}$.

4. ***(i)*** Scale factor of $\frac{4}{10} = \mathbf{\frac{2}{5}}$.

   (ii) $x = \frac{2}{5}(15) = \mathbf{6}$. $y = \frac{2}{5}(13) = \frac{26}{5} = \mathbf{5.2}$.

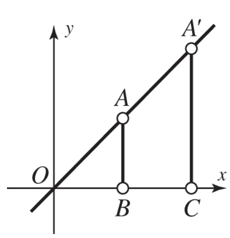
5. $\frac{3\text{ cm}}{10\text{ cm}} = \frac{BA\text{ cm}}{40\text{ cm}} \Rightarrow 10 \cdot BA = 120 \Rightarrow BA =$ the height of the candle $=$ **12 cm**.

6. Given $A(2, 3)$ and $B(^-2, 3)$:
   - **(a)** $A' = (2 \cdot 3 \cdot 2, 3 \cdot 3 \cdot 2) = \mathbf{(12, 18)}$.

     $B' = (^-2 \cdot 3 \cdot 2, 3 \cdot 3 \cdot 2) = \mathbf{(^-12, 18)}$.
   - **(b)** **Same as in (a)**. Order of dilation is irrelevant.

7. The dilation with center O and scale factor $\frac{1}{r}$ (equivalent to using division to reverse multiplication).

8. **(a)** Answers may vary. One explanation is shown by the figure below:



Let $A'(x', y')$ be the image of A under the dilation. The definition of a size transformation yields $\frac{OA'}{OA} = r$. Since $\Delta OA'C \sim \Delta OAB$, $\frac{OC}{OB} = r$, which implies that $\frac{x'}{x} = r$ and thus $x' = rx$. Similarly, $y' = ry$.

   - **(b)** The dilation with center at the origin and scale factor 2, followed by a half turn on the origin.
   - **(c)** $(x, y) \to (rx, ry)$, which implies:

     (i) $y = 2x \to \frac{1}{2}y = \frac{1}{2}(2x) \Rightarrow \frac{1}{2}y = x \Rightarrow \boldsymbol{y = 2x}$.

     (ii) $y = 2x \to 2y = 2(2x) \Rightarrow 2y = 4x \Rightarrow \boldsymbol{y = 2x}$.

     (iii) $P(1, 3) \to P'\left(\frac{1}{3}, 1\right)$ and $P(0, 1) \to P'\left(0, \frac{1}{3}\right)$.

     Thus $m = \frac{\frac{1}{3} - 1}{0 - \frac{1}{3}} = 2$ and $b = \frac{1}{3}$. The equation of the image is $\boldsymbol{y = 2x + \frac{1}{3}}$.

     (iv) A dilation about the origin with scale factor 3 would keep the same slope, but move the point $(0, ^-1)$ to $(0, ^-3)$. The final equation would then be $\boldsymbol{y = {}^-x - 3}$.

9. A translation taking $\boldsymbol{O_1}$ **to** $\boldsymbol{O_2}$ followed by a dilation with **center at** $\boldsymbol{O_2}$ and **scale factor** $\mathbf{\frac{3}{2}}$.

10. The set of images of the set of integers on a number line would be the set of points with coordinates multiples of 3; i.e., $\mathbf{\{\ldots, {}^-6, {}^-3, 0, 3, 6, \ldots\}}$.

11. ***(i)*** Scale factor of $\mathbf{\frac{7}{15}}$.

    (ii) $\boldsymbol{x} = \frac{7}{15}(14) = \mathbf{\frac{98}{15}}$. $\boldsymbol{y} = 6 \div \frac{7}{15} = \mathbf{\frac{90}{7}}$.

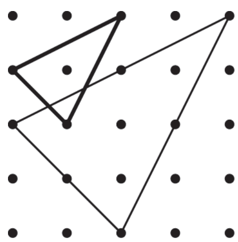
12. **Yes**. The enlargement can be achieved by dilation with scale factor 2, but there are many potential centers; e.g., one might be the lower left corner of the 2 inch $\times$ 3 inch rectangle.

Assessment 13-3B

1. **(a)** Rotate the smaller triangle 90° counterclockwise with its lower-right vertex as the center of rotation. Then complete a dilationwith scale factor 2 using the same point as center.

(b) Assume the lower left corner of the grid has coordinates (0, 0). Translate the smaller triangle using $(x, y) \rightarrow (x - 5, y - 1)$. Then use a dilation with (1, 2) as center and scale factor of 2.

2. Each vertex moves one-half the distance to O:



3. Answers may vary.

(a) Rotate $90°$ counterclockwise using center B. Then translate to take B to B'. Finally complete a dilation with center B' and scale factor $\frac{A'B'}{AB} = \frac{1}{2}$.

(b) Rotate a half turn about C followed by dilation with center C and scale factor $\frac{A'B'}{AB} = \frac{3}{2}$.

4. **(*i*)** $\frac{4}{x} = \frac{3}{9} \Rightarrow \mathbf{x = 12}$. $\frac{5+y}{9} = \frac{5}{3} \Rightarrow \mathbf{y = 10}$.

(*ii*) Scale factor: $\frac{x}{4} = \frac{12}{4} = \mathbf{3}$.

5. **(*i*)** Scale factor of $\frac{6}{3} = \mathbf{2}$.

(*ii*) $x + 5 = 2x \Rightarrow \mathbf{x = 5}$.
$y + 4 = 2y \Rightarrow \mathbf{y = 4}$.

6. **(a)** Slide arrow $\Rightarrow (x + 1, y + 3)$:

$A' = (2 \cdot 2 + 1, 3 \cdot 2 + 3) = \mathbf{(5, 9)}$.

$B' = (^-2 \cdot 2 + 1, 3 \cdot 2 + 3) = \mathbf{(^-3, 9)}$.

(b) Slide arrow $\Rightarrow (x + 1, y + 3)$:

$A' = [(2 + 1) \cdot 2, (3 + 3) \cdot 2] = \mathbf{(6, 12)}$.

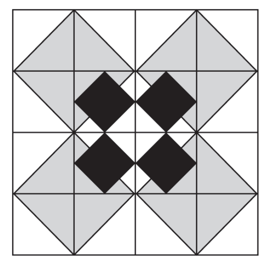
$B' = [(^-2 + 1) \cdot 2, (3 + 3) \cdot 2] = \mathbf{(^-2, 12)}$.

(c) The coordinates of the image depend on the order of the transformations.

7. The dilation with **center at the origin** and **scale factor $\mathbf{\frac{3}{4}}$**.

8. Answers vary; a translation taking O_1 to O_2 followed by a dilation with center O_2 and a scale factor 6. Alternatively, use a center on the line $\overline{O_1 O_2}$ placed $\frac{O_2 - O_1}{5}$ units from O_1 on the opposite side of O_2.

9.



10. **(*i*)** The scale factor is $\frac{A'B'}{AB} = \mathbf{\frac{3}{4}}$.

(*ii*) The center is the intersection of line segments $\overline{BB'}$ and $\overline{CC'}$. No other point would produce the image $A'B'C'$.

11. The center would remain at O; the undoing scale factor would be the reciprocal of $\frac{3}{4}$, or $\mathbf{\frac{4}{3}}$.

12. This is not possible because
$$\frac{4.5}{2} \neq \frac{6.5}{3} \Rightarrow 2.25 \neq 2.1\overline{6}\,.$$

Mathematical Connections 13-3: Review Problems

14. **(a)** The translation given by slide arrow from N to M.

(b) A counterclockwise rotation of $75°$ about O.

(c) A clockwise rotation of $45°$ about A.

(d) A reflection in m and translation from B to A.

(e) A second reflection in line n.

15. The x-coordinate is unchanged when reflecting about m; the y-coordinate is unchanged when reflecting about n.

(a) (4, 3) reflects about m to (4, 1), then (4, 1) reflects about n to (2, 1).

(b) $(0, 1) \rightarrow (0, 3) \rightarrow (6, 3)$.

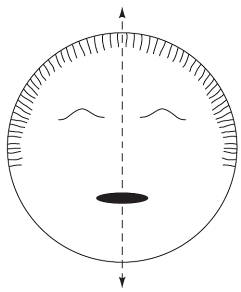
(c) $(^{-}1, 0) \rightarrow (^{-}1, 4) \rightarrow (7, 4)$.

(d) $(0, 0) \rightarrow (0, 4) \rightarrow (6, 4)$.

16. **(a)** The angle will reflect to the angle itself.

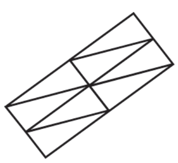
(b) The square will reflect to the square itself.

17. Reflect each part of the figure across the reflecting line:

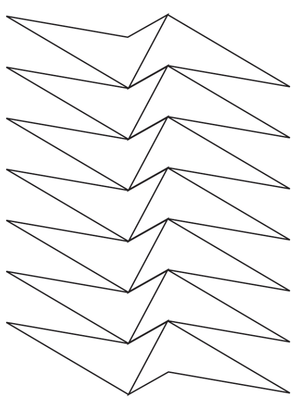


Assessment 13-4A: Tessellations of the Plane

1. Forming rectangles will tessellate the plane.

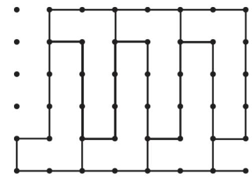


2. **(a)** Perform half-turns about the midpoints of all sides.



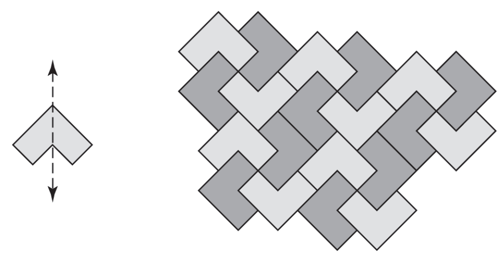
(b) **Yes**. If a polygon tessellates the plane, the sum of the angles around every vertex must be 360°. Successive 180° turns of a quadrilateral about the midpoint of its sides will produce four congruent quadrilaterals around a common vertex, with each of the quadrilaterals angles being represented at each vertex. These angles must add to 360°, as do angles of any quadrilateral.

3. Experimentation by cutting shapes out and moving them about is one way to learn about these tessellations.



4. Answers may vary. Consider figures such as a pentagon formed by combining a square and an equilateral triangle, or use hexagons and triangles sized as in the statement of the problem and squares with sides equal to the sum of the hexagon and triangle side. Shapes can be mixed, or a row of squares can be placed between hexagon/ triangle rows.

5. The shape will tessellate a plane as shown below, provided that it has a symmetry line as shown below:



6. **(a)** Construct as indicated.

(b) It combines a translation and a reflection.

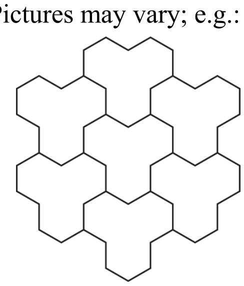
(c) **Yes**. The figures will overlap with no gaps.

7. **(a)** The dual is another tessellation of squares (congruent to those given).

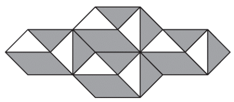
(b) A tessellation of equilateral triangles.

(c) The tessellation of equilateral hexagons is illustrated in the statement of the problem.

8. Pictures may vary; e.g.:

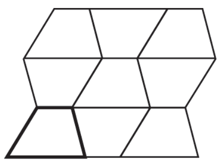


9. The hexagon will tessellate the plane as follows:

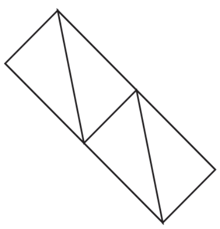


Assessment 13-4B

1. Rotate the trapezoid 180° and place to form a parallelogram. Place these together to cover the plane.



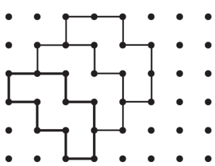
2. **(a)** Perform half-turns about the midpoints of all sides and continue the strips.



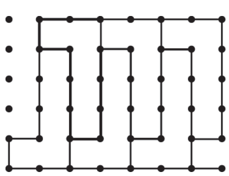
(b) **Yes**. Any two congruent triangles can form a parallelogram and any parallelogram can tessellate a plane.

3. Experimentation by cutting shapes out and moving them about is one way to learn about these tessellations.

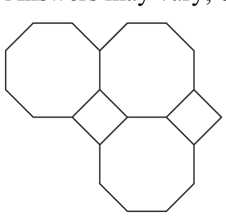
(a)



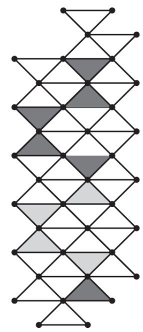
(b)



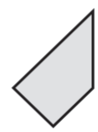
4. Answers may vary; one such example is below:



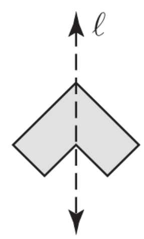
5. The shape shown is a "bow-tie" which can tessellate the plane using only copies of that shape and translations. See below:



6. **(a)** Answers vary. Consider the figure below.

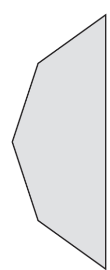


Reflect it as shown.



The original figure adjoined with its reflection creates the figure shown in exercise 5 from Assessment 13-4A, which tessellates a plane.

(b) Answers vary. Consider the figure below.

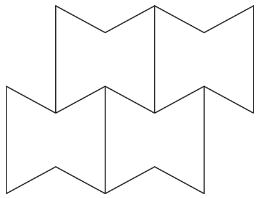


Reflecting about a vertical line as done in part (a) forms an octagon, which will not tessellate a plane.

7. **(a)** Figure 17: A translation from *A* to *B*.

(b) Figure 18: A translation would move lizard *C* to lizard *D*; a rotation would move lizard *C* to lizard *E*.

8. All sides fit; there are no gaps.



9 **No**, not as drawn. The head fits the tail, but there would be gaps elsewhere.

Mathematical Connections 13-4: Review Questions

18. **(a)** M slides to N:

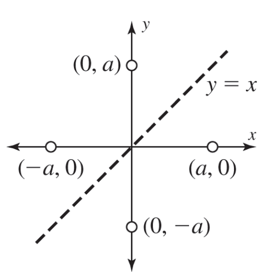
(b) 90° counterclockwise about M:

19. **(a)** **90°, 180°** (point), and **270°** symmetries.

(b) 180° rotation about the center (half-turn about that center); reflections across vertical and horizontal lines through the center.

20. 180° rotation about the center of the dark-colored square.

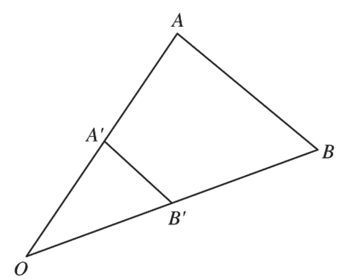
21.



(a) $d(a)$ from the origin, or intersection of the diagonals, is the same to all vertices. Thus the figure must be a **square**. You could also say that each vertex is a 90° rotation of a previous vertex.

(b) $P(x, y) \to P(y, x)$ under reflection in the line $y = x$. Thus $(a, 0) \to \mathbf{(0, a)}$, $(0, a) \to \mathbf{(a, 0)}$, $(^{-}a, 0) \to \mathbf{(0, {}^{-}a)}$, and $(0, {}^{-}a) \to \mathbf{({}^{-}a, 0)}$.

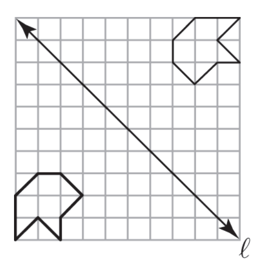
22. If O is on $\overline{AB}$, then the points of $\overline{A'B'}$ lie on $\overleftrightarrow{AB}$ and thus the segments are parallel. If O is not on $\overline{AB}$, then $OA' = r(OA)$; $OB' = r(OB)$; and $\angle O \cong \angle O$. Thus, $\Delta A'OB' \sim \Delta AOB$, which implies $\angle OA'B' \cong \angle OAB$. These are corresponding angles for lines $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{AB}$ and transversal $\overline{OA}$. Thus, $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ implying $\overline{AB} \parallel \overline{A'B'}$.



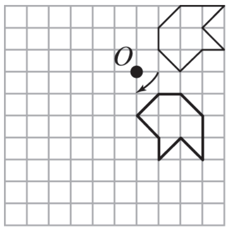
23. Such dilation does not exist because the image of a line is a parallel line (exercise 22) but the lines intersect at (0,0).

Chapter 13 Review

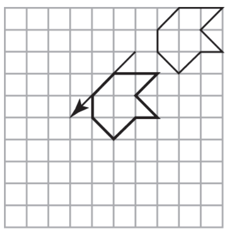
1. **(a)** Reflect across ℓ:



(b) Rotate $90°$ clockwise with O as a center:

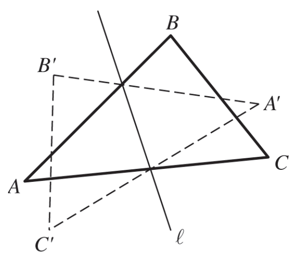


(c) Translate three units down and three units left:

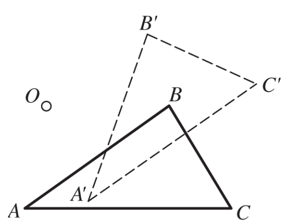


2. In each part find the images of the vertices:

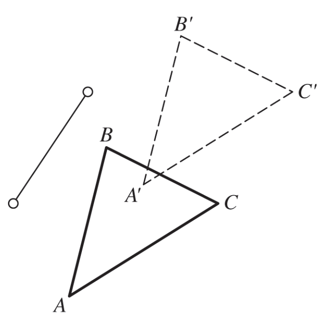
(a) Reflect about ℓ:



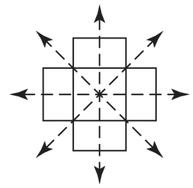
(b) Rotate counterclockwise in O:



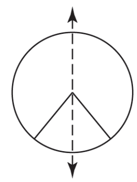
(c) Translate through the arrow pictured:



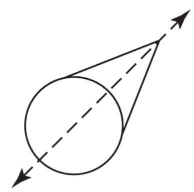
3. **(a)** **Four reflections**, two diagonals, one horizontal, and one vertical, as shown below. Rotations of $90°$, $180°$, and $270°$ also work.



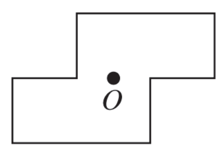
(b) **One**, the diameter bisecting the central angle.



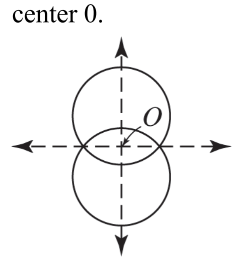
(c) **One**, the bisector of the point angle.



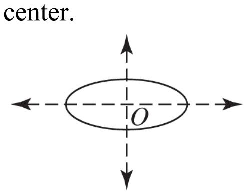
(d) **One:** a half turn through center 0.



(e) **Three**: A horizontal reflection through the points of intersection, a vertical reflection through the centers, and a half turn through center 0.



(f) **Three**: Vertically and horizontally reflections through the center and a half turn through the center.



4. $A = A'$, B is the midpoint of $\overline{A'B'}$, and C is the midpoint of $\overline{A'C'}$.

5. **(a)** A **half-turn** about x.
(b) A **half-turn** about x.

6. **(a)** **Rotation by 120°** clockwise about the center of the hexagon.
(b) A **reflection** in the perpendicular bisector of $\overline{BY}$ or a **rotation by 60°** counterclockwise around the center and then a reflection about the perpendicular bisector of $\overline{B'Y'}$.

7. A **reflection** in $\overleftrightarrow{SO}$.

8. Let $\Delta H'O'R'$ be the image of ΔHOR under a half-turn about R. Then ΔSER is the image of $\Delta H'O'R'$ under dilation with center R and scale factor $\frac{2}{3}$. Thus ΔSER is the image of ΔHOR under a half-turn about R followed by a $\frac{2}{3}$ size transformation.

9. Rotate ΔPIG $180°$ (i.e., a half-turn) about the mid-point of $\overline{PT}$, then perform a size transformation with scale factor 2 and center $P'(=T)$.

10. **(a)** In each case, $(x, y) \rightarrow (x' - 3, y' + 5)$.
$A'(0, 7.91) \rightarrow (0 - 3, 7.91 + 5) =$
$\boldsymbol{A(^-3, 12.91)}$.
$B'(^-5, ^-4.93) \rightarrow (^-5 - 3, ^-4.93 + 5) =$
$\boldsymbol{B(^-8, 0.07)}$.
$C'(4.83, 0) \rightarrow (4.83 - 3, 0 + 5) =$
$\boldsymbol{C(1.83, 5)}$.
(b) Under the translation $\boldsymbol{(x, y) \rightarrow (x - 3,\ y + 5)}$.

11. $\boldsymbol{(x, y) \rightarrow (x, y)}$. This is an "identity" translation; i.e., $(x - 3, y + 2)$ reverses $(x + 3, y - 2)$ so that every point is its own image.

12. **(a)** The translation from $\Delta A''B''C''$ to ΔABC would be under the translation $(x, y) \rightarrow (x - 1, y - 3)$ followed by the translation $(x, y) \rightarrow (x - 2, y + 1)$.
$\boldsymbol{A} = (0 - 1 - 2, 0 - 3 + 1) = \boldsymbol{(^-3, ^-2)}$.
$\boldsymbol{B} = (1 - 1 - 2, 5 - 3 + 1) = \boldsymbol{(^-2, 3)}$.
$\boldsymbol{C} = (^-2 - 1 - 2, 7 - 3 + 1) = \boldsymbol{(^-5, 5)}$.
(b) $(x, y) \rightarrow (x + 2 + 1, y - 1 + 3) =$ $\boldsymbol{(x + 3, y + 2)}$.

13. **(a)** **(*i*)** A translation from ***A* to *C***.
(*ii*) **Same** as (*i*). Translations with the same center are commutative.
(b) A rotation about O by $90° - 30° = \boldsymbol{60°}$ **counterclockwise**.
(c) **(*i*)** A dilation with center O and scale factor $3 \cdot 2 = \boldsymbol{6}$.
(*ii*) **Same** as (*i*). Dilations are commutative as long as the center does not change.

14. If $m_\ell = 2, m_\perp = \frac{^-1}{2}$. Then $3 = \frac{^-1}{2}(^-1) + b$ $\Rightarrow b = \frac{5}{2}$. The equation of the line is
$\boldsymbol{y = \frac{^-1}{2}x + \frac{5}{2}}$.

15. **(a)** If $(x, y) \rightarrow (x + 2, y - 3)$ then each point on the line is translated. Thus $(0, 3) \rightarrow (2, 0)$ and $(3, 0) \rightarrow (5, ^-3)$.
$m = \frac{^-3-0}{5-2} = {}^-1$ and $0 = {}^-1(2) + b \Rightarrow$ $b = 2$. **The equation of the image is** $\boldsymbol{y = {}^-x + 2}$.
(b) $P(x, y) \rightarrow P(x, ^-y)$ under reflection in the x-axis. Thus $P(0, 3) \rightarrow P'(0, ^-3)$ and $P(3, 0) \rightarrow P'(3, 0)$. The equation of the reflected line is $\boldsymbol{y = x - 3}$.
(c) $P(x, y) \rightarrow P'(^-x, y)$ under reflection in the y-axis. Thus $P(0, 3) \rightarrow P'(0, 3)$ and $P(3, 0) \rightarrow P'(^-3, 0)$. The equation of the reflected line is $\boldsymbol{y = x + 3}$.
(d) $P(x, y) \rightarrow P'(y, x)$ under reflection in the line $y = x$. Thus $P(0, 3) \rightarrow P'(3, 0)$ and $P(3, 0) \rightarrow P'(0, 3)$. The equation of the reflected line is $\boldsymbol{y = {}^-x + 3}$.

(e) $P(x, y) \to P'({}^{-}x, {}^{-}y)$ under a half-turn about the origin. Thus $P(0, 3) \to P'(0, {}^{-}3)$ and $P(3, 0) \to P'({}^{-}3, 0)$. The equation of the reflected line is $\mathbf{y = {}^{-}x - 3}$.

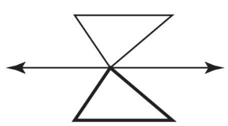
(f) $P(x, y) \to P(2x, 2y)$ under a scale factor of 2. Thus $P(0, 3) \to P'(0, 6)$ and $P(3, 0) \to P'(6, 0)$. The equation of the scaled line is $\mathbf{y = {}^{-}x + 6}$.

16. **(a)** The translation must be reversed: $(x, y) \to$ $\mathbf{(x - 3, y + 5)}$.

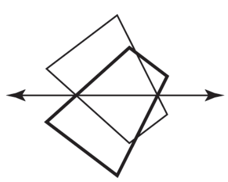
(b) This becomes the identity translation: $\mathbf{(x, y) \to (x, y)}$.

17. The measure of each exterior angle of a regular octagon is $\frac{360^\circ}{8} = 45^\circ$. Thus the measure of each interior angle is $180^\circ - 45^\circ = 135^\circ$, and $135^\circ \nmid 360^\circ$. Therefore a regular octagon does not tessellate the plane.

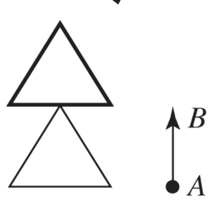
18. **(a)**



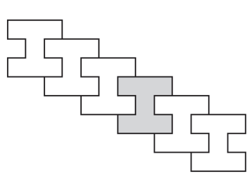
(b)



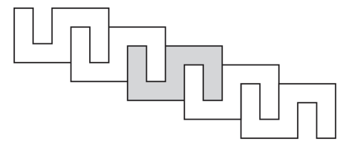
(c)



19. **(a)** **Yes**, it will tessellate. Below is a start.



(b) **Yes**, it will tessellate. Below is a sptart.



(c) **No**, it will not tessellate.

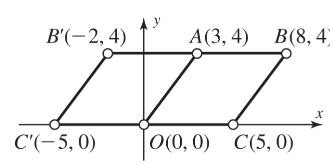
20. For parts (i) and (ii) assume OA forms one side of possible rhombi—not a diagonal For part (iii) assume OA forms the diagonal.

(a) **(*i*)** $OA = \sqrt{(3-0)^2 + (4-0)^2} = 5.$ Thus, possible coordinates for point C would be $(0 + 5, 0) = \mathbf{(5, 0)}$ or $(0 - 5, 0) = \mathbf{({}^{-}5, 0)}$.

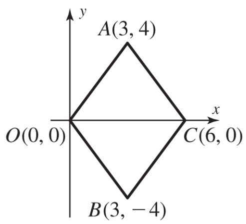
(*ii*) OA, or the length of each side, is 5. Thus $5 = \sqrt{(a-3)^2 + (4-0)^2} = \sqrt{a^2 - 6a + 9 + 16} = \sqrt{a^2 - 6a + 25}.$ Squaring each side of the equation (i.e., if $a = b$, then $a^2 = b^2$): $25 = a^2 - 6a + 25$ $\Rightarrow a^2 - 6a = 0 \Rightarrow a(a-6) = 0.$ So $a = 0$ or $a = 6$. If $a = 0$, then $C = O$, a contradiction. So possible coordinates for point C are $\mathbf{(6, 0)}$.

(iii) If OA is the diagonal, then the sides form by OC and CA must be of equal length. $\sqrt{(a-0)^2 + (0-0)^2} = \sqrt{(3-a)^2 + (4-0)^2} \Rightarrow$ $a^2 = (3-a)^2 + 4 \Rightarrow a = \frac{25}{6}$. Thus, a possible value for C would be $\left(\mathbf{\frac{25}{6}, 0}\right)$

(b) **(*i*)** The possible values for point B would be $(3 + 5, 4) = \mathbf{(8, 4)}$ or $(3 - 5, 4) = \mathbf{({}^{-}2, 4)}$.



(*ii*) If $O = (0,0)$, $A = (3,4)$, and $C = (6,0)$ then another possible value for point B is **$(3, {}^-4)$.** See the following figure:



(iii) Reflecting $\left(\frac{25}{6}, 0\right)$ across the diagonal OA will help us estimate the coordinates. Then note that the other vertex (x, y) is equal distance from O and A.
Thus,
$\sqrt{(x-0)^2+(y-0)^2} = \sqrt{(x-3)^2+(y-4)^2} \Rightarrow$
$x^2+y^2 = (x-3)^2+(y-4)^2 \Rightarrow 6x+8y = 25.$
Using the visual estimation and guess and check in the last equation, the solution $\left(\frac{{}^-7}{6}, 4\right)$ can be found.

(c) (i) Translating the point (3,4) five units to the right or left will give the coordinates (-2,4) and (8,4).

(ii) Reflecting the point (3,4) across the x-axis yields the point $(3, {}^-4)$.

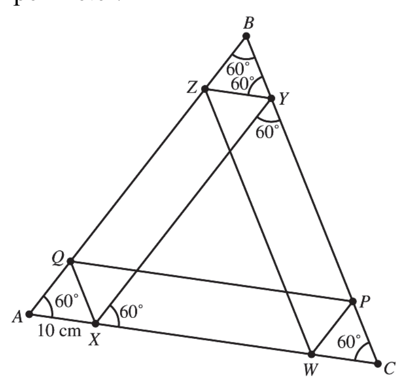
(iii) Reflecting the point $\left(\frac{25}{6}, 0\right)$ across the diagonal OA yields the point $\left(\frac{{}^-7}{6}, 4\right)$.

(d) (*i*) $m_{\overline{OB}} = \frac{4-0}{8-0} = \frac{1}{2}$. $m_{\overline{AC}} = \frac{4-0}{3-5} = {}^-2.$
$\boldsymbol{m_{\overline{OB}} \cdot m_{\overline{AC}} = {}^-1},$ so the diagonals are perpendicular. Similarly, the slopes of diagonals $B'O$ and $C'A$ are also perpendicular to each other.

(*ii*) One diagonal is the x-axis and the other is parallel to the y-axis $\Rightarrow$ the diagonals are perpendicular to each other.

21. Find the image O' of the center of the circle O under the dilation with center A and a scale factor $\frac{1}{2}$. Then construct the circle with center O' and radius $\frac{1}{2}$ of the radius of the circle.

22. As the path is traced, the lengths of $XY + YZ$ is the same as the length AC. We know this because of the 60° angles formed as the ball bounces at Y making $\angle BYZ$ have a measure of 60°. Thus $AXYZ$ is a parallelogram making $\overline{AX} \cong \overline{ZY}$ and $\overline{XY} \cong \overline{AZ}$. Thus $XY + YZ = AX + AZ$. Also ΔBYZ is equilateral so $\overline{BZ} \cong \overline{ZY}$. Thus $XY + YZ = AZ + ZB$. Hence $XY + YZ = 1\text{ m}$. Similarly $ZW + WP = BC = 1\text{ m}$, and $PQ + QX = AC = 1\text{ m}$. Thus the length of the path is 3(1 m) = 3 m. The perimeter of $\triangle ABC$ is also 3 m, so the ball travels the length equal to the perimeter.



CHAPTER 14

AREA, PYTHAGOREAN THEOREM, AND VOLUME

Assessment 14-1A: Linear Measure

1. **(a)** $AB = 1.0 \text{ cm} - 0 \text{ cm} = \mathbf{1.0\ cm}$.

   (b) $DE = 4.5 \text{ cm} - 3.5 \text{ cm} = \mathbf{1.0\ cm}$.

   (c) $CJ = 10.0 \text{ cm} - 2.0 \text{ cm} = \mathbf{8.0\ cm}$.

   (d) $EF = 5.0 \text{ cm} - 4.5 \text{ cm} = \mathbf{0.5\ cm}$.

   (e) $IJ = 10.0 \text{ cm} - 9.3 \text{ cm} = \mathbf{0.7\ cm}$.

   (f) $AF = 5.0 \text{ cm} - 0 \text{ cm} = \mathbf{5.0\ cm}$.

   (g) $IC = 9.3 \text{ cm} - 2.0 \text{ cm} = \mathbf{7.3\ cm}$.

   (h) $GB = 6.2 \text{ cm} - 1.0 \text{ cm} = \mathbf{5.2\ cm}$.

2. Answers will vary.

   (a) About **3 cubits**, assuming a 60-inch desk and a 20-inch measurement from elbow to fingertips.

   (b) About **2 pencil-lengths,** assuming a mechanical pencil.

   (c) About **25 pencil-widths,** assuming the book is closed.

3. **(a)** $100 \text{ inches} = \frac{100 \text{ inches}}{1} \times \frac{1 \text{ yard}}{36 \text{ inches}} = \frac{25}{9} = \mathbf{2\frac{7}{9}}$ yd.

   (b) $400 \text{ yards} = \frac{400 \text{ yards}}{1} \times \frac{36 \text{ inches}}{1 \text{ yard}} = \mathbf{14{,}400}$ in.

   (c) $300 \text{ feet} = \frac{300 \text{ feet}}{1} \times \frac{1 \text{ yard}}{3 \text{ feet}} = \mathbf{100}$ yd.

   (d) $372 \text{ inches} = \frac{372 \text{ inches}}{1} \times \frac{1 \text{ foot}}{12 \text{ inches}} = \mathbf{31}$ ft.

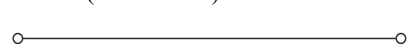
4. The following lines are one-half scale (e.g., a 10 cm line will measure 5 cm below):

   (a) 10 mm (or 1 cm):

   (b) 100 mm (or 10 cm):

   (c) 1 cm (or 10 mm):

   (d) 10 cm (or 100 mm):

5. Answers may vary depending on your estimate; e.g.,

   (a) About **81 mm**.

   (b) About **8.1 cm**.

6. **(a)** **Centimeters**. A new pencil measures about 19 cm.

   (b) **Millimeters** or **centimeters**. The diameter is about 21 mm or 2 cm.

   (c) **Centimeters** or **meters**. A desk is normally about 120 cm or 1.2 meters wide.

7. **(a)** **Inches**. 19 cm is about 7.5 inches.

   (b) **Inches**. 21 mm is about 0.8 or $\frac{13}{16}$ inches.

   (c) **Feet**. 1.2 m is about 3 feet 11 inches.

8. In each case, note that:

   From m to cm move the decimal point two places to the right;
   From cm to mm move the decimal point one place to the right;
   From mm to cm move the decimal point one place to the left; and
   From cm to m move the decimal point two places to the left.

Item	m	cm	mm
(a) Length of a piece of paper	**0.35**	35	**350**
(b) Height of a woman	1.63	**163**	**1630**
(c) Width of a filmstrip	**0.035**	**3.5**	35
(d) Length of a cigarette	**0.1**	**10**	100
(e) Length of two meter sticks laid end to end	2	**200**	**2000**

9. **(a)** Dime = \$ 0.10 = Decidollar

   (b) Penny = \$ 0.01 = Centidollar

   (c) \$10 = Dekadollar

   (d) \$100 = Hectodollar

   (e) \$1000 = Kilodollar

10. **(a)** **13.50 mm**. 10 mm is about 0.4 inch.
(b) **0.770 m**. 0.77 m is about 30 inches.
(c) **10.0 m**. 10 m is about 33 feet.
(d) **15.5 cm**. 15.5 cm is about 6 inches.

11. Convert each measure to cm to list in decreasing order. **6 m** $=$ 600 cm $>$ **5218 mm** $=$ 521.8 cm $>$ **245 cm** $>$ **700 mm** $=$ 70 cm $>$ **91 mm** $=$ 9.1 cm $>$ **8 cm**.

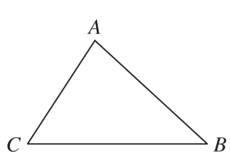
12. Answers may vary; e.g.: A circle with radius about $\frac{5}{8}$ inch. $\left(r = \frac{C}{2\pi} = \frac{2}{\pi} \approx \frac{5}{8}.\right)$

13. **(a)** Where $r = 1$ unit and a semicircle is one-half the circumference, $\frac{1}{2}C = \frac{1}{2}(2\pi r) = \boldsymbol{\pi}$ **units**.
(b) Where $r = \frac{1}{2}$ unit, $\frac{1}{2}C = \frac{1}{2}\left(2\pi \cdot \frac{1}{2}\right) =$ $\boldsymbol{\frac{\pi}{2}}$ **units**.

14. Listing lengths of sides starting clockwise from the top of each figure (measurements are approximate):
(a) About $1.8 + 1.8 + 1.8 + 1.8 =$ **7.2 cm**.
(b) About $3.6 + 0.8 + 1.8 + 1.0 + 1.8 + 1.9 =$ **10.9 cm**.

15. See Figure 13-2. In each case below, to move from a smaller unit to a larger move the decimal point to the left, and from a larger unit to a smaller move the decimal point to the right.
(a) 10 mm $=$ **<u>1 cm</u>**.
(b) 262 m $=$ **<u>0.262 km</u>**.
(c) 3 km $=$ **<u>3000 m</u>**.
(d) 30 mm $=$ **<u>0.03 m</u>**.

16. Consider the following triangle, in which $AB \approx$ 20 mm. $BC \approx 24$ mm, and $AC \approx 17$ mm. The sum of the lengths of any two sides of the triangle is greater than the length of the third side:

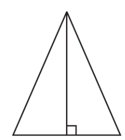


(a) $\boldsymbol{AB + BC} \approx 44 \text{ mm} > \boldsymbol{AC} \approx 17$ mm.
(b) $\boldsymbol{BC + AC} \approx 41 \text{ mm} > \boldsymbol{AB} \approx 20$ mm.
(c) $\boldsymbol{AB + AC} \approx 37 \text{ mm} > \boldsymbol{BC} \approx 24$ mm.

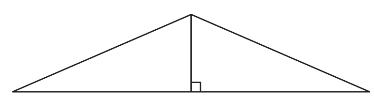
17. Use the triangle inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
(a) **Yes**. $23 + 50 > 60$.
(b) **No**. $10 + 40 \not> 50$.

18. **(a)** 31 + 85 = 116, so the third side must be less than or equal to 115 cm.
(b) 85 – 31 = 54, so the third side must be greater than or equal to 55 cm.

19. The hypotenuses of the resultant right triangles are $\sqrt{4.25^2 + 11^2} \approx$ **11.8 inches**.
(a) Form an isosceles triangle with the two congruent sides formed by the diagonals and the base formed by the two short $\left(4\frac{1}{4}\text{ inch}\right)$ sides. The perimeter is approximately **32.1 inches**.



(b) Form the base with the two long (11 inch) sides. The perimeter is about **45.6 inches**.



20. Circumference $(C) = 2\pi \cdot$ radius $(r) \Rightarrow r = \frac{C}{2\pi}$.
(a) $r = \frac{12\pi}{2\pi} =$ **6 cm**.
(b) $r = \frac{6}{2\pi} = \boldsymbol{\frac{3}{\pi}}$ **m**.

21. Circumference $(C) = \pi \cdot$ diameter (d) or $C = 2\pi \cdot$ radius (r).
(a) $C = \pi \cdot 6 =$ $\boldsymbol{6\pi}$ **cm**.
(b) $C = 2\pi \cdot \frac{2}{\pi} =$ **4 cm**.

22. The relationship between the two measures is linear; i.e., when the radius increases, the circumference will increase by the same factor. Thus the circumference will **double**.

23. **(a)** Mach $2.5 \times \frac{0.344 \text{ km}}{\text{sec}} \times \frac{3600 \text{ sec}}{\text{hr}} =$ **3096 km/hr**.

(b) Mach $3 \times \frac{344 \text{ m}}{\text{sec}} =$ **1032 m/sec**.

(c) $\text{M} = \frac{\text{speed of aircraft}}{\text{speed of sound}} = \frac{\frac{5000 \text{ km}}{\text{hr}}}{\frac{0.344 \text{ km}}{\text{sec}} \cdot \frac{3600 \text{ sec}}{\text{hr}}}$

$\approx$ **Mach 4.04**.

24. Greatest possible error is one-half the whole-number measurement unit used.

(a) 23 m implies GPE of **0.5 m**.

(b) 3.6 cm = 36 mm implies GPE of 0.5 mm = **0.05 cm**.

(c) 3.12 m = 312 cm implies GPE of 0.5 cm = **0.005 m**.

25. **(a)** Length of an arc = $2\pi r \cdot \frac{\textit{central angle}}{360^\circ}$, so if the arc is 36° and the radius is 6 cm, then the length of the arc is

$$\left(2\pi\left(6 \text{ cm}\right)\right) \cdot \frac{36^\circ}{360^\circ} = 12\pi \cdot \frac{1}{10} \text{ cm} = \frac{6\pi}{5} \text{ cm}.$$

(b) The arc has to be longer because 8 cm is greater than 6 cm, and the radius is in the numerator of the fraction in the computation.

(c) The arc length is

$$\left(2\pi\left(8 \text{ cm}\right)\right) \cdot \frac{36^\circ}{360^\circ} = 16\pi \cdot \frac{1}{10} \text{cm} = \frac{8\pi}{5} \text{cm}$$

26. If the arc length is 14π m then the radius of the circle containing an 80° arc is

$$2\pi r \text{ m} \cdot \frac{80^\circ}{360^\circ} = 14\pi$$

$$r = \frac{360^\circ}{80^\circ} \cdot \frac{14\pi \text{ m}}{2\pi} \Rightarrow r = \frac{9}{2} \cdot 7 \text{ m} = \frac{63}{2} \text{ m}$$

27. Use 2, 3, and 4 as the radii and find the arc lengths:

$$2\pi(2) \cdot \frac{60^\circ}{360^\circ} : 2\pi(3) \cdot \frac{60^\circ}{360^\circ} : 2\pi(4) \cdot \frac{60^\circ}{360^\circ}$$

$$4\pi \cdot \frac{1}{6} : 6\pi \cdot \frac{1}{6} : 8\pi \cdot \frac{1}{6}$$

$$\frac{4}{6}\pi : \frac{6}{6}\pi : \frac{8}{6}\pi \Rightarrow \frac{2}{3}\pi : \frac{3}{3}\pi : \frac{4}{3}\pi$$

The ratio of the arc lengths is 2:3:4.

Assessment 14-1B

1. **(a)** AB = 1 in. − 0 in. = **1 in**.

(b) $DE = 3\frac{1}{4}$ in. − $2\frac{1}{2}$ in. = $\mathbf{\frac{3}{4}}$ **in**.

(c) CJ = 6 in. − 2 in. = **4 in**.

(d) EF = 4 in. − $3\frac{1}{4}$ in. = $\mathbf{\frac{3}{4}}$ **in**.

(e) IJ = 6 in. − $5\frac{1}{2}$ in. = $\mathbf{\frac{1}{2}}$ **in**.

(f) AF = 4 in. − 0 in. = **4 in**.

(g) $IC = 5\frac{1}{2}$ in. − 2 in. = $\mathbf{3\frac{1}{2}}$ **in**.

(h) $GB = 4\frac{5}{8}$ in. − 1 in. = $\mathbf{3\frac{5}{8}}$ **in**.

2. Answers will vary.

(a) About **six** book lengths, assuming Billstein's 12th edition and a 60-inch desktop.

(b) About **four** dollar-widths, assuming a ten-inch page. A paper dollar is about $2\frac{1}{2}$ inches wide.

(c) About $\mathbf{1\frac{1}{3}}$ dollar-lengths, assuming an eight-inch wide page. A paper dollar is about 6 inches long.

3. **(a)** 100 inches $= \frac{100 \text{ inches}}{1} \times \frac{1 \text{ foot}}{12 \text{ inches}} = \frac{50}{6} =$ $\mathbf{8\frac{1}{3}}$ **ft**.

(b) 400 yards $= \frac{400 \text{ yards}}{1} \times \frac{3 \text{ feet}}{1 \text{ yard}} =$ **1200 ft**.

(c) 300 feet $= \frac{300 \text{ feet}}{1} \times \frac{12 \text{ inches}}{1 \text{ foot}} =$ **3600 in**.

(d) 372 inches $= \frac{372 \text{ inches}}{1} \times \frac{1 \text{ yard}}{36 \text{ inches}} =$ $\mathbf{10\frac{1}{3}}$ **yd**.

4. The following lines are one-half scale (e.g., a 10 cm line will measure 5 cm below):

(a) 0.01 m (or 1 cm, or 10 mm):

(b) 15 cm (or 150 mm, or 0.15 m):

(c) 35 mm (or 3.5 cm):

(d) 150 mm (or 15 cm):

5. Answers may vary depending on your estimate; e.g.,

(a) about **40 mm**.

(b) about **4 cm**.

6. **(a)** **Millimeters** or **centimeters**. A desktop is normally about 20 mm or 2 cm thick.

(b) **Centimeters**. A page in this book is about 25 cm long.

(c) **Centimeters** or **meters**. A normal door is about 190 cm or 1.9 m high.

7. **(a)** **Inches**. 20 mm is about 0.79 or $\frac{25}{32}$ inches.

(b) **Inches**. 25 cm is about $9\frac{13}{16}$ inches.

(c) **Feet**. 1.9 m is about 6 feet 3 inches.

8. In each case, note that:

From m to cm move the decimal point two places to the right;
From cm to mm move the decimal point one place to the right;
From mm to cm move the decimal point one place to the left; and
From cm to m move the decimal point two places to the left.

Item	m	cm	mm
(a) Width of a piece of paper	**0.20**	20	**200**
(b) Height of a woman	1.52	**152**	**1520**
(c) Length of a pencil	**0.09**	**9**	90
(d) Length of a baseball bat	1.1	**110**	**1100**

9. **(a)** **Decicarton = 1**

(b) **Dekacarton = 100**

(c) **Hectocarton = 1000**

(d) **Kilocarton = 10,000**

10. **(a)** **195.0 cm**. 195 cm is about 6 feet 5 inches.

(b) **8.100 cm**. 8.1 cm is about 3 inches.

(c) **40.0 km/hr**. 40 km/hr is about 25 mi/hr.

11. Convert each measure to cm to list in decreasing order:

91 m $=$ 9100 cm $>$ **5218 cm** $>$ **8 m**
$=$ 800 cm $>$
6 m $=$ 600 cm $>$ **245 cm** $>$ **925 mm**
$=$ 92.5 cm

12. Answers may vary; e.g.:

(a) A triangle with 1, $1\frac{1}{2}$, and $1\frac{1}{2}$ inch sides.

(b) A concave kite with 3, 3, 1, and 1 cm sides.

13. The unshaded half circle on the left can be flipped and translated to cover the shaded half circle on the right. Together they form a whole circle whose circumference is equal to the length of the curve that separates the two regions. Thus, the total arc length is $2\pi r = 2\pi\left(\frac{2\text{ cm}}{2}\right) = 2\pi$ cm.

14. Listing lengths of sides starting clockwise from the top of each figure (measurements are approximate):

(a) About $3 + 3 + 3 =$ **9 cm**.

(b) About $4 + 1 + 3 + 1 + 3 + 1 + 4 + 3 =$ **20 cm**.

15. In each case below, to move from a smaller unit to a larger move the decimal point to the left, and from a larger unit to a smaller move the decimal point to the right.

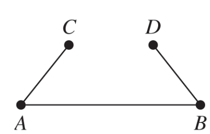
(a) 35 m $=$ **3500 cm**.

(b) 359 mm $=$ **0.359 m**.

(c) 647 mm $=$ **64.7 cm**.

(d) 0.1 cm $=$ **1 mm**.

16. A casual explanation is that the shortest distance between two points is a straight line. But this response assumes what is to be argued or explained. For a better explanation places four points *A*, *B*, *C*, and *D* so that *C* and *D* are not collinear with *A* and *B* and that *AC* plus *BD* is less than *AB*. Assume the segments are hinged at *A* and *B*. Push *AC* and *BD* down toward *AB* and swing the segments to note that *C* and *D* never meet. Thus, a triangle is never formed nor is a line segment.



17. Use the triangle inequality: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

(a) **Yes**. $20 + 40 > 50$.

(b) **No**. $20 + 40 \not> 60$.

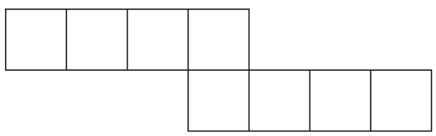
(c) Change 250 mm to 25 cm. Then **no**; $12 + 25 \not> 41$.

18. **(a)** 21 + 75 = 96, so the third side must be less than or equal to 95.

(b) 75 – 21 = 54, so the third side must be greater than or equal to 55 cm.

19. **(a)** Answers may vary. One way would be to form a seven-square by two-square rectangle; thus $7 + 2 + 7 + 2 = 18$; another would be to add four squares to the end of the top row.

(b) **Eight**. The minimum number of squares for a fixed perimeter occurs when the squares have the maximum number of sides exposed, such as in the figure below.



(c) **Twenty**. The maximum number of squares for a fixed perimeter occurs when the squares have the minimum number of sides exposed. I.e., from a four-squares by five-square rectangle, where the perimeter is $5 + 4 + 5 + 4 = 18$.

20. Circumference $(C) = 2\pi \cdot$ radius $(r) \Rightarrow r = \frac{C}{2\pi}$.

(a) $r = \frac{0.67}{2\pi} = \mathbf{\frac{0.335}{\pi}}$ **m**.

(b) $r = \frac{92\pi}{2\pi} = \mathbf{46}$ **cm**.

21. Circumference $(C) = \pi \cdot$ diameter (d) or $C = 2\pi \cdot$ radius (r).

(a) $C = 2\pi \cdot 3 = \mathbf{6\pi}$ **cm**.

(b) $C = \pi \cdot 6\pi = \mathbf{6\pi^2}$ **cm**.

22. $C = 2\pi r$. If we double the circumference $2C = 4\pi r$. The new circle has radius r_2. Thus, $2\pi r_2 = 2C = 4\pi r \Rightarrow 2\pi r_2 = 4\pi r \Rightarrow r_2 = 2r$. The radius is also **doubled**.

23. **(a)** $\frac{300{,}000 \text{ km}}{\text{sec}} \times \frac{60 \text{ sec}}{\text{min}} \times \frac{60 \text{ min}}{\text{hr}} \times \frac{24 \text{ hr}}{\text{day}} \times \frac{365 \text{ days}}{\text{yr}} \approx \mathbf{9.5 \cdot 10^{12}}$ **km**.

(b) 4.34 light years $\times\ 9.5 \cdot 10^{12}$ km/yr $\approx$ $\mathbf{4.1 \cdot 10^{13}}$ **km**.

(c) $\frac{4.1 \cdot 10^{13} \text{ km}}{6 \cdot 10^{4} \text{ km/hr}} \approx 6.8 \cdot 10^{8}$ hours, or **about 78,000 years** at 8760 hours per year.

(d) Light travels $(8 \cdot 60 + 19)$ sec $\times$ 300,000 km/sec $= 1.497 \cdot 10^8$ km in 8 minutes 19 sec. $\frac{1.497 \cdot 10^8 \text{ km}}{6 \cdot 10^4 \text{ km/hr}} \approx 2495$ hours, or **about 104 days**.

24. Greatest possible error is one-half the whole-number measurement unit used.

(a) The actual measurement would be between 135.5 m and 136.5 m, so the GPE would be **0.5 m**.

(b) The actual measurement would be between 3.45 ft and 3.55 ft, so the GPE would be **0.05 ft**.

(c) The actual measurement would be between 3.615 cm and 3.625 cm, so the GPE would be **0.005 cm**.

25. **(a)** Length of an arc $= 2\pi r \cdot \dfrac{\textit{central angle}}{360^\circ}$, so if the arc is 48° and the radius is 6 cm, then the length of the arc is

$$\left(2\pi\left(6 \text{ cm}\right)\right) \cdot \frac{48^\circ}{360^\circ} = 12\pi \cdot \frac{2}{15} \text{ cm} = \frac{8\pi}{5} \text{ cm}.$$

(b) The arc has to be longer because 8 cm is greater than 6 cm, and the radius is in the numerator of the fraction in the computation.

(c) The arc length is

$$\left(2\pi\left(8 \text{ cm}\right)\right) \cdot \frac{48^\circ}{360^\circ} = 16\pi \cdot \frac{2}{15} \text{ cm} = \frac{32\pi}{15} \text{ cm}$$

26. If the arc length is 12π m then the radius of the circle containing an 60° arc is

$$2\pi r \cdot \frac{60^\circ}{360^\circ} = 12\pi \text{ m}$$

$$r = \frac{360^\circ}{60^\circ} \cdot \frac{12\pi \text{ m}}{2\pi} \Rightarrow r = 6 \cdot 6 \text{ m} = 36 \text{ m}$$

27. Use 3, 4, and 5 as the radii and find the arc lengths:

$$2\pi(3) \cdot \frac{60^\circ}{360^\circ} : 2\pi(4) \cdot \frac{60^\circ}{360^\circ} : 2\pi(5) \cdot \frac{60^\circ}{360^\circ}$$

$$6\pi \cdot \frac{1}{6} : 8\pi \cdot \frac{1}{6} : 10\pi \cdot \frac{1}{6}$$

$$\frac{6}{6}\pi : \frac{8}{6}\pi : \frac{10}{6}\pi \Rightarrow \frac{3}{3}\pi : \frac{4}{3}\pi : \frac{5}{3}\pi$$

The ratio of the arc lengths is 3:4:5.

Assessment 14-2A: Areas of Polygons and Circles

1. Answers may vary. E.g., for a 30-inch by 60-inch desktop, and if the $8\frac{1}{2}$ by 11-inch notebook papers were to be oriented in portrait fashion, the desktop would be about 7 papers by $2\frac{3}{4}$ papers $= 19\frac{1}{4}$ square notebook papers.

2. **(a)** (*i*) $\mathbf{cm^2}$; about 21.6 cm by 28 cm.

   (*ii*) $\mathbf{in.^2}$; $8\frac{1}{2}$ in. by 11 in.

   (b) (*i*) $\mathbf{mm^2}$ **or** $\mathbf{cm^2}$; diameter about 2.3 cm or 23 mm.

   (*ii*) $\mathbf{in.^2}$; diameter about $\frac{15}{16}$ in.

   (c) (*i*) $\mathbf{cm^2}$ **or** $\mathbf{m^2}$; about $61\text{cm} = 0.61\text{m}$ by $152 \text{ cm} = 1.52 \text{ m}$.

   (*ii*) $\mathbf{in.^2}$ **or** $\mathbf{yd^2}$; about 24 in. $= \frac{2}{3}$ yd by 60 in. $= 1\frac{2}{3}$ yd.

   (d) (*i*) $\mathbf{m^2}$; assume about 4.5 m by 9 m.

   (*ii*) $\mathbf{yd^2}$; assume about 5 yd by 10 yd.

3. Answers may vary. Some possible approximate measures are:

   (a) A 30 in. by 78 in. door (approximately 0.8 m by 2.0 m), **about 1.6** $\mathbf{m^2}$.

   (b) A 30 in. by 60 in. desktop (approximately 0.76 m by 1.5 m), **about 1.1** $\mathbf{m^2}$.

4. In each case, note that:

   From m^2 to cm^2, move the decimal point four places to the right;

   From cm^2 to mm^2, move the decimal point two places to the right;

   From mm^2 to cm^2, move the decimal point two places to the left; and

   From cm^2 to m^2, move the decimal point four places to the left;

Item	m^2	cm^2	mm^2
(a) Area of sheet of paper	**0.0588**	588	**58,800**
(b) Area of a cross section of a crayon	**0.000192**	**1.92**	192
(c) Area of a desktop	1.5	**15,000**	**1,500,000**
(d) Area of a dollar bill	**0.01**	100	**10,000**
(e) Area of a postage stamp	**0.0005**	5	**500**

5. **(a)** $\frac{4000 \text{ ft}^2}{1} \cdot \frac{1 \text{ yd}^2}{9 \text{ ft}^2} = \mathbf{444\frac{4}{9}\ yd^2}$.

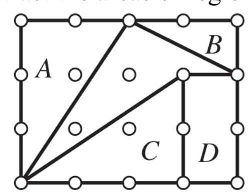
   (b) $\frac{10^6 \text{ yd}^2}{1} \cdot \frac{1 \text{ mi}^2}{3.0976 \cdot 10^6 \text{ yd}^2} \approx \mathbf{0.32\ mi^2}$.

   (c) $\frac{10 \text{ mi}^2}{1} \cdot \frac{640 \text{ acre}}{1 \text{ mi}^2} = \mathbf{6400\ acres}$.

   (d) $\frac{3 \text{ acre}}{1} \cdot \frac{4840 \text{ yd}^2}{1 \text{ acre}} \cdot \frac{9 \text{ ft}^2}{1 \text{ yd}^2} = \mathbf{130{,}680\ ft^2}$.

6. **(a)** Triangle with base 3 and height 2: $A = \frac{1}{2} \cdot 3 \cdot 2 = \mathbf{3\ units^2}$.

   (b) Construct a rectangle around the figure and subtract the areas of region A, B, C, and D:



   Total area $= 4 \cdot 3 = 12 \text{ units}^2$.

   Area of $A = \frac{1}{2} \cdot 3 \cdot 2 = 3 \text{ units}^2$;

   Area of $B = \frac{1}{2} \cdot 2 \cdot 1 = 1 \text{ unit}^2$;

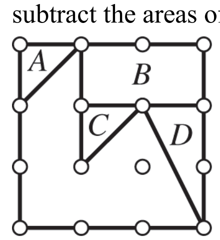
   Area of $C = \frac{1}{2} \cdot 3 \cdot 2 = 3 \text{ units}^2$; and

   Area of $D = 2 \cdot 1 = 2 \text{ units}^2$.

   Area of figure $= 12 - (3 + 1 + 3 + 2) = \mathbf{3\ units^2}$.

   (c) Triangle with base 2 and height 2: $A = \frac{1}{2} \cdot 2 \cdot 2 = \mathbf{2\ units^2}$.

   (d) Construct a rectangle around the figure and subtract the areas of regions A, B, C, and D:



   Total area $= 3 \cdot 3 = 9 \text{ units}^2$.

   Area of $A = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} \text{ unit}^2$;

   Area of $B = 2 \cdot 1 = 2 \text{ units}^2$;

Area of $C = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ unit^2; and
Area of $D = \frac{1}{2}\cdot 2\cdot 1 = 1$ unit^2.

Area of figure $= 9 - \left(\frac{1}{2} + 2 + \frac{1}{2} + 1\right) =$ $\mathbf{5\ units^2}$.

7. **(a)** $49\text{ m}\cdot 100\text{ m} = \mathbf{4900\ m^2}$.

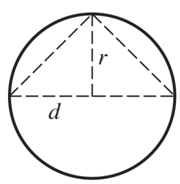
(b) $\frac{2\text{ fields}}{1}\cdot\frac{4900\text{ m}^2}{1\text{ field}}\cdot\frac{1\text{ a}}{100\text{ m}^2} = \mathbf{98\ a}$.

(c) $\frac{98\text{ a}}{2\text{ fields}}\cdot\frac{1\text{ ha}}{100\text{ a}} = \mathbf{0.98\ ha}$.

8. **(a)** $A = \frac{1}{2}\cdot 10\cdot 4 = \mathbf{20\ cm^2}$.

(b) $A = \frac{1}{2}\cdot 5\cdot 3 = \mathbf{7\frac{1}{2}\ m^2}$.

9. The greatest area of the triangle would occur if drawn as below:



In this case, $A = \frac{1}{2}(2r)(r) = \boldsymbol{r^2}$.

10. **(a)** **Yes**. All squares are similar.

(b) Measurement of area is in square units, so the ratio of the areas is $\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, or $\boldsymbol{a^2 : b^2}$.

11. **(a)** $A = 3\cdot 3 = \mathbf{9\,cm^2}$.

(b) $A = 8\cdot 12 = \mathbf{96\,cm^2}$.

(c) $A = 5\cdot 4 = \mathbf{20\ cm^2}$.

(d) $A = \frac{1}{2}\cdot 7\cdot(14+10) = \mathbf{84\ \ cm^2}$.

12. **(a)** **(*i*)** $A = 1.3\text{ km}\times 1.5\text{ km} = \mathbf{1.95\,km^2}$.

(*ii*) $A = \frac{1.95\text{ km}^2}{1}\times\frac{10^6\text{m}^2}{1\text{ km}^2}\times\frac{1\text{ ha}}{10^4\text{m}^2} =$ **195 ha**.

(b) **(*i*)** $A = 1300\text{ yd}\times 1500\text{ yd} = 1.95\cdot 10^6\text{ yd}^2$.
$\frac{1.95\cdot 10^6\text{ yd}^2}{1}\times\frac{1\text{ mi}^2}{3.0976\cdot 10^6\text{ yd}^2}\approx \mathbf{0.63\,mi^2}$.

(*ii*) $\frac{0.63\text{ mi}^2}{1}\times\frac{640\text{ acres}}{1\text{ mi}^2}\approx$ **403 acres**.

(c) Answers may vary. The metric system is easier because it is necessary only to move the decimal point to convert units.

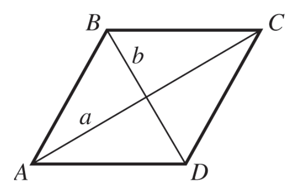
13. **(a)** **True**. It is not possible to determine a height when only side lengths are known.

(b) The area could be $60\,\text{cm}^2$ if the parallelogram is a rectangle; however the assertion is **false**.

(c) **False**. The area cannot be greater than $60\,\text{cm}^2$ since the maximum is $60\,\text{cm}^2$ if the parallelogram is a rectangle.

(d) **False**. The area could equal $60\,\text{cm}^2$.

14. A rhombus is a parallelogram with all sides congruent, thus the diagonals are perpendicular. The height of ΔABC in the rhombus below is then $\frac{b}{2}$. Area of $\Delta ABC = \frac{1}{2}\cdot a\cdot\frac{b}{2} = \frac{ab}{4}$; since there are two such triangles the area of the rhombus is $2\cdot\frac{ab}{4} = \frac{ab}{2} = \frac{12\text{cm}\cdot 5\text{cm}}{2} = \mathbf{30\,cm^2}$.



15. **(a)** $6.5\text{ m}\times 4.5\text{ m} = 29.25\text{ m}^2$. $29.25\text{ m}^2\times$ \$13.85 per $\text{m}^2 =$ **\$405.11**.

(b) $15\text{ ft}\times 11\text{ ft} = 165\text{ ft}^2$. $\left(\frac{165\text{ ft}^2}{1}\right)\cdot\left(\frac{1\text{ yd}^2}{9\text{ ft}^2}\right) = \frac{165}{9}\text{ yd}^2$. $18\frac{1}{3}\text{ yd}^2\times\$30$ per $\text{yd}^2 =$ **\$550**.

16. **(a)** $A = \pi\cdot 5^2 = \mathbf{25\pi\ cm^2}$.

(b) $A = \frac{60°}{360°}\cdot\pi\cdot 4^2 = \mathbf{\frac{8}{3}\pi\ cm^2}$.

17. Bathroom area $= 300\text{ cm}\times 400\text{ cm} = 120{,}000\text{ cm}^2$. Each tile is $10\text{ cm}\times 10\text{ cm} = 100\text{ cm}^2$, thus $\frac{120{,}000\text{ cm}^2}{100\text{ cm}^2} =$ **1200 tiles** (assuming no waste).

18. The area of a regular polygon is $\frac{1}{2}ap$, where a is the apothem (height of one of the triangles of a regular polygon) and p is the perimeter.

(a) $a = 2\sqrt{3}$; $p = 6 \cdot 4 = 24$.
$A = \frac{1}{2} \cdot 2\sqrt{3} \cdot 24 = \mathbf{24\sqrt{3}\ cm^2}$.

(b) The area of a regular triangle is $\frac{1}{2}$ as, where s is the length of a side. $a = 3\sqrt{3}$; $s = 6$, so
$A = \frac{1}{2} \cdot 3\sqrt{3} \cdot 6 = \mathbf{9\sqrt{3}\ cm^2}$.

19. **(a)** $C = 2\pi r \Rightarrow r = \frac{C}{2\pi} = \frac{8\pi}{2\pi} = 4$.
$A = \pi r^2 = \pi(4)^2 = \mathbf{16\pi\ cm^2}$.

(b) $A_{circle} = A_{square} \Rightarrow \pi r^2 = s^2 \Rightarrow r^2 = \frac{s^2}{\pi}$.
$\boldsymbol{r = \frac{s}{\sqrt{\pi}}}$.

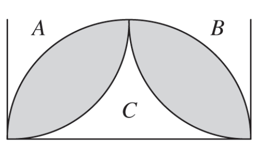
20. **(a)** Radius of large circle $= 2$ cm; $A_{large\ circle} = \pi(2)^2 = 4\pi\ \text{cm}^2$. $A_{each\ small\ circle} = \pi(1)^2 = \pi\ \text{cm}^2$. $A_{shaded} = 4\pi - 2 \cdot \pi = \mathbf{2\pi\ cm^2}$.

(b) $A_{semicircle} = \frac{1}{2}(\pi \cdot 1^2) = \frac{1}{2}\pi\ \text{cm}^2$.
$A_{triangle} = \frac{1}{2} \cdot 1 \cdot 2 = 1\ \text{cm}^2$.
$A_{shaded} = \left(\frac{1}{2}\pi + 1\right)\mathbf{cm^2}$.

(c) If the 1 cm radius were to be extended it would be the diameter of the large circle, cutting off a small shaded semicircle the same size as the small white semicircle. The shaded area is thus equal to half the large circle with radius 2 cm.

$A_{shaded} = \frac{1}{2}(\pi \cdot 2^2) = \mathbf{2\pi\ cm^2}$.

(d) Consider half the figure, as shown below. Areas $A + B =$ area C:



10 cm square

Areas $A + B =$ rectangle $-$ semicircle $=$
$5 \cdot 10 - \frac{1}{2}(\pi \cdot 5^2) = 50 - \frac{25}{2}\pi$.
$A + B + C$ is twice this, or $100 - 25\pi$. Considering both halves of the figure, total unshaded area is
$2(100 - 25\pi) = (200 - 50\pi)\ \text{cm}^2$.
The shaded area is that of the square less the unshaded area. $A_{shaded} = 10^2 - (200 - 50\pi) = \mathbf{(50\pi - 100)\ cm^2}$.

21. The flower bed with its encircling sidewalk forms a circle with radius $(3 + 1) = 4$ m.

$A_{encircled\ bed} = \pi \cdot 4^2 = 16\pi\ \text{m}^2$.

$A_{flower\ bed} = \pi \cdot 3^2 = 9\pi\ \text{m}^2$.

$A_{side\ walk} = A_{encircled\ bed} - A_{flower\ bed}$
$= 16\pi - 9\pi = \mathbf{7\pi\ m^2}$.

22. **(a)** $A_{square} = 144\ \text{cm}^2$. Length per side $= \sqrt{144} = 12$ cm. Perimeter $= 4 \cdot 12 = \mathbf{48\ cm}$.

(b) $P_{square} = 32$ cm. Length per side $= \frac{32}{4} = 8$ cm. Area $= 8^2 = \mathbf{64\ cm^2}$.

23. **(a)** Area is **quadrupled**. Sides of length s mean area $= s^2$. Sides of length $2s$ mean area $= (2s)^2 = 4s^2$, which is quadruple the original area.

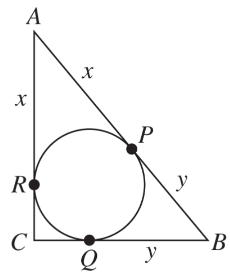
(b) **1 : 25**. Sides of first square $= 1s$, so area $= (1s)^2 = s^2$. Sides of second square $= 5s$, so area $= (5s)^2 = 25s^2$—or a ratio of $1 : 25$.

24. Draw diameters connecting points of tangency. The shaded area is the area of a 20 m by 16 m rectangle minus that of two half-circles, each with radius 8.

$A_{shaded} = 20 \cdot 16 - 2 \cdot \frac{1}{2} \cdot \pi(8)^2 =$
$\mathbf{(320 - 64\pi)\ m^2}$.

25. The radius of the total area is $148\pi = \pi r^2 \Rightarrow r = \sqrt{148} = 2\sqrt{37}$. If the radius of the sidewalk is 0.5 m then the radius of the flower bed is $2\sqrt{37} - 0.5 = \left(2\sqrt{37} - \frac{1}{2}\right)$ m.

26. Label the triangle as shown:



If two tangents are drawn to a circle from the same point on the exterior of the circle, the distances from the common point to the points of tangency are equal. Thus $RC = 4 - x$ and $QC = 3 - y$. Since $QC = RC$, then $4 - x = 3 - y$, or $x - y = 1$. $AB = 5 \Rightarrow x + y = 5$.

Solving this system of equations:

$$\begin{aligned} x - y &= 1 \\ \underline{x + y} &\underline{= 5} \\ 2x \quad &= 6 \\ x &= 3 \end{aligned}$$

Since $x = 3$, then RC, and the radius of the inscribed circle, $= 4 - 3 = 1$. The radius of the circular glass is **1 ft**.

27. **(a)** $s = \frac{3+4+5}{2} = 6$.

$A = \sqrt{6(6-3)(6-4)(6-5)} = \mathbf{6\ cm^2}$.

(b) $s = \frac{5+12+13}{2} = 15$.

$A = \sqrt{15(15-5)(15-12)(15-13)} =$ $\mathbf{30\ cm^2}$.

28. The area of the cross section is $\pi r^2 = \pi(6378)^2 = 40{,}678{,}884\pi \text{ km}^2$ or $127{,}796{,}483.1 \text{ km}^2 \approx 128{,}000{,}000 \text{ km}^2$.

29. **(a)** The area of the sector is

$$\pi r^2 \cdot \frac{\textit{central angle}}{360^\circ} = \pi\left(1^2\right)\cdot\frac{1^\circ}{360^\circ} = \frac{\pi}{360^\circ}\text{ m}^2$$

(b) The area of the circle is $\pi\left(1^2\right) = \pi \text{ m}^2$

(c) 360 sectors $= 360 \cdot \frac{\pi}{360^\circ}\text{ m}^2 = \pi \text{ m}^2$

(d) $\pi \text{ m}^2 - \pi \text{ m}^2 = 0$

30. The shaded portion is $\frac{1}{4}$ of the difference of the outer circle and the inner circle =

$$\frac{1}{4}\left(\pi\left(4^2\right) - \pi\left(2^2\right)\right) = \frac{1}{4}(16\pi - 4\pi) = \frac{1}{4}12\pi = 3\pi\text{ cm}^2.$$

31. **(a)** If each of the nine squares is divided into 4 equal parts then 24 out of 36 parts are shaded. $\frac{24}{36} = \frac{2}{3}$ of the square is shaded. If the area of the square is $6^2 = 36\text{ cm}^2$ then the area of the shaded quilt pattern is $\frac{2}{3}\cdot 36 = 24\text{ cm}^2$.

(b) If each of the nine squares is divided into 4 equal parts then 28 out of 36 parts are shaded. $\frac{28}{36} = \frac{7}{9}$ of the square is shaded. If the area of the square is $6^2 = 36\text{ cm}^2$ then the area of the shaded quilt pattern is $\frac{7}{9}\cdot 36 = 28\text{cm}^2$.

(c) If each of the nine squares is divided into 2 equal parts then 8 out of 18 parts are shaded. $\frac{8}{18} = \frac{4}{9}$ of the square is shaded. If the area of the square is $6^2 = 36\text{ cm}^2$ then the area of the shaded quilt pattern is $\frac{4}{9}\cdot 36 = 16\text{cm}^2$.

(d) If each of the nine squares is divided into 2 equal parts then 6 out of 18 parts are shaded. $\frac{6}{18} = \frac{1}{3}$ of the square is shaded. If the area of the square is $6^2 = 36\text{ cm}^2$ then the area of the shaded quilt pattern is $\frac{1}{3}\cdot 36 = 12\text{ cm}^2$.

32. Half of the parallelogram is shaded. The area of the parallelogram is $b \cdot h$, so the area of the white staircase is $\frac{b\cdot h}{2}$ square units.

33. **(a)** The area of the rectangle is $xy = 24\text{ cm}^2$.

(b) Answers vary. The graph does not lie along a line because the equation is not linear. Solving for y: $y = \dfrac{24}{x} \Rightarrow Y = 24x^{-1}$ which is not a polynomial and not linear. The graph lies along a curve known as a hyperbola.

34. The area of the trapezoid is
$\dfrac{(b_1 + b_2)h}{2} = \dfrac{(15+35)20}{2} = (50)(10) = 500\ \text{cm}^2$.
The side of the square with the same area is
$s^2 = 500\ \text{cm}^2 \Rightarrow s = \sqrt{500\ \text{cm}^2} = 10\sqrt{5}\ \text{cm}$.

35. If the hexagon is regular then the side of the hexagon is equal to the radius of the circle because the hexagon consist of six equilateral triangles that all have equal sides. The area is then
$\pi 4^2 = 16\pi\ \text{cm}^2$.

Assessment 14-2B

1. Answers may vary. E.g., for a 60-inch by 30-inch desktop and a hand measuring 4 inches by 8 inches (thumb closed) and oriented in portrait fashion, the desktop would be about 15 hands by $3\frac{3}{4}$ hands, or about $56\frac{1}{4}$ hands.

2. **(a)** (*i*) $\mathbf{m^2}$; assume 2.5 m by 7.5 m.

(*ii*) $\mathbf{yd^2}$; assume $2\frac{2}{3}$ yd by $8\frac{1}{3}$ yd.

(b) (*i*) $\mathbf{m^2}$; assume 30 m by 300 m.

(*ii*) yd^2; assume 35 yd by 3500 yd.

(c) (*i*) $\mathbf{cm^2}$; assume 21 cm by 26 cm.

(*ii*) $\mathbf{in^2}$; assume $8\frac{1}{4}$ in by $10\frac{1}{4}$ in.

3. Answers may vary. Some possible approximate measures are:

(a) An 14 in. by 15.5 in. chair seat (approximately 35 cm by 40 cm), **about** 1400 $\mathbf{cm^2}$.

(b) A 5 ft by 6.5 ft white-or chalkboard (approximately 1.5 m by 2 m), **about** $3\,\mathbf{m^2}$.

4. In each case, note that:

From m^2 to cm^2, move the decimal point four places to the right;

From cm^2 to mm^2, move the decimal point two places to the right;

From mm^2 to cm^2, move the decimal point two places to the left; and

From cm^2 to m^2, move the decimal point four places to the left;

	m^2	cm^2	mm^2
(a)	52	**520,000**	**52,000,000**
(b)	**0.000105**	**1.05**	105
(c)	**0.0086**	86	**8600**
(d)	**0.01**	**100**	10,000
(e)	8.2	**82,000**	**8,200,000**

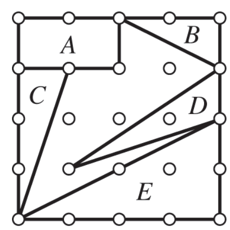
5. **(a)** $\frac{99\ \text{ft}^2}{1} \cdot \frac{1\ \text{yd}^2}{9\ \text{ft}^2} = \mathbf{11\,yd^2}$.

(b) $\frac{10^6\ \text{yd}^2}{1} \cdot \frac{9\ \text{ft}^2}{1\ \text{yd}^2} \approx \mathbf{9 \cdot 10^6\ ft^2}$.

(c) $\frac{6.5\ \text{mi}^2}{1} \cdot \frac{640\ \text{acre}}{1\ \text{mi}^2} = \mathbf{4160\ acres}$.

(d) $\frac{3\ \text{acre}}{1} \cdot \frac{4840\ \text{yd}^2}{1\ \text{acre}} = \mathbf{14{,}520\,yd^2}$.

6. **(a)** Construct a rectangle around the figure and subtract the areas of regions A, B, C, D, and E:



Total area $= 4 \cdot 4 = 16$ units^2.

Area of $A = 2 \cdot 1 = 2$ units^2;

Area of $B = \frac{1}{2} \cdot 2 \cdot 1 = 1$ unit^2;

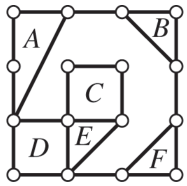
Area of $C = \frac{1}{2} \cdot 3 \cdot 1 = 1\frac{1}{2}$ units^2;

Area of $D = \frac{1}{2} \cdot 1 \cdot 3 = 1\frac{1}{2}$ units^2; and

Area of $E = \frac{1}{2} \cdot 4 \cdot 2 = 4$ units^2.

Area of figure $= 16 - (2 + 1 + 1\frac{1}{2} + 1\frac{1}{2} + 4) = \mathbf{6\ units^2}$.

(b) Construct a rectangle around the figure and subtract the areas of regions A, B, C, D, E, and F:



Total area $= 3 \cdot 3 = 9$ units^2.

Area of $A = \frac{1}{2} \cdot 1 \cdot 2 = 1$ unit^2;

Area of $B = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ unit^2;

Area of $C = 1 \cdot 1 = 1$ unit^2;

Area of $D = 1 \cdot 1 = 1$ unit^2;

Area of $E = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ unit^2; and

Area of $F = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ unit^2.

Area of figure $= 9 - (1 + \frac{1}{2} + 1 + 1 + \frac{1}{2} + \frac{1}{2}) = \mathbf{4\frac{1}{2}}$ **units^2**.

7. **(a)** $I = 3; B = 8.\ A = 3 + \frac{1}{2} \cdot 8 - 1 = \mathbf{6}$ **units^2**.

(b) $I = 0; B = 11.\ A = 0 + \frac{1}{2} \cdot 11 - 1 = \mathbf{4\frac{1}{2}}$ **units^2**.

8. **(a)** (*i*) $A = \frac{1}{2} \cdot 600 \cdot 3 = \mathbf{900\,cm^2}$.

(*ii*) $A = \frac{1}{2} \cdot 6 \cdot 0.03 = \mathbf{0.09\ m^2}$.

(b) Place point D at the intersection of the two dashed lines.

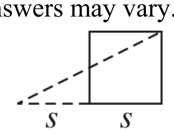
Area $\Delta ABD = \frac{1}{2} \cdot 8 \cdot 6 = 24\text{ cm}^2$ and

Area $\Delta CBD = \frac{1}{2} \cdot 10 \cdot 3 = 15\text{ cm}^2$.

Area $\Delta ABC =$ area $(\Delta ABD + \Delta CBD)$

$= 24 + 15 = \mathbf{39\,cm^2}$.

9. Answers may vary. One possibility is as below:



Where s is the length of a side of the square. Then $A_{triangle} = \frac{1}{2}(2s)(s) = \mathbf{s^2}$, which is the area of the square.

10. **(a)** Measurement of the sides is linear, so the ratio of the heights is also $\frac{2}{3}$.

(b) Measurement of area is in square units, so the ratio of the areas is $\left(\frac{2}{3}\right)^2 = \mathbf{\frac{4}{9}}$.

11. **(a)** $A = 9 \cdot 9 = \mathbf{81\,cm^2}$.

(b) $A = \frac{1}{2} \cdot 6 \cdot (27 + 8) = \mathbf{105\,cm^2}$.

12. **(a)** (*i*) $1.2\text{ km} \cdot 900\text{ m} \cdot \frac{1\text{ km}}{1000\text{ m}} = \mathbf{1.08\,km^2}$.

(*ii*) $1.08\text{ km}^2 \cdot \frac{1{,}000{,}000\text{ m}^2}{1\text{ km}^2} \cdot \frac{\text{hm}^2}{10{,}000\text{ m}^2} = 108\text{ hm}^2 = \mathbf{108\ ha}$.

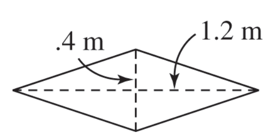
(b) (*i*) $1.2\text{ mi} \cdot 900\text{ yd} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \cdot \frac{3\text{ ft}}{\text{yd}} \approx \mathbf{0.61\,mi^2}$.

(*ii*) $\frac{(1.2)(900)(3)}{5280}\text{ mi}^2 \cdot \frac{640\text{ acres}}{1\text{ mi}^2} \approx 393$ acres.

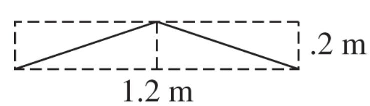
13. **(a)** Area $= L \cdot W$. If the perimeter is increased for a constant area, length must increase while width decreases. Thus the rectangle will elongate and become less like a square.

(b) For the area to change, the dimensions must change. For the perimeter to remain constant, as one side increases the adjacent side must decrease. If we start with a rectangle that is not a square, then the area will increase as we change the dimensions toward a square.

14. The rhombus



can be rearranged with translations and rotations to form a rectangle.



So the area is $.2\text{ m} \cdot 1.2\text{ m} = \mathbf{0.24\,m^2}$.

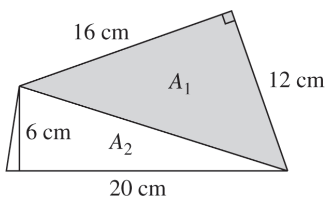
15. The plot is $22 \text{ m} \times 28 \text{ m} = 616 \text{ m}^2$. $\frac{616 \text{ m}^2}{85 \text{ m}^2 \text{ per bag}}$ ≈ 7.25 bags, so **8 bags** must be purchased.

16. **(a)** $A = \frac{180°}{360°} \cdot \pi \cdot 3^2 = \mathbf{\frac{9}{2}\pi \textbf{ cm}^2}$.

(b) $\frac{\theta}{360°} = \frac{\text{arc length}}{2\pi r} \Rightarrow$
$\theta = \frac{360° \times 20 \text{ cm}}{2\pi \cdot 10 \text{ cm}} = \left(\frac{360}{\pi}\right)°$.
$A = \frac{\left(\frac{360}{\pi}\right)°}{360°} \cdot \pi \cdot 10^2 = \mathbf{100 \textbf{ cm}^2}$.

17. $A_{rectangle} = 64 \cdot 25 = 1600 \text{ m}^2$. Length of each side of a square with area of $1600 \text{ m}^2 = \sqrt{1600} = \mathbf{40 \textbf{ m}}$.

18. **(a)** Divide the figure into two triangles as shown:



$A_1 + A_2 = \frac{1}{2} \cdot 16 \cdot 12 + \frac{1}{2} \cdot 6 \cdot 20$
$= \mathbf{156 \textbf{ cm}^2}$.

(b) $\frac{1}{2} \cdot 30 \text{ m} \cdot 18 \text{ m} = \mathbf{270 \textbf{ m}^2}$.

19. **(*i*)** Square peg inside circular hole: Diagonal of square $= 2r$. Then $(2r)^2 = s^2 + s^2$ (by the Pythagorean theorem), where r is the radius of the circle and s is the length of a side of the square, so $s^2 = 2r^2$. $A_{square} = s^2 = 2r^2$.

Percentage of wasted space $= \frac{A_{circle} - A_{square}}{A_{circle}} =$
$\frac{\pi r^2 - 2r^2}{\pi r^2} = \frac{\pi - 2}{\pi} \approx 36.34\%$.

(*ii*) Circular peg inside square hole: Length of side of square $= 2r$. $A_{square} = (2r)^2 = 4r^2$.

Percentage of wasted space $= \frac{A_{square} - A_{circle}}{A_{square}} =$
$\frac{4r^2 - \pi r^2}{4r^2} = \frac{4 - \pi}{4} \approx 21.46\%$.

The **circular peg inside the square hole** has less wasted space.

20. **(a)** The two shaded areas form a circle with radius $\frac{r}{2}$. $A_{shaded} = \pi\left(\frac{r}{2}\right)^2 = \mathbf{\frac{\pi r^2}{4}}$ **units²**.

(b) The four shaded areas form two circles each having radius $\frac{\pi}{4}$.

$A_{shaded} = 2\pi(\frac{r}{4})^2 = \mathbf{\frac{\pi r^2}{8}}$ **units²**.

(c) The eight shaded areas form four circles each having radius $\frac{r}{8}$.

$A_{shaded} = 4\pi\left(\frac{r}{8}\right)^2 = \mathbf{\frac{\pi r^2}{16}}$ **units²**.

(d) In (a) the unshaded part has an area of $\frac{1}{2}\pi r^2 - \frac{\pi r^2}{4} = \frac{2\pi r^2}{4} - \frac{\pi r^2}{4} = \frac{\pi r^2}{4}$. The area of the unshaded and shaded parts are equal and their ratio is 1:1.

In (b) the unshaded part has an area of: $\frac{1}{2}\pi r^2 - \frac{\pi r^2}{8} = \frac{4\pi r^2}{8} - \frac{\pi r^2}{8} = 3 \cdot \frac{\pi r^2}{8}$. The area of the unshaded part is three times the area of the shaded part, so their ratio is 3:1.

In (c)the unshaded part has an area of $\frac{1}{2}\pi r^2 - \frac{\pi r^2}{16} = \frac{8\pi r^2}{16} - \frac{\pi r^2}{16} = 7 \cdot \frac{\pi r^2}{16}$. The area of the unshaded part is seven times the area of the shaded part, so their ratio is 7:1.

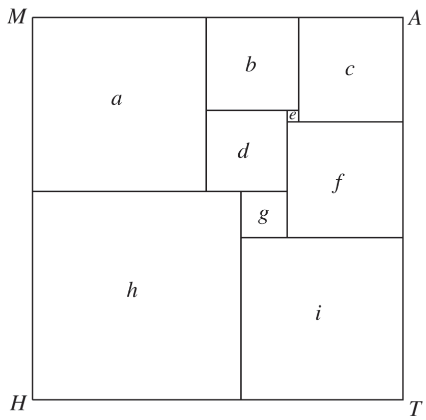
21. $A_{complete\ target} = \pi(5^2) = 25\pi \text{ in.}^2$;
$A_{out\,side\,shaded\ region} = 25\pi - \pi(4^2) = 9\pi \text{ in.}^2$; and
$A_{in\,side\,sheded\ region} = \pi(3^2) = 9\pi \text{ in.}^2$.
The areas of the inside and outside shaded regions are **both** $\mathbf{9\pi}$ **in.²**

22. Each side has length $\sqrt{169} = 13$ in.
The perimeter is $4s =$ **52 in**.

23. **(a)** Area is **quadrupled**. Diameter doubled means radius doubled. Area $= \pi(2r)^2 = 4\pi r^2$, which is quadruple the original area.

(b) Area is 1.1^2, or **1.21 times** as great. New radius is 110% of old radius $= 1.1r$, so area $= \pi(1.1r)^2 = 1.21\pi r^2$ (or area is increased by 21%).

(c) Area will **increase by** factor of **9**. $C = 2\pi r$ $\Rightarrow r = \frac{C}{2\pi} \Rightarrow$ area $= \pi\left(\frac{C}{2\pi}\right)^2$. If circumference is increased by a factor of 3, then area $= \pi\left(\frac{3C}{2\pi}\right)^2$, or area will increase by 3^2, a factor of 9.

24. 

Let the letter in each of the squares represent the length on one side of the square.

Then: $b + c = 17$;
$e = c - b = 1$;
$d = b - e = 7$;
$f = b + c - d = 10$;
$g = (c + f) - (b + d) = 4$;
$a = b + d = 15$; and
$h = (a + b + c) - (f + g) = 18$.

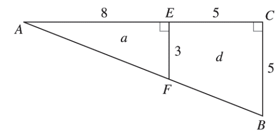
Thus: $MA = a + b + c =$ **32 units**.
$MH = a + h =$ **33 units**.
$A_{MATH} = 32 \cdot 33 =$ **1056 units^2**.

25. The area of the large rectangle is $(a + b)(c + d)$ square units. The areas of the four small rectangles that make up the large rectangle are ac, ad, bc, and bd square units. The sum of the areas of the four small rectangles equals the area of the large rectangle they construct. Thus, $(a + b)(c + d) = ac + ad + bc + bd$.

26. **(a)** The area is $(3+5)^2 = 8^2 = 64$ units^2.

(b) The area is $(5+8)\cdot 5 = 13\cdot 5 = 65$ units^2.

(c) Although the pieces look like they should fit together, they do not really fit. To see this, assume the pieces do fit. We then obtain this figure:



Since $\Delta AEF \sim \Delta ACB$, we have $\frac{8}{13} = \frac{3}{5}$, which is a contradiction. This implies that pieces like those in the figure cannot fit together to form a triangle. In order for the pieces to fit together, the measures of $\overline{EF}$ must satisfy $\frac{8}{13} = \frac{EF}{5}$; hence, $EF = \frac{40}{13} = 3\frac{1}{13}$. Since $3\frac{1}{13}$ is close to 3, the discrepancy is so small that the pieces appear to fit.

27. ***(i)*** First package: 3 rolls $\times\ 2\frac{1}{2}$ ft $\times$ 8 ft $= 60$ ft^2. $\frac{\$6.00}{60 \text{ ft}^2} = 10¢$ per ft^2 (or 10 ft^2 per dollar).

(ii) Second package: 5 rolls $\times\ 2\frac{1}{2}$ ft $\times$ 6 ft $=$ 75 ft^2. $\frac{\$8.00}{75 \text{ ft}^2} = 10\frac{2}{3}¢$ per ft^2 (or 9.375 ft^2 per dollar).

The **first package** is the better buy.

28. Let s be the length of a side of the square. Decompose the shaded quadrilateral into two triangles by drawing a line from the center of square A to the vertex of A that lies in B. Both of the triangles have height $\frac{s}{2}$ and the sum of their bases is s, so the sum of their areas is $\left(\frac{1}{2}\right)(s)\left(\frac{s}{2}\right) = \frac{s^2}{4}$. The shaded area is then $\frac{\mathbf{1}}{\mathbf{4}}$ the area of square A.

29. The parallel lines produce similar triangles. ΔBDE has sides $\frac{1}{3}$ the length of ΔABC. The segments are congruent, so the height of ΔBDE is $\frac{1}{3}$ the height of ΔABC.

$$\text{Area}_{\Delta BDE} = \frac{bh}{2}$$
$$= \frac{\frac{1}{3}\text{ the base of } \Delta ABC \cdot \frac{1}{3}\text{ the height of } \Delta ABC}{2}$$
$$= \frac{1}{9}\,\frac{\text{the base of } \Delta ABC \cdot \text{the height of } \Delta ABC}{2}.$$

Thus $\mathbf{Area}_{\Delta BDE}. = \frac{\mathbf{1}}{\mathbf{9}}\,\mathbf{Area}_{\Delta BAC}$

30. In the figure of problem 29, let h be the height between each pair of parallel lines (and between $\overline{DE}$ and B). Since the segments are congruent, $FG = 2DE$ and $AC = 3DE$.

(a) $\dfrac{\text{Area } DBE}{\text{Area } DEGF} = \dfrac{\frac{1}{2}(DE)h}{\frac{1}{2}(DE+FG)h} = \dfrac{DE}{DE+2DE} = \mathbf{\dfrac{1}{3}}.$

(b) $\dfrac{\text{Area } DBE}{\text{Area } FGCA} = \dfrac{\frac{1}{2}(DE)h}{\frac{1}{2}(FG+CA)h} = \dfrac{DE}{2DE+3DE} = \mathbf{\dfrac{1}{5}}.$

(c) $\dfrac{\text{Area } DEGF}{\text{Area } FGCA} = \dfrac{\frac{1}{2}(DE+FG)h}{\frac{1}{2}(FG+CA)h} = \dfrac{DE+2DE}{2DE+3DE} = \mathbf{\dfrac{3}{5}}.$

(d) $\dfrac{\text{Area } DEGF}{\text{Area } ABC} = \dfrac{\frac{1}{2}(DE+FG)h}{\frac{1}{2}(AC)3h} = \dfrac{DE+2DE}{3(3DE)}$

$= \frac{3}{9} = \mathbf{\dfrac{1}{3}}.$

(e) $\dfrac{\text{Area } FGCA}{\text{Area } ABC} = \dfrac{\frac{1}{2}(FG+AC)h}{\frac{1}{2}(AC)3h} = \dfrac{2DE+3DE}{3(3DE)} = \mathbf{\dfrac{5}{9}}.$

(f) $\dfrac{\text{Area } ABC}{\text{Area } DECA} = \dfrac{\frac{1}{2}(AC)3h}{\frac{1}{2}(DE+AC)2h} = \dfrac{3(3DE)}{2(DE+3DE)} = \mathbf{\dfrac{9}{8}}.$

31. The radius of the total area is $148\pi = \pi r^2 \Rightarrow r = \sqrt{148} = 2\sqrt{37}$. If the radius of the sidewalk is 1 m then the radius of the flower bed is $2\sqrt{37} - 1 = \left(2\sqrt{37} - 1\right)$ m.

32. If the diameter is 12,756 km then the radius is $\dfrac{12,756}{2} = 6378$ km. The area of the cross section is $\pi r^2 = \pi(6378)^2 = 40,678,884\pi \text{ km}^2$ or $127,796,483.1 \text{ km}^2 \approx 128,000,000 \text{ km}^2$.

33. (a) The area of the sector is

$$\pi r^2 \cdot \frac{\text{central angle}}{360^\circ} = \pi\left(2^2\right)\cdot\frac{1^\circ}{360^\circ}.$$
$$= \frac{4\pi}{360} = \frac{\pi}{90} \text{ m}^2$$

(b) The area of the circle is $\pi\left(2^2\right) = 4\pi \text{ m}^2$.

(c) 360 sectors $= 360\cdot\dfrac{\pi}{90} \text{ m}^2 = 4\pi \text{ m}^2$.

(d) $4\pi \text{ m}^2 - 4\pi \text{ m}^2 = 0$.

34. The outer shaded portion is $\dfrac{3}{4}$ of the difference of the outer circle and the inner circle =

$$\frac{3}{4}\left(\pi\left(4^2\right) - \pi\left(2^2\right)\right) = \frac{3}{4}(16\pi - 4\pi)$$
$$= \frac{3}{4}12\pi = 9\pi \text{ cm}^2$$

The inner shaded portion is $\dfrac{1}{4}$ of the area of the inner circle: $\dfrac{1}{4}\left(\pi\left(2^2\right)\right) = \dfrac{1}{4}4\pi = \pi \text{ cm}^2$. The total area shaded is $9\pi + \pi = 10\pi \text{ cm}^2$.

35. (a) If each of the four squares is divided into 4 equal parts then 16 out of 32 parts are shaded. $\dfrac{16}{32} = \dfrac{1}{2}$ of the square is shaded. If the area of the square is $6^2 = 36 \text{ cm}^2$ then the area of the shaded quilt pattern is $\dfrac{1}{2}\cdot 36 = 18 \text{ cm}^2$.

(b) If each of the four squares is divided into 8 equal parts then 8 out of 32 parts are shaded. $\dfrac{8}{32} = \dfrac{1}{4}$ of the square is shaded. If the area of the square is $6^2 = 36 \text{ cm}^2$ then the area of the shaded quilt pattern is $\dfrac{1}{4}\cdot 36 = 9 \text{ cm}^2$.

(c) If each of the four squares is divided into 4 equal parts then 16 out of 32 parts are shaded. $\dfrac{16}{32} = \dfrac{1}{2}$ of the square is shaded. If the area of the square is $6^2 = 36 \text{ cm}^2$ then the area of the shaded quilt pattern is $\dfrac{1}{2}\cdot 36 = 18 \text{ cm}^2$.

(d) If each of the four squares is divided into 4 equal parts then 16 out of 32 parts are shaded. $\dfrac{16}{32} = \dfrac{1}{2}$ of the square is shaded. If the area of the square is $6^2 = 36 \text{ cm}^2$ then the area of the shaded quilt pattern is $\dfrac{1}{2}\cdot 36 = 18 \text{ cm}^2$.

36. **(a)** The area of the rectangle is $xy = 36\ \text{cm}^2$.

(b) Answers vary. The graph does not lie along a line because the equation is not linear. Solving for y: $y = \frac{36}{x} \Rightarrow y = x^{-1}36$ which is not a polynomial and not linear. The graph lies along a curve known as a hyperbola.

37. The area of the trapezoid is $\frac{(b_1+b_2)h}{2} = \frac{(12+36)12}{2} = (48)(6) = 288\ \text{cm}^2$. The side of the square with the same area is $s^2 = 288\ \text{cm}^2 \Rightarrow s = \sqrt{288\ \text{cm}^2} = 12\sqrt{2}\ \text{cm}$.

38. If the hexagon is regular then the side of the hexagon is equal to the radius of the circle because the hexagon consist of six equilateral triangles that all have equal sides. The area is then $\pi 6^2 = 36\pi\ \text{cm}^2$.

Mathematical Connections 14-2: Review Problems

20. Each rounded circular corner has perimeter $\frac{1}{4}(2\pi \cdot 3) = \frac{3}{2}\pi$ in. Thus the total perimeter $= 4\left(\frac{3}{2}\pi\right) + 4(24) = (6\pi + 96)$ inches $\approx$ 114.8 inches or **about 9.6 feet**.

21. **(a)** $C = 2\pi r \Rightarrow r = \frac{C}{2\pi} = \frac{39750}{2\pi}$, about **6330 km**.

(b) This arc length is $\frac{1}{4}$ the circumference, or $\frac{39750}{4} =$ **9937.5 km**.

22. The perimeter of a regular hexagon is $6s$, where s is the length of each side. In a regular hexagon the radius r of the circumscribed circle equals s and its circumference is thus $2\pi s$. The ratio of the circumference to the perimeter is then $\frac{2\pi s}{6s} = \frac{\pi}{3} \approx$ 1.04720, making the circumference about **4.7%** longer. Alternatively, the difference in length is $2\pi s - 6s \approx$ **0.28 *s***.

23. **(a)** Using sides 1 and 3 their respective perimeters are (1 + 1 + 1) = 3 and (3 + 3 + 3) = 9. The ratio of their perimeters is 3:9 or 1:3.

(b) $\frac{1}{3} = \frac{x}{36\pi} \Rightarrow 36\pi = 3x \Rightarrow x = 12\pi\ \text{cm}$.

24. The sector is $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the circle. $\frac{1}{6}$ of the arc is $\frac{1}{6}$ of the circumference: $\frac{1}{6}2\pi 6 = 2\pi$. Add the sides of the sector (=2r) and the perimeter of the sector is $(12 + 2\pi)$ cm.

25. **(a)** $2\text{ m} = 2\text{ m}\cdot\frac{100\text{ cm}}{1\text{ m}} = 200\text{ cm}$

(b) $2\text{ in.} = \frac{2}{12} = \frac{1}{6}\text{ ft.}$

(c) $250\text{ cm}\cdot\frac{1\text{ km}}{100000\text{ cm}} = 0.0025\text{ km}$

(d) $2500\text{ yd}\cdot\frac{3\text{ ft}}{1\text{ yd}}\cdot\frac{1\text{ mi}}{5280\text{ ft}} \approx 1.42\text{ mi}$.

Assessment 14-3A: The Pythagorean Theorem, Distance Formula, and Equation of a Circle

1. **(a)** $d = \sqrt{4^2 + 2^2} = \boldsymbol{\sqrt{20} \approx 4.5}$.

(b) $d = \sqrt{4^2 + 1^2} = \boldsymbol{\sqrt{17} \approx 4.1}$.

2. **(a)** $x^2 + 8^2 = 10^2 \Rightarrow x^2 = 10^2 - 8^2 \Rightarrow x = \sqrt{100 - 64} = \sqrt{36}$. $\boldsymbol{x = 6}$.

(b) $x^2 = (4a)^2 + (3a)^2 = 16a^2 + 9a^2 \Rightarrow x = \sqrt{a^2(16 + 9)} = \sqrt{25a^2}$. $\boldsymbol{x = 5a}$.

(c) $x^2 + 5^2 = 13^2 \Rightarrow x^2 = 13^2 - 5^2 \Rightarrow x = \sqrt{169 - 25} = \sqrt{144}$. $\boldsymbol{x = 12}$.

(d) $x^2 + \left(\frac{s}{2}\right)^2 = s^2 \Rightarrow x^2 = s^2 - \left(\frac{s}{2}\right)^2 = s^2 - \frac{s^2}{4} = \frac{4s^2 - s^2}{4} = \frac{3s^2}{4} \Rightarrow x = \sqrt{\frac{3s^2}{4}}$. $\boldsymbol{x = \frac{s\sqrt{3}}{2}}$.

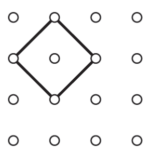
(e) $x^2 + 1^2 = y^2$ and $x^2 = 1^2 + 1^2$. Since $\boldsymbol{x = \sqrt{2}}$, $y^2 = (\sqrt{2})^2 + 1^2$. Thus, $y^2 = 2 + 1 \Rightarrow \boldsymbol{y = \sqrt{3}}$.

(f) $x^2 = 10^2 + 15^2$ and $y^2 = x^2 + 7^2$. Since $x = \sqrt{100+225} = \sqrt{\mathbf{325}}$, $y^2 = \sqrt{325}^2 + 7^2 = 374$. Thus, $y = \sqrt{374}$.

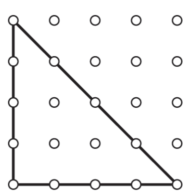
3. **(a)** Make a right triangle with **sides 2 and 3**. $d = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(b) Make a right triangle with **sides 1 and 2**. $d = \sqrt{1^2 + 2^2} = \sqrt{5}$.

4. Answers may vary. If the distance between dots is one unit, d each side $= \sqrt{2}$ units. $P = 4\sqrt{2}$ units.



5. The hypotenuse has length $\sqrt{4^2 + 4^2} = 4\sqrt{2}$.



$P = 8 + 4\sqrt{2} \approx$ **13.66 units**.

6. Let x and $2x$ be the lengths of the two legs. $x^2 + (2x)^2 = 30^2 \Rightarrow 5x^2 = 900 \Rightarrow$ $\boldsymbol{x} = \sqrt{180} = \sqrt{36 \cdot 5} = \mathbf{6\sqrt{5}}$ and $\mathbf{2x} = \mathbf{12\sqrt{5}}$.

7. For the answer to be yes, the number must satisfy the Pythagorean theorem.

(a) **No**. $24^2 \neq 10^2 + 16^2$.

(b) **Yes**. $34^2 = 16^2 + 30^2$.

(c) **Yes**. $2^2 = (\sqrt{2})^2 + (\sqrt{2})^2$.

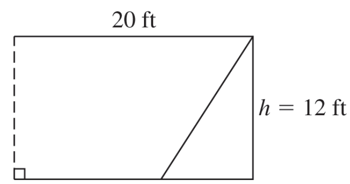
8. Let x be the diagonal of a face. Then $x^2 = 9^2 + 12^2 \Rightarrow x = \sqrt{81 + 144} = \sqrt{225}$. $x = 15$.
Let d be the diagonal of the prism. Then
$d^2 = 15^2 + 15^2 \Rightarrow$
$d = \sqrt{225+225} = \sqrt{225(1+1)}$. $\boldsymbol{d} = \mathbf{15\sqrt{2}}$.

9. The boat is 10 miles south and 5 miles east of A. The distance from A is $d = \sqrt{10^2 + 5^2} = \sqrt{125}$, or **about 11.2 miles**.

10. Let h be the height of the ladder's top. $h^2 + 3^3 = 15^2 \Rightarrow h = \sqrt{15^2 - 3^2} = \sqrt{216}$, or **about 14.7 feet**.

11. The tall pole stands 10 m above the short pole. Draw a horizontal line from the top of the short pole to form a right triangle, where d is the distance between the poles. $d^2 + 10^2 = 14^2 \Rightarrow$ $d^2 = 14^2 - 10^2 \Rightarrow d = \sqrt{196 - 100} = \sqrt{96} = \sqrt{16 \cdot 6} = 4\sqrt{6}$, or **about 9.8 m**.

12. **(a)** $h = \sqrt{15^2 - 9^2} = 12$.



Thus, the area is $20 \cdot 12 = \mathbf{240\ ft^2}$.

(b) $A = \frac{1}{2}bh = \frac{1}{2} \cdot 2\sqrt{40^2 - 30^2} \cdot 30$
$\approx 793.7\ \text{ft}^2$

13. **(a)** $A = \frac{1}{2} \cdot$ (length of diagonal 1) $\cdot$ (length of diagonal 2) $= \frac{1}{2}(8)\left(2 \cdot \sqrt{10^2 - 4^2}\right) = \mathbf{8\sqrt{84}}$.
Alternatively, observe that the figure is made of 4 triangles, each of which have area $\frac{1}{2}(4) \cdot \sqrt{10^2 - 4^2} = 2\sqrt{84}$.

(b) The figure is made of four triangles, each having area $\frac{1}{2}(2)\sqrt{10^2 - 2^2} = \sqrt{96}$. Thus, the area is $\mathbf{4\sqrt{96}}$.

14. **(a)** **(*i*)** In the large triangle: $x^2 = 4^2 + (4\sqrt{3})^2 = 16 + 16 \cdot 3 \Rightarrow$
$x = \sqrt{16+48} = \sqrt{64}$.
$x = 8$.

(*ii*) Area$_{large\ triangle}$ may be expressed in two ways: $A = \frac{1}{2}(4)(4\sqrt{3}) = 8\sqrt{3}$ and $A = \frac{1}{2}xy = \frac{1}{2}(8)y = 4y$

Equating: $4y = 8\sqrt{3} \Rightarrow y = 2\sqrt{3}$.

(b) Use the special relationship for 45°-45°-90° right triangles. If the length of each leg is a, the hypotenuse has length $a\sqrt{2}$.

(*i*) Hypotenuse $= 2\sqrt{2}$, so $a = 2$. $\boldsymbol{x} = 2a = \mathbf{4}$.

(*ii*) $y = \frac{1}{2}x$. $\boldsymbol{y} = \mathbf{\frac{1}{2}} \cdot 4 = \mathbf{2}$.

15. Let ℓ be the length of the lake. $\ell^2 + 150^2 = 180 \Rightarrow \ell^2 = 180^2 - 150^2 \Rightarrow \ell = \sqrt{9900} = \sqrt{100 \cdot 99} = 10\sqrt{99}$, or **about 99.5 ft**.

16. The area of the trapezoid is equal to the sum of the areas of the three triangles:

$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$;

$\frac{1}{2}(a^2 + 2ab + b^2) = ab + \frac{1}{2}c^2$; so

$\frac{a^2}{2} + ab + \frac{b^2}{2} = ab + \frac{c^2}{2}$.

Subtracting ab from both sides and multiplying both sides by 2 yields $a^2 + b^2 = c^2$ (note: the angle formed by the two sides of length c is supplementary to the sum of the other two complementary angles, thus the angle $= 90°$).

17. The area of the large square equals the sum of the areas of the small square and the four right triangles:

$(a + b)^2 = c^2 + 4\left(\frac{ab}{2}\right) \Rightarrow a^2 + 2ab + b^2 = c^2 + 2ab \Rightarrow a^2 + b^2 = c^2$.

The inside quadrilateral is in fact a square, since each side is the hypotenuse of a triangle with the same length sides, and since at each of its vertices there are three angles whose measures sum to 180º— two of which are complementary. Thus the angles of the quadrilateral are right angles.

18. **(a)** The equilateral triangle of base 4 cm has height $= \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$ cm. Thus, the area is $\frac{1}{2}(4)(2\sqrt{3}) = \mathbf{4\sqrt{3}}$ **cm**. Working clockwise, the area of the triangles are

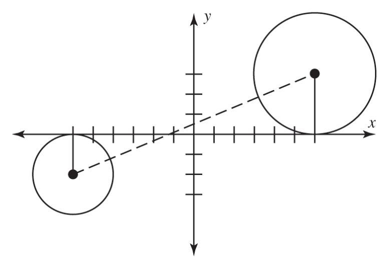
$$\frac{1}{2} \cdot 5 \cdot \sqrt{5^2 - \left(\frac{5}{2}\right)^2} = \frac{1}{2} \cdot 5 \cdot \sqrt{25 - \frac{25}{4}}$$
$$= \frac{5}{2} \cdot \sqrt{\frac{75}{4}} = \frac{5}{2} \cdot \frac{5\sqrt{3}}{2}$$
$$= \mathbf{\frac{25\sqrt{3}}{4}}\ \textbf{cm}$$

and

$$\frac{1}{2} \cdot 3 \cdot \sqrt{3^2 - \left(\frac{3}{2}\right)^2} = \frac{3}{2} \cdot \sqrt{9 - \frac{9}{4}}$$
$$= \frac{3}{2} \cdot \sqrt{\frac{27}{4}} = \mathbf{\frac{9}{4}\sqrt{3}}\ \textbf{cm}.$$

(b) The areas of the smaller triangles (those attached to the legs of the shaded triangle) sum to the area of the largest triangle (attached to the hypotenuse). $4\sqrt{3} + \frac{9}{4}\sqrt{3} = \frac{16 + 9}{4}\sqrt{3} = \frac{25}{4}\sqrt{3}$.

19. We can position the figure on a coordinate system with the figure positioned so that the origin is at the midpoint of the two right angles, as shown below.



Thus, the distance is $\sqrt{(6 - (-6))^2 + (3 - (-2))^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \mathbf{13}$.

20. **(a)** $AB = \sqrt{(0-0)^2 + (7-3)^2} = \sqrt{0+16} = \mathbf{4}$.

(b) $AB = \sqrt{(4-0)^2 + (0-3)^2} = \sqrt{16+9} = \mathbf{5}$.

(c) $AB = \sqrt{(3 - {}^-1)^2 + ({}^-4 - 2)^2} = \sqrt{16+36} = \sqrt{52} = \mathbf{2\sqrt{13}}$, or **about 7.2.**

(d) $AB = \sqrt{\left(\frac{1}{2} - 4\right)^2 + \left(\frac{^{-}7}{4} - {^{-}5}\right)^2}$

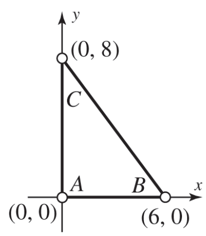
$= \sqrt{\left(\frac{1-8}{2}\right)^2 + \left(\frac{^{-}7+20}{4}\right)^2}$

$= \sqrt{\frac{365}{16}} = \mathbf{\frac{\sqrt{365}}{4}}$,

or **about 4.78**.

21. From the special properties of a 30°-60°-90° right triangle, the short leg (opposite the 30° angle) is half the hypotenuse, or $\frac{1}{2} \cdot \frac{c}{2} = \frac{c}{4}$. The longer leg (opposite the 60° angle) is $\sqrt{3}$ times the short leg. Thus the side opposite the 60° angle $= \sqrt{3} \cdot \frac{c}{4} = \mathbf{\frac{c\sqrt{3}}{4}}$.

22. Given the triangle:



(a) Altitude is the line segment from each vertex perpendicular to the base.

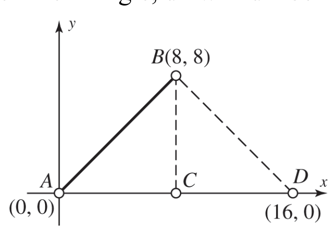
(i) Altitude of $\overline{AC}$ is $\boldsymbol{x = 0}$.

(ii) Altitude of $\overline{AB}$ is $\boldsymbol{y = 0}$.

(iii) $m_{BC} = \frac{0-8}{6-0} = \frac{^{-}4}{3}$, thus $m_{\perp} = \frac{3}{4}$. Then $0 = \frac{3}{4}(0) + b \Rightarrow b = 0$. Therefore the altitude is $\boldsymbol{y = \frac{3}{4}x}$.

(b) All altitudes intersect at **(0, 0)**.

23. Given the triangle, draw and label it as follows:



(a) Draw the line segment $\overline{BC}$ so that it is perpendicular to the x-axis. Reflect the triangle ABC about $\overline{BC}$ to form isosceles triangle ABC with third vertex at **(16, 0)**.

(b) $AB^2 = AC^2 + BC^2 \Rightarrow AB = \sqrt{8^2 + 8^2} = 8\sqrt{2}$. Thus the sides are $\mathbf{8\sqrt{2}, 8\sqrt{2},}$ and **16**.

(c) $(8\sqrt{2})^2 + (8\sqrt{2})^2 = 128 + 128 = 256 = \mathbf{16^2}$.

24. Given that the equation of a circle with (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$:

(a) $h = {^{-}3}$ and $k = 4 \Rightarrow (x - {^{-}3})^2 + (y - 4)^2 = 4^2$, or $\boldsymbol{(x + 3)^2 + (y - 4)^2 = 16}$.

(b) $h = {^{-}3}$ and $k = {^{-}2} \Rightarrow (x - {^{-}3})^2 + (y - {^{-}2})^2 = (\sqrt{2})^2$, or $\boldsymbol{(x + 3)^2 + (y + 2)^2 = 2}$.

25. **(a)** If $(x - 0)^2 + (y - 0)^2 = 4^2$, then the center of the circle is **(0, 0)** and the radius is **4**.

(b) If $(x - 3)^2 + (y - 2)^2 = 10^2$, then the center of the circle is **(3, 2)** and the radius is **10**.

(c) If $(x - {^{-}2})^2 + (y - 3)^2 = (\sqrt{5})^2$, then the center of the circle is $(\mathbf{^{-}2, 3})$ and the radius is $\mathbf{\sqrt{5}}$.

(d) If $(x - 0)^2 + (y - {^{-}3})^2 = 3^2$, the center of the circle is $(\mathbf{0, {^{-}3}})$ and the radius is **3**.

26. AB forms the hypotenuse of a $3 - 1 = 2$, $2 + 4 = 6$, h right triangle. Then $h = \sqrt{2^2 + 6^2} = 2\sqrt{10} \approx$ **6.3 km**.

27. The area of the square is $s^2 = (2r)^2 = 4r^2$. The area of the circle is πr^2. The percent that will be watered is the percent the circle area is of the square area. $\frac{\pi r^2}{4r^2} \approx 0.7854 \approx 78.54\%$ will be watered.

The area of the square is $s^2 = (6r)^2 = 36r^2$. The area of the nine circles is $9\pi r^2$. The percent that will be watered is the percent the nine circles' area is of the square area. $\frac{9\pi r^2}{36r^2} \approx 0.7854 \approx 78.54\%$ will be watered.

Both sprinkler systems cover the same percentage of the square field and it does not matter which system is used if the only selection criterion is the amount of land covered by the system.

28. **(a)** The distance from first base to third base is the diagonal of the square. Using Pythagorean Theorem:
$90^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{16200} = 90\sqrt{2}\text{ ft} \approx 127\text{ ft}.$

(b) The distance from halfway between first and second base and third base can be found using Pythagorean Theorem:
$45^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{10125} = 45\sqrt{5}\text{ ft} \approx 101\text{ ft}.$

(c) The distance from halfway between first and second base and home base can be found using Pythagorean Theorem:
$45^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{10125} = 45\sqrt{5}\text{ ft} \approx 101\text{ ft}.$

29. One side of the rhombus creates a right triangle with half of each of the diagonals. One side is $s^2 = 4^2 + 6^2 \Rightarrow s = \sqrt{52} = 2\sqrt{13} \approx 7.2$ in.

30. Similar triangles will have sides $3k, 4k,$ and $5k$ and $(3k)^2 + (4k)^2 = (5k)^2$.

31. In the right triangle $AC^2 + BC^2 = AB^2$. The area of the triangle is $\frac{1}{2}AC \cdot BC$. The area of the shaded part is the area of the triangle plus the area of the two semicircles with centers O and P minus the area of the semicircle with center E.

$$\left[\left(\frac{1}{2}AC\cdot BC\right) + \left(\frac{1}{2}\pi\left(\frac{1}{2}AC\right)^2\right) + \left(\frac{1}{2}\pi\left(\frac{1}{2}BC\right)^2\right)\right] - \left(\frac{1}{2}\pi\left(\frac{1}{2}AB\right)^2\right)$$

$$= \frac{1}{2}AC\cdot BC + \frac{1}{8}\pi AC^2 + \frac{1}{8}\pi BC^2 - \frac{1}{8}\pi AB^2$$

$$= \frac{1}{2}AC\cdot BC + \frac{1}{8}\pi\left(AC^2 + BC^2 - AB^2\right)$$

$$= \frac{1}{2}AC\cdot BC + \frac{1}{8}\pi(AB^2 - AB^2)$$

$$= \frac{1}{2}AC\cdot BC + \frac{1}{8}\pi(0) = \frac{1}{2}AC\cdot BC$$

Assessment 14-3B

1. **(a)** $d = \sqrt{2^2 + 2^2} = \boldsymbol{\sqrt{8} \approx 2.8}$.

(b) $d = \sqrt{3^2 + 2^2} = \boldsymbol{\sqrt{13} \approx 3.6}$.

2. **(a)** Let the base of the large right triangle be represented by a; the base of the small right triangle by b. Then
$a^2 + 8^2 = 17^2 \Rightarrow a^2 = 17^2 - 8^2 \Rightarrow$
$a = \sqrt{289 - 64} = \sqrt{225} = 15.$
$b^2 + 8^2 = 10^2 \Rightarrow b^2 = 10^2 - 8^2 \Rightarrow$
$b = \sqrt{100 - 64} = \sqrt{36} = 6.$
$x = a - b = 15 - 6.$ $\boldsymbol{x = 9}$.

(b) $x^2 = 5^2 + 12^2 \Rightarrow x = \sqrt{25 + 144} = \sqrt{169}.$
$\boldsymbol{x = 13}$.

(c) $(2x)^2 = 4^2 + 4^2 \Rightarrow 4x^2 = 32 \Rightarrow$
$x^2 = 8 \Rightarrow x = \sqrt{8} = \sqrt{4\cdot 2}.$ $\boldsymbol{x = 2\sqrt{2}}$.

(d) $x^2 = 3^2 + 6^2 \Rightarrow$
$x = \sqrt{9 + 36} = \sqrt{45} = \sqrt{9\cdot 5}.$ $\boldsymbol{x = 3\sqrt{5}}$.

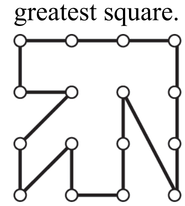
(e) Let d be the diagonal on one face of the cube.
$d^2 = 3^2 + 3^2 = 18.$
Then $x^2 = d^2 + 3^2 \Rightarrow x = \sqrt{18 + 9} =$
$\sqrt{27} = \sqrt{9\cdot 3}.$ $\boldsymbol{x = 3\sqrt{3}}$.

(f) Let y be the hypotenuse of the large right triangle. Then $y^2 = 3^2 + 4^2 \Rightarrow y =$
$\sqrt{9 + 16} = 5$ m.
Using similar triangles: $\frac{x}{y} = \frac{1}{3} \Rightarrow \frac{x}{5} =$
$\frac{1}{3} \Rightarrow 3x = 5\cdot 1.$ $\boldsymbol{x = \frac{5}{3}}$ **m**.

3. **(a)** Make a right triangle with **sides 1 and 3**.
$d = \sqrt{1^2 + 3^2} = \sqrt{10}.$

(b) Make a right triangle with **sides 1 and 4**.
$d = \sqrt{1^2 + 4^2} = \sqrt{17}.$

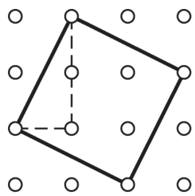
4. Answers may vary. Any polygon with interior vertices would have a greater perimeter than the greatest square. For example:



where $P = 13 + 2\sqrt{2} + \sqrt{5} \approx 18.1 > 12.$

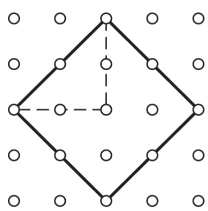
5. The sides of a square made in this fashion (i.e. the sides not unit numbers) will always be hypotenuses of right triangles [see part (a)]. As such, the relationship $s^2 = a^2 + b^2$ must hold, with a and b whole numbers (i.e. they are spaces between dots). Only number representable in this way can be areas of squares, or s^2.

(a) $s^2 = 1^2 + 2^2 = 5.$



(b) **Not possible.** No sums of squares of whole numbers equals 7.

(c) $s^2 = 2^2 + 2^2 = 8.$



(d) **Not possible.** No sum of squares of whole numbers equals 14.

(e) **Not possible.** No sum of squares of whole numbers equals 15.

6. **Yes**. The diagonal opening of the door, in inches, is $\sqrt{78^2 + 36^2} = \sqrt{7380} \approx 85.9$. If the plywood is turned so that the 7 ft dimension goes through the door diagonally, it will fit with 1.9 inches to spare (assuming a reasonable plywood thickness).

7. For the answer to be yes, the numbers must satisfy the Pythagorean theorem.

(a) **Yes**. $\left(\frac{5}{2}\right)^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{4}{2}\right)^2.$

(b) **Yes**. $(\sqrt{5})^2 = (\sqrt{2})^2 + (\sqrt{3})^2.$

(c) **Yes**. $30^2 = 18^2 + 24^2.$

8. Form a right triangle with a diameter (twice the radius, or 4 inches), height (10 inches), and length of spaghetti (the hypotenuse) that just fits. $\ell^2 = 4^2 + 10^2 \Rightarrow \ell = \sqrt{16 + 100} = \sqrt{116} = \sqrt{4 \cdot 29} = 2\sqrt{29}$, or **about 10.77 inches**.

9. Let d be the distance along the highway intersected by a 6.1 mile circle with center C. $\left(\frac{1}{2}d\right)^2 + 3^2 = 6.1^2 \Rightarrow \frac{1}{4}d^2 = 6.1^2 - 3^2 \Rightarrow d^2 \approx 4(37.21 - 9)$ $\Rightarrow d = \sqrt{112.84}$, or **about 10.6 mi**.

10. $\ell^2 = 3^2 + 1^2 \Rightarrow \ell = \sqrt{9 + 1} = \sqrt{10}$, or **about 3.16 m**.

11. Label the lower right corner D. Then $BD = CD = d$; $d^2 + d^2 = 12^2 \Rightarrow 2d^2 = 144 \Rightarrow d = \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2}$. Each side $= 2d = 12\sqrt{2}, \approx 16.97$ in., or **17 in.** to the nearest $\frac{1}{10}$ in.

12. **(a)** Let h be the altitude of the triangle.

$$h^2 + \left(\frac{s}{2}\right)^2 = s^2 \Rightarrow$$
$$h^2 = s^2 - \frac{s^2}{4} = \frac{4s^2 - s^2}{4} \Rightarrow$$
$$h = \sqrt{\frac{3s^2}{4}} = \frac{s\sqrt{3}}{2}.$$
$$A = \frac{1}{2} \cdot s \cdot \frac{s\sqrt{3}}{2} = \mathbf{\frac{s^2\sqrt{3}}{4}}.$$

(b) The triangle is isosceles with height = base = s.
$$A = \frac{1}{2} \cdot s \cdot s = \mathbf{\frac{s^2}{2}}.$$

13. **(a)** Let d be the distance from the intersection of the diagonals to the vertical apex.
$d = \sqrt{5^2 - 2^2} = \sqrt{21}.$
The area of the top triangle is: $A = \frac{1}{2} \cdot (2 + 2) \cdot \sqrt{21} = 2\sqrt{21}$. There are two triangles, thus total area is
$2 \cdot 2\sqrt{21} = \mathbf{4\sqrt{21}\ cm^2}$.

(b) $d = \sqrt{5^2 - 1^2} = \sqrt{24} = 2\sqrt{6}$. The area of each triangle is: $A = \frac{1}{2} \cdot 2 \cdot 2\sqrt{6} = 2\sqrt{6}$, thus the total of the two triangles is:
$A = 2 \cdot 2\sqrt{6} = \mathbf{4\sqrt{6}\ cm^2}$.

14. Draw radius OK to form a 30°-60°-90° right triangle, ΔOKC. $OC = \frac{1.3}{2} = 0.65\,\text{m}$,
$CK = \frac{0.65}{\sqrt{3}}\,\text{m}$, and
$OK = OB = 2\left(\frac{0.65}{\sqrt{3}}\right) \approx 0.75\,\text{m}.$
$AB = 2 \cdot OB$, or **about 1.5 m**.

15. The distance from home plate to third base is the same as the distance from home plate to first base (90 feet).The third base - home plate - first base triangle is a 45°-45°-90° right triangle with 90-feet legs. Then the distance from third base to first base $= \sqrt{90^2 + 90^2} = \sqrt{90^2(1+1)} = 90\sqrt{2}$, or **about 127.28 feet**.

16. $\Delta ACD \sim \Delta ABC$, so $\frac{AC}{AB} = \frac{AD}{AC} \Rightarrow \frac{b}{c} = \frac{x}{b}$.

$b^2 = cx$.

$\Delta CBD \sim \Delta ABC$, so $\frac{AB}{CB} = \frac{CB}{DB} \Rightarrow \frac{c}{a} = \frac{a}{y}$.

$a^2 = cy$.

Therefore $a^2 + b^2 = cx + cy = c(x+y) = c \cdot c = c^2$.

17. The outside quadrilateral is a square of area c^2. The inside quadrilateral is a square of area $(b-a)^2 = b^2 - 2ab + a^2$.

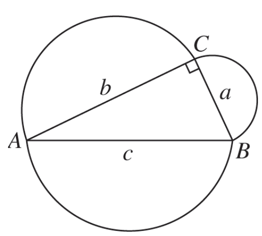
Each of the four triangles have are $\frac{1}{2}ab$.

Thus $c^2 = (b^2 - 2ab + a^2) + 4\left(\frac{1}{2}ab\right)$;

$c^2 = b^2 - 2ab + a^2 + 2ab$; so

$c^2 = a^2 + b^2$.

18. **Yes**. See below:



Let $c = AB$, $a = BC$, and $b = AC$. The area of the semicircle with diameter a, for example, is $\frac{1}{2}\pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{8}$. Similarly, the area of the semicircle of the other leg is $\frac{\pi b^2}{8}$.. So the sum of the areas of the semicircles on the legs is $\frac{\pi a^2}{8} + \frac{\pi b^2}{8}$. Now, note that the area of the semicircle on the hypotenuse is $\frac{\pi c^2}{8}$. Since $a^2 + b^2 = c^2$, in the right triangle, we have $\frac{\pi a^2}{8} + \frac{\pi b^2}{8} = \frac{\pi c^2}{8} \Rightarrow$ the Pythagorean relationship.

19. (*i*) $AB = \sqrt{(^-4 - 0)^2 + (^-3 - 0)^2} = \sqrt{16 + 9} = 5$.

(*ii*) $BC = \sqrt{(^-5 - ^-4)^2 + (0 - ^-3)^2} = \sqrt{1 + 9} = \sqrt{10}$.

(*iii*) $CA = \sqrt{(^-5 - 0)^2 + (0 - 0)} = \sqrt{25 + 0} = 5$.

Perimeter $= 5 + 5 + \sqrt{10} = \mathbf{10 + \sqrt{10}}$.

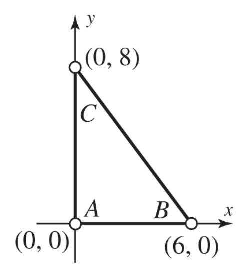
20. (*i*) $AB = \sqrt{(1 - ^-2)^2 + (^-1 - ^-5)^2} = \sqrt{9 + 16} = 5$.

(*ii*) $BC = \sqrt{(5 - 1)^2 + (2 - ^-1)^2} = \sqrt{16 + 9} = 5$.

(*iii*) $AC = \sqrt{(^-2 - 5)^2 + (^-5 - 2)^2} = \sqrt{49 + 49} = 7\sqrt{2}$.

$AB = BC$, so the triangle is isosceles.

21. Given the triangle:



(a) (*i*) Midpoint of $\overline{AC}$ is (0, 4). The perpendicular bisector is on $\boldsymbol{y = 4}$.

(*ii*) Midpoint of $\overline{AB}$ is (3, 0). The perpendicular bisector is on $\boldsymbol{x = 3}$.

(*iii*) Midpoint of $\overline{BC}$ is $\left(\frac{6+0}{2}, \frac{0+8}{2}\right) = (3,4)$.

$m_{BC} = \frac{0-8}{6-0} = \frac{^-4}{3}$, thus $m_\perp = \frac{3}{4}$.

Then $4 = \frac{3}{4}(3) + b \Rightarrow b = \frac{7}{4}$.

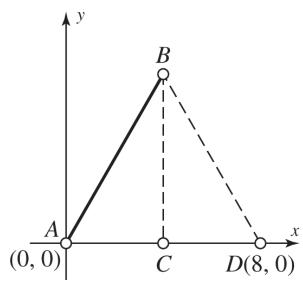
The perpendicular bisector is on $\boldsymbol{y = \frac{3}{4}x + \frac{7}{4}}$.

(b) The perpendicular bisectors meet where the y-coordinates are equal, or $4 = \frac{3}{4}x + \frac{7}{4} \Rightarrow x = 3$. If $x = 3$, $y = \frac{3}{4}(3) + \frac{7}{4} = 4$. The coordinates are **(3, 4)**, the midpoint of $\overline{BC}$.

(c) The center of the circumcircle is the point of intersection of the perpendicular bisectors, or (3, 4). The radius is the distance between (3, 4) and any vertex. Choose point A. Then $d = \sqrt{(3-0)^2 + (4-0)^2} = 5$. Thus $\boldsymbol{r = 5}$.

(d) **(3, 4)**. See part (b).

22. Given the triangle, draw and label it as follows:



$AB^2 = AC^2 + BC^2 \Rightarrow$
$BC = \sqrt{8^2 + 4^2} = \sqrt{48} = 4\sqrt{3}$.
Thus the third vertex is at $\mathbf{(4, 4\sqrt{3})}$ **or** $\mathbf{(4, {}^-4\sqrt{3})}$.

23. Given that the equation of a circle with center (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$:

(a) $h = 0$ and $k = 0 \Rightarrow (x-0)^2 + (y-0)^2 = (\sqrt{5})^2$, or $\boldsymbol{x^2 + y^2 = 5}$.

(b) $h = {}^-6$ and $k = {}^-7 \Rightarrow (x - {}^-6)^2 + (y - {}^-7)^2 = 6^2$, or $\boldsymbol{(x+6)^2 + (y+7)^2 = 36}$.

24. **(a)** If $(x-3)^2 + (y - {}^-2)^2 = 3^2$, then the center of the circle is $\mathbf{(3, {}^-2)}$ and the radius is **3**.

(b) $3x^2 + 3y^2 = 9 \Rightarrow x^2 + y^2 = 3$. Then if $(x-0)^2 + (y-0)^2 = (\sqrt{3})^2$, then the center of the circle is **(0, 0)** and the radius is $\boldsymbol{\sqrt{3}}$.

25. Both circles have the same center at $({}^-4, 8)$ and radii of 3 and 8, respectively.

(*i*) The area of the larger circle is $\pi(8^2) = 64\pi$.

(*ii*) The area of the smaller circle is $\pi(3^2) = 9\pi$. The area between the circles is the difference in area between the two, or $(64 - 9)\pi = \boldsymbol{55\pi}$ **units^2**.

26. The intersection of the radius of the circle with the point of tangency is the right angle APO, so ΔAPO is a right triangle.

Thus $AP = \boldsymbol{\sqrt{d^2 - r^2}}$.

27. **(a)** The distance from second base to home base is the diagonal of the square. Using Pythagorean Theorem:
$90^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{16200} = 90\sqrt{2}$ ft ≈ 127 ft.

(b) The distance from halfway between second and third base and first base can be found using Pythagorean Theorem:
$45^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{10125} = 45\sqrt{5}$ ft ≈ 101 ft.

(c) The distance from halfway between second and third base and home base can be found using Pythagorean Theorem:
$45^2 + 90^2 = d^2 \Rightarrow$
$d = \sqrt{10125} = 45\sqrt{5}$ ft ≈ 101 ft.

28. One side of the rhombus creates a right triangle with half of each of the diagonals. One side is $s^2 = 3^2 + 5^2 \Rightarrow s = \sqrt{34} \approx 5.8$ in.

29. Similar triangles will have sides $5k, 12k,$ and $13k$ and $(5k)^2 + (12k)^2 = (13k)^2$.

30. The area of square $CDOE = x^2$. The diagonal of square $CDOE$ is $r^2 = x^2 + x^2 \Rightarrow r = \sqrt{2x^2}$. The area of square $BFAO$ is $r^2 = \left(\sqrt{2x^2}\right)^2 = 2x^2$. The area of square $CDOE$ is half the area of square $BFAO$.

31. The area of the trapezoid is $\frac{(15+25)12}{2} = (40)(6) = 240 \text{ in.}^2$. The radius of the circle with same area is

$\pi r^2 = 240 \Rightarrow r = \sqrt{\frac{240}{\pi}}$ in.

32. Draw altitudes $\overline{BE}$ and $\overline{DF}$ of triangles BCP and DCP respectively. $\triangle ABE \cong \triangle CDF$ by AAS

$A: \angle E \cong \angle F \cong 90^\circ; A: \angle BAE \cong \angle DCF; S: \overline{AB} \cong \overline{CD}$

Thus altitudes $\overline{BE} \cong \overline{DF}$. Because $\overline{CP}$ is a base of $\triangle BCP$ and $\triangle DCP$, and because the altitudes are the same, the areas must be equal.

Mathematical Connections 14-3: Review Problems

18. Change all values to meters. Then $0.032 \text{ km}^2 = 32000 \text{ m}^2 > 3.2 \text{ m}^2 > 322 \text{ cm}^2 = 0.0322 \text{ m}^2 > 3020 \text{ mm}^2 = .00302 \text{ m}^2$.

19. **(a)** Change all measurements to mm and draw horizontal lines to form three rectangles: 75 mm by 25 mm, 25 mm by 30 mm, and 35 mm by 20 mm. The respective areas are

$75 \cdot 25 = 1875 \text{ mm}^2$,

$25 \cdot 30 = 750 \text{ mm}^2$, and $35 \cdot 20 = 700 \text{ mm}^2$.

$1875 + 750 + 700 = 3325 \text{ mm}^2 = \mathbf{33.25 \text{ cm}^2}$.

(b) $A = \frac{1}{2} \cdot 10 \cdot 6 = \mathbf{30 \text{ cm}^2}$.

(c) Change 600 cm to 6 m. $A = \frac{1}{2} \cdot (6 + 10) \cdot 4 = \mathbf{32 \text{ m}^2}$.

20. In each case: circumference $= \pi d$ or $2\pi r$; area $= \pi r^2$. If given A, $r = \sqrt{\frac{A}{\pi}}$ and if given C, $r = \frac{C}{2\pi}$.

	Radius	Diameter	Circumference	Area
(a)	5 cm	**10 cm**	$\mathbf{10\pi}$ **cm**	$\mathbf{25\pi \text{ cm}^2}$
(b)	**12 cm**	24 cm	$\mathbf{24\pi}$ **cm**	$\mathbf{144\pi \text{ cm}^2}$
(c)	$\mathbf{\sqrt{17}}$ **m**	$\mathbf{2\sqrt{17}}$ **m**	$\mathbf{2\pi\sqrt{17}}$ **m**	$17\pi \text{ m}^2$
(d)	**10 cm**	**20 cm**	20π cm	$\mathbf{100\pi \text{ cm}^2}$

21. Given $C = 10$, $2\pi r = 10 \Rightarrow r = \frac{5}{\pi}$. $A = \pi \cdot \left(\frac{5}{\pi}\right)^2 = \mathbf{\frac{25}{\pi} \text{ m}^2}$.

22. Mary paid $\frac{\$3.49}{1 \text{ yd}}$ and Samuel paid $\frac{\$3.76}{1 \text{ m}} \cdot \frac{1 \text{ m}}{1.09361 \text{ yd}} = \frac{\$3.44}{1 \text{ yd}}$. Samuel got the better buy by \$0.05 per yard.

23. The U.S. penny has a diameter of 0.75 in. = 19.05 mm. The circumference of the penny is $2\pi r = \pi d = \pi(19.05) \approx 59.85 \text{ mm} \approx 5.985 \text{ cm}$ or about 6 cm.

Assessment 14-4A: Surface Areas

1. (b) and (d) can form cubes. The other figures have one of the faces out of order.

2. Where SA represents surface area:

(a) SA of a cube $= 6e^2$ (where e is the length of each edge). $SA = 6(4 \text{ cm})^2 = \mathbf{96 \text{ cm}^2}$.

(b) SA of a right cylinder $= 2\pi r^2 + 2\pi rh$ (where r is the radius of the top and bottom circles and h is the height of the cylinder). $SA = 2\pi(6 \text{ cm})^2 + 2\pi(6 \text{ cm})(12 \text{ cm}) = 72\pi + 144\pi = \mathbf{216\pi \text{ cm}^2}$.

(c) SA of a right rectangular prism is the sum of the lateral surface area and the area of the bases. The lateral surface area is ph (where p is the lateral perimeter), or $2\ell + 2w$, and h is the height. The sum of the bases is $2B$ (where B is length times width. $SA = (2 \cdot 8 \text{ cm} + 2 \cdot 5 \text{ cm}) \cdot 6 + 2(8 \text{ cm} \cdot 5 \text{ cm}) = 26 \cdot 6 + 2 \cdot 40 = 156 + 80 = \mathbf{236 \text{ cm}^2}$.

(d) SA of a sphere $= 4\pi r^2$ (where r is the radius of the sphere). $SA = 4\pi(4 \text{ cm})^2 = \mathbf{64\pi \text{ cm}^2}$.

(e) SA of a right circular cone $= \pi r^2 + \pi r\ell$ (where r is the radius of the base and ℓ is the slant height from any point of the base to the vertex of the cone). To find ℓ, use the Pythagorean theorem, i.e., $\ell = \sqrt{r^2 + h^2}$ (where h is the height of the vertex above the base). $SA = \pi(6 \text{ cm})^2 + \pi(6 \text{ cm})\sqrt{(6 \text{ cm})^2 + (8 \text{ cm})^2} = 36\pi + 60\pi = \mathbf{96\pi\ cm^2}$.

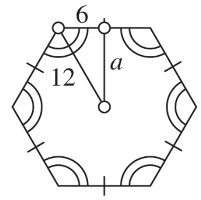
3. Area of walls $= 2(6 \text{ m})(2.5 \text{ m}) + 2(4 \text{ m})(2.5 \text{ m}) = 50 \text{ m}^2$. Paint needed is $\left(\frac{50 \text{ m}^2}{1}\right) \cdot \left(\frac{1 \text{ L}}{20 \text{ m}^2}\right) =$ **2.5 liters**, so buy 3 liters.

4. Change all units to mm. The ring then has an inner radius of 20 mm and a height of 30 mm. The outer ring has a radius of 22 mm and a height of 30mm.
$LSA = 2\pi rh = 2\pi(22)(30) = 1320\pi \text{ mm}^2$.
The inner ring has a radius of 20 mm and a height of 30 mm. $LSA = 2\pi(20)(30) = 1200\pi \text{ mm}^2$.
The area of the top and bottom rings is the area of a circle with radius 22 mm minus the area of a circle radius 20 mm. There are two base rings.
$A = 2\pi(r_{outer}^2 - r_{inner}^2) = 2\pi(22^2 - 20^2) = 168\pi \text{ mm}^2$.
Total $SA = (1320 + 1200 + 168)\pi \text{ mm}^2 = \mathbf{2688\pi\ mm^2}$.

5. $SA = 4\pi(6370 \text{ km})^2 = \mathbf{162{,}307{,}600\pi\ km^2}$.

6. $SA_{large\ cube} = 6e^2 = 6(6 \text{ cm})^2 = 216 \text{ cm}^2$.
$SA_{small\ cube} = 6(4 \text{ cm})^2 = 96 \text{ cm}^2$.
Ratio of surface areas $= 96 : 216 = \mathbf{4 : 9}$.

7. SA of a right pyramid is $B + \frac{1}{2}p\ell$, where B is the area of the base, p is the perimeter of the base, and ℓ is the slant height from the base to the apex. $B = \frac{1}{2}ap$, where a is the apothem (the height of the triangles forming the base of a regular polygon). Since a hexagon is composed of equilateral triangles about the centre, the distance from each vertex to the center is the same as the length of each edge (see below).



Regular hexagon

$a = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}$ and $B = \frac{1}{2}(6\sqrt{3})(6 \cdot 12) = 216\sqrt{3}$.
The apothem $= 6\sqrt{3}$ and altitude $= 9$.
$\ell = \sqrt{9^2 + (6\sqrt{3})^2} = \sqrt{189} = 3\sqrt{21}$.
Thus $SA = 216\sqrt{3} + \frac{1}{2}(6 \cdot 12)(3\sqrt{21}) = \mathbf{(216\sqrt{3} + 108\sqrt{21})\,m^2}$.

8. $LSA = 2\pi rh = \pi dh$
$= \pi\left(2\frac{5}{8} \text{ in.}\right)(4 \text{ in.}) = 10.5\pi \text{ in.}^2$,
or **about 32.99 in.²**.

9. **(*i*)** The top could be $(1 \cdot 88) \text{ cm}^2, (2 \cdot 44) \text{ cm}^2, (4 \cdot 22) \text{ cm}^2$, or $(8 \cdot 11) \text{ cm}^2$.

(*ii*) One side could be $(1 \cdot 32) \text{ cm}^2, (2 \cdot 16) \text{ cm}^2$ or $(4 \cdot 8) \text{ cm}^2$. The other side could be $(1 \cdot 44) \text{ cm}^2, (2 \cdot 22) \text{ cm}^2$, or $(4 \cdot 11) \text{ cm}^2$.

The only dimensions shared by two groups each are $(8 \cdot 11) \text{ cm}^2, (4 \cdot 8) \text{ cm}^2$, and $(4 \cdot 11) \text{ cm}^2$. Thus the box is **4 cm by 8 cm by 11 cm**.

10. **(a)** Lateral surface area is proportional to slant height. If slant height is tripled, lateral surface area is **tripled**.

(b) Lateral surface area is proportional to the radius of the base. If radius is tripled, lateral surface area is **tripled**.

(c) Lateral surface area is proportional to the product of slant height and base radius. If both slant height and radius are tripled, lateral surface area is **multiplied by 9**.

11. $SA = B + \frac{1}{2}p\ell$, where $B = 100$, $p = 4\sqrt{100} = 40$ (i.e., the length of each of the four sides of the square base is the square root of the area), and $\ell = \sqrt{20^2 + 5^2} = \sqrt{425} = 5\sqrt{17}$.

$SA = \left(100 + \frac{1}{2} \cdot 40 \cdot 5\sqrt{17}\right)$
$= \mathbf{(100 + 100\sqrt{17})\ cm^2}$,
or about 512.3 cm^2.

12. $\ell = 1.5$ m.
Circumference of base $= \frac{240}{360}[2\pi(1.5)]$ $= 2\pi$ m. $2\pi = 2\pi r$ (where r is the radius of the base of the cone), thus $r = 1$ m.

(a) Lateral surface area $= \pi r\ell = \pi(1)(1.5) =$ $\mathbf{1.5\pi\ m^2}$.

(b) $SA = \pi r^2 + \pi r\ell = \pi(r^2 + r\ell) =$ $\pi(1^2 + 1 \cdot 1.5) = \mathbf{2.5\pi\, m^2}$.

13. **(a)** $2\pi r = 6\pi \Rightarrow \frac{6\pi}{2\pi} =$ **3 units**.

(b) $\ell =$ sector radius $=$ **5 units**.

(c) $h = \sqrt{\ell^2 - r^2} = \sqrt{25 - 9} =$ **4 units**.

(d) $C_{full\ circle} = 2\pi(5) = 10\pi$ units. Then $m(\angle_{sector}) = \frac{6\pi}{10\pi}(360°) = \mathbf{216°}$.

14. $C = \pi d = 2.5\pi$ in. Then $SA = 4(2.5\pi) = \mathbf{10\pi\ in^2}$.

15. Surface area is proportional to the square of the radius, thus their ratio would be $\left(\frac{1}{2}\right)^2 = \mathbf{\frac{1}{4}}$.

16. The earth's surface area is approximately $4\pi\left(\frac{13000}{2}\right)^2 = 169{,}000{,}000\pi\text{ km}^2$. 70% of this area is $.7(169{,}000{,}000\pi) = \mathbf{118{,}300{,}000\pi\ km^2} \approx$ $371{,}650{,}410.9\text{ km}^2$.

17. $A_{cube} = 6s^2$, where s is the length of a side. Thus $s = \sqrt{\frac{A}{6}}$. The ratio of the edges of the cubes would be $\sqrt{\frac{\left(\frac{64}{6}\right)}{\left(\frac{36}{6}\right)}} = \sqrt{\frac{16}{9}} = \mathbf{\frac{4}{3}}$.

18. **(a)** $SA = 6s^2$ (where s is the length of a side of the cube). Then $s = \sqrt{\frac{SA}{6}} = \sqrt{\frac{10{,}648\text{ cm}^2}{6}}$, or about **42.13 cm**.

(b) Consider the base square with sides s. Based on the properties of 45°-45°-90° triangles, the length of the diagonal of the base is $s\sqrt{2}$. Using the Pythagorean theorem, take the length of the diagonal and the length of a vertical side, s, to find $d = \sqrt{s^2 + (s\sqrt{2})^2} =$ $\sqrt{3s^2} = s\sqrt{3} = \sqrt{\frac{10648\text{ cm}^2}{6}} \cdot \sqrt{3} =$ $\sqrt{5324\text{ cm}^2} \approx$ **72.97 cm**.

19. If h is the height of the completed cone, $\frac{h}{100} = \frac{h-40}{60} \Rightarrow h = 100$ cm.
Slant height $= \sqrt{100^2 + 100^2} = 100\sqrt{2}$ cm.
Slant height of the missing piece $= \sqrt{60^2 + 60^2} =$ $60\sqrt{2}$ cm.

SA of the total cone $= \pi r^2 + \pi r\ell$
$= \pi(100^2) + \pi(100)(100\sqrt{2})$
$= 10{,}000(1 + \sqrt{2})\pi\text{ cm}^2$.

Cutting off the top eliminates the lateral surface area $\pi r\ell = \pi(60)(60\sqrt{2}) = 3600\sqrt{2}\pi\text{ cm}^2$ and adds the circular area $\pi r^2 = \pi(60^2) = 3600\pi\text{ cm}^2$. Thus total $SA = 10{,}000(1 + \sqrt{2})\pi - 3600\sqrt{2}\pi +$ $3600\pi = \mathbf{(6400\sqrt{2} + 13{,}600)\pi\ cm^2}$.

20. The surface area of the square pyramid is the area of the base $10^2 = 100\text{ cm}^2$ plus the area of the lateral faces. If the height of the pyramid is 10 cm then the slant height of the lateral face is $10^2 + 5^2 = h^2 \Rightarrow h = \sqrt{125} = 5\sqrt{5}$ cm and the area is $\frac{1}{2}10 \cdot 5\sqrt{5} = 25\sqrt{5}\text{ cm}^2$ and the total surface area is $100 + 4 \cdot 25\sqrt{5} = 100 + 100\sqrt{5}\text{ cm}^2$.

The surface area of the right circular cone is the area of the base $\pi 5^2 = 25\pi \text{ cm}^2$ and the area of the "lateral face" $\pi(5)\cdot 5\sqrt{5} = 25\pi\sqrt{5} \text{ cm}^2$. The total surface area of the cone is $25\pi + 25\pi\sqrt{5}$. The numerical difference in the surface area of the square pyramid and the cone is
$100 + 100\sqrt{5} - (25\pi + 25\pi\sqrt{5}) \approx 69.44 \text{ cm}^2$.

21. The surface area of a hexagon minus the bases is the perimeter times the height. The perimeter of the hexagon is $4\cdot 6 = 24$ cm and the height is 48 cm. The surface area of the inside and outside is $2\cdot 24\cdot 48 = 2304 \text{ cm}^2$.

22. Each glass plate has a surface area of
$$2(48\cdot 96) + 2(48\cdot \frac{1}{4}) + 2(96\cdot \frac{1}{4})$$
$$= 9216 + 24 + 48 = 9288 \text{ in.}^2$$
and the total surface area is
$$54\cdot 9288 = 501{,}552 \text{ in.}^2 = 3483 \text{ ft}^2.$$

Assessment 14-4B

1. (a) and (b) can form rectangular prisms. (c) when folded would be missing some faces.

2. **(a)** *SA* of a right square pyramid $= B + 4\left(\frac{1}{2}b\ell\right)$ (where 4 is the number of faces, B is the area of the base, b is the length of one side of the base, and ℓ is slant height from the base to the apex).
$$SA = (5\text{ cm})^2 + 4\left(\tfrac{1}{2}\right)(5\text{ cm})(6.5\text{ cm})$$
$$= 25 + 65 = \mathbf{90\ cm^2}.$$

(b) ***(i)*** Hemispherical $SA = \frac{1}{2}(4\pi r^2) = \frac{1}{2}(4\pi)(10\text{ ft})^2 = 200\pi \text{ ft}^2$.

(ii) Cylindrical $SA = \frac{1}{2}(2\pi r^2) + 2\pi rh = \frac{1}{2}(2\pi)(10\text{ ft})^2 + 2\pi(10\text{ ft})(60\text{ ft}) = 100\pi + 1200\pi = 1300\pi \text{ ft}^2$.

Total $SA = 200\pi + 1300\pi = \mathbf{1500\pi\ ft^2}$.

(c) ***(i)*** Conical $SA = \pi r\ell = \pi(4\text{ cm})\sqrt{(4\text{ cm})^2 + (8\text{ cm})^2} = 4\pi\sqrt{80} = 4\pi(4\sqrt{5}) = 16\pi\sqrt{5}\text{ cm}^2$.

(ii) Hemispherical $SA = \frac{1}{2}(4\pi r^2) = \frac{1}{2}(4\pi)(4\text{ cm})^2 = 32\pi\text{ cm}^2$.

Total $SA = 16\pi\sqrt{5} + 32\pi = \mathbf{(32 + 16\sqrt{5})\pi\ cm^2}$.

3. Area of walls $= 2(8\text{ m})(2.5\text{ m}) + 2(5\text{ m})(2.5\text{ m}) = 65\text{ m}^2$. Paint needed is $\left(\frac{65\text{ m}^2}{1}\right)\cdot\left(\frac{1\text{ L}}{20\text{ m}^2}\right) =$ **3.25 liters**, so buy 4 liters.

4. **(a)** Lateral surface area$_{first}$ $= 2\pi rh$
$= 2\pi(2\text{ m})(6\text{ m})$
$= 24\pi\text{ m}^2$.

Lateral surface area$_{second}$ $= 2\pi(6\text{ m})(2\text{ m})$
$= 24\pi\text{ m}^2$.

The lateral *SA*'s are **equal**.

(b) $SA_{first} = 2\pi r^2 + 2\pi rh$
$= 2\pi(2\text{ m})^2 + 2\pi(2\text{ m})(6\text{ m})$
$= (8 + 24)\pi\text{ m}^2 = 32\pi\text{ m}^2$.

$SA_{second} = 2\pi(6\text{ m})^2 + 2\pi(6\text{ m})(2\text{ m})$
$= (72 + 24)\pi\text{ m}^2 = 96\pi\text{ m}^2$.

The cylinder with **radius 6 m** has greater total surface area.

5. **(a)** *SA* of a sphere is proportional to the square of the radius. If radius is doubled, surface area is **multiplied by 4**.

(b) *SA* of a sphere is proportional to the square of the radius. If radius is tripled, surface area is **multiplied by 9**.

6. For any box, if all dimensions have the same percentage change the change in area is the square of the change in a side. I.e., if the dimensions of a box are w, h, and ℓ its surface area is $2wh + 2h\ell + 2w\ell = 2(wh + h\ell + w\ell)$.

If these dimensions are doubled to $2w$, $2h$, and 2ℓ, then $SA = 2(2w\cdot 2h + 2h\cdot 2\ell + 2w\cdot 2\ell) = 8(wh + h\ell + w\ell)$, which is 2^2 times the original surface area.

(a) Surface area is **multiplied by 4**.

(b) Surface area is **multiplied by 9**.

(c) Surface area is **multiplied by k^2**.

7. Slant heights are given by $s = \sqrt{\left(2\frac{1}{2}\right)^2 + 3^2} = \sqrt{\frac{61}{4}} = \frac{\sqrt{61}}{2}$ ft.

Area of each side $= 6\left(\frac{\sqrt{61}}{2}\right)$ ft^2. Area of bottom $= 5 \cdot 6 = 30$ ft^2. Area of each end $= \frac{1}{2} \cdot 5 \cdot 3 = 7\frac{1}{2}$ ft^2.

Amount of material $=$ total area $= 2\left[6\left(\frac{\sqrt{61}}{2}\right)\right] + 30 + 2\left(7\frac{1}{2}\right) = 12\left(\frac{\sqrt{61}}{2}\right) + 45$, or **about 91.86 ft^2**.

8. Lateral surface area $=$ area of paper square $= 10 \cdot 10 = 100$ cm^2.

Circumference of the cylinder $=$ length of a side of the paper $= 10$ cm. $10 = 2\pi r \Rightarrow r(\text{radius}) = \frac{10}{2\pi} = \frac{5}{\pi}$ cm.

Top plus bottom areas $= 2\pi\left(\frac{5}{\pi}\right)^2 = \frac{50}{\pi}$ cm^2.

$SA = \left(100 + \frac{50}{\pi}\right)$ cm^2, or **about 115.9 cm^2**.

9. **(a)** SA of the given structure is 40 units^2 (i.e., each face has surface area of 1 unit^2 and there are 40 faces showing). Adding one cube could increase surface area by 4 units^2 (i.e., adding five faces but eliminating one). Thus the maximum would be **44 units^2**.

(b) Placing a cube in the center hole eliminates four faces while adding only two, for **38 units^2**.

(c) **Yes**. Answers may vary. E.g., arrange five cubes in the shape of a *C*. Filling the hole with a sixth cube would add no surface area.

10. **(a)** Let $A_o = 2\pi rh$ be the lateral surface area of the original cylinder. If we double the radius the area is now $2\pi(2r)h = 2(2\pi rh) = 2A_o$, which is **doubled**.

(b) If we double the height the area is now $2\pi r(2h) = 2(2\pi rh) = 2A_o$, which is also **doubled**.

11. The surface area is the sum of the area of the four triangle faces and the area of the base. It is $4\left(\frac{1}{2} \cdot \sqrt{169} \cdot 13\right) + 169 = 2 \cdot 169 + 169 = 3 \cdot 169 =$ **507 cm^2**.

12. **(a)** **(*i*)** The figure will be a **right circular cone** with base radius 10 cm and height 20 cm.

(*ii*) $SA = \pi r^2 + \pi r\ell = \pi(10^2) + \pi(10)\left(\sqrt{20^2 + 10^2}\right) = 100\pi + 10\pi(10\sqrt{5})$ $= \mathbf{100\pi(1 + \sqrt{5})\ cm^2}$.

(b) **(*i*)** The figure will be a **right circular cylinder** with base radius 15 cm and height 30 cm.

(*ii*) $SA = 2\pi r^2 + 2\pi rh = 2\pi(15^2) + 2\pi(15)(30) = 450\pi + 900\pi = \mathbf{1350\pi\ cm^2}$.

(c) **(*i*)** The figure will be a **truncated right circular cone,** with large end radius 25 cm and small end radius 15 cm.

(*ii*) Extend the 35 cm line to form a right triangle and use similar triangles (where x is the hypotenuse of the extended triangle). $\frac{x-35}{15} = \frac{x}{25} \Rightarrow 25(x - 35) = 15x \Rightarrow x = 87.5$ cm.
The triangle, when rotated, will form a cone with slant height 87.5 cm and radius 25 cm.

$$\begin{aligned} SA &= \pi r^2 + \pi r\ell \\ &= \pi(25^2) + \pi(25)(87.5) \\ &= 2812.5\pi \text{ cm}^2. \end{aligned}$$

Cutting off the top eliminates the lateral surface area $\pi r\ell = \pi(15)(87.5 - 35) = 787.5\pi$ cm^2 and adds the circular area $\pi r^2 = \pi(15^2) = 225\pi$ cm^2.
Thus

$$\begin{aligned} SA &= (2812.5 - 787.5 + 225)\boldsymbol{\pi} \\ &= \mathbf{2250\pi\ cm^2}. \end{aligned}$$

13. **(a)** $SA = n\left(\frac{1}{2}b\ell\right) + B$, where n is the number of faces $= 4$, b is the length of each side of the base $= 10$, ℓ is slant height $= \sqrt{5^2 + 10^2} = 5\sqrt{5}$, and B is the area of the base $= 10 \cdot 10 = 100$. $SA = 4\left(\frac{1}{2} \cdot 10 \cdot 5\sqrt{5}\right) + 100 = 100(\sqrt{5} + 1)\text{in.}^2$, or **about 323.6 in.^{2}**.

(b) $SA = \pi r^2 + \pi r\ell$, where $r = 5$ and $\ell = 5\sqrt{5}$. $SA = \pi(5^2) + \pi(5)(5\sqrt{5}) = 25(1 + \sqrt{5})\pi\ \text{in.}^2$, or **about 254.2 in.^{2}**.

14. The surface area of the tank is $2\pi(3)^2 + 2\pi(3)(8) = 18\pi + 48\pi = 66\pi \approx 207.3\ \text{ft}^2$.
Yes, the farmer will have enough paint.

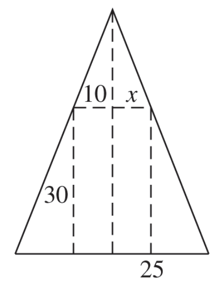
15. One way to address the problem is by noting that the ratio of lengths in the smaller cone to lengths in the larger cone is $\frac{2}{3}$. If we denote lengths in the smaller cone by a subscript of 1 and lengths in the larger cone by a subscript of 2, we have $\ell_1 = \frac{2}{3}\ell_2$. Thus, $S.A._1 = \pi r_1^2 + \pi r_1\ell_1 = \pi\left(\frac{2}{3}r_2\right)^2 + \pi\left(\frac{2}{3}r_2\right)\left(\frac{2}{3}\ell_2\right) = \frac{\mathbf{4}}{\mathbf{9}}\left[\pi r_2^2 + \pi r_2\ell_2\right]$.

16. Surface area of a sphere is $4\pi r^2$; i.e., surface area is proportional to the square of the radius. If the diameter of Jupiter, and thus its radius, is 11 times greater than that of Earth, the surface area will be multiplied by $11^2 =$ **121 times**.

17. $S.A._1 = 6(2\ \text{ft})^2 = 24\ \text{ft}^2$
$S.A._2 = 6(4\ \text{ft})^2 = 96\ \text{ft}^2$.
$24\ \text{ft}^2 : 96\ \text{ft}^2 =$ **1 : 4**.

18. SA of outer cube $= 6(12^2) = 864\ \text{ft}^2$. The SA of the faces of the section removed $= 2(3^2) = 18\ \text{ft}^2 \Rightarrow SA$ of exterior $= (864 - 18)\ \text{ft}^2 = 846\ \text{ft}^2$.
SA of interior $= 4(3)(12) = 144\ \text{ft}^2 \Rightarrow$ total $SA = (846 + 144)\ \text{ft}^2 =$ **990 ft^2**.

19. The cross section is shown below:



Using similar triangles: $\frac{10}{x} = \frac{40}{25} \Rightarrow x = 6.25$ cm (i.e., the radius of the cylinder).
Lateral surface area $= 2\pi rh = 2\pi(6.25)(30) =$ **375π cm^2**.

20. The surface area of a cylinder is $2\pi rh + 2\pi r^2$. The surface area of the cylinder with the base inscribed in the square is $2\pi(5 \cdot 10) + 2\pi(5^2) = 100\pi + 50\pi = 150\pi$. To find the radius of the base of the cylinder circumscribed about the square base, use Pythagorean Theorem: $5^2 + 5^2 = r^2 \Rightarrow r = \sqrt{50} \Rightarrow r = 5\sqrt{2}$. The surface area of the cylinder circumscribed about the square base is $2\pi 5\sqrt{2} \cdot 10 + 2\pi(5\sqrt{2})^2 = 100\pi\sqrt{2} + 100\pi$. The difference between the surface area of the cylinder with the base circumscribed about the square base and the cylinder with the base inscribed in the square base is

$$100\pi\sqrt{2} + 100\pi - 150\pi = 100\pi\sqrt{2} - 50\pi \approx 287\ \text{cm}^2$$

21. The surface area of a hexagon minus the bases is the perimeter times the height. The perimeter of the hexagon is $5 \cdot 6 = 30$ cm and the height is 60 cm. The surface area of the inside and outside is $2 \cdot 30 \cdot 60 = 3600\ \text{cm}^2$.

22. The height of the pyramid is 321 feet, the Pythagorean Theorem is used to find the height of the face of the triangle, which will be the hypotenuse of the triangle. This yields

$$a^2+b^2=c^2 \Rightarrow \left(\frac{591}{2}\right)^2+321^2=c^2; c=\sqrt{\left(\frac{591}{2}\right)^2+321^2}$$

Each triangular face has an area of

$$\frac{591}{2}\sqrt{\left(\frac{591}{2}\right)^2+321^2}\text{ ft}^2.$$

The four faces together have an area of

$$4\cdot\frac{591}{2}\sqrt{\left(\frac{591}{2}\right)^2+321^2}\approx 515{,}711\text{ ft}^2.$$

Mathematical Connections 14-4: Review Problems

15. **(a)** $10\text{ m}^2=\mathbf{100{,}000\,cm^2}$ $(1\text{ m}^2=10{,}000\text{ cm}^2)$.

(b) $13{,}680\text{cm}^2=\mathbf{1.368\,m^2}(10{,}000\text{ cm}^2=1\text{ m}^2)$.

(c) $5\text{ cm}^2=\mathbf{500\ mm^2}$ $(1\text{ cm}^2=100\text{ mm}^2)$.

(d) $2\text{ km}^2=\mathbf{2{,}000{,}000\,m^2}(1\text{ km}^2=1{,}000{,}000\text{ m}^2)$.

(e) $10^6\text{ m}^2=1{,}000{,}000\text{ m}^2=\mathbf{1\ km^2}$ $(1{,}000{,}000\text{ m}^2=1\text{ km}^2)$.

(f) $10^{12}\text{ mm}^2=\mathbf{10^6\ m^2}(1{,}000{,}000\text{ mm}^2=1\text{m}^2)$.

16. $d=\sqrt{10^2+20^2}=\sqrt{500}=\mathbf{10\sqrt{5}\ cm}$.

17. $\left(\frac{1}{2}d\right)^2+20^2=30^2\Rightarrow d=\mathbf{20\sqrt{5}\ cm}$.

18. **(a)** Change 0.6m to 60 cm.

Hypotenuse = $\sqrt{60^2+80^2}=100$ cm.

(i) Perimeter = 60 +80 + 100 = **240 cm.**

(ii) Area = $\frac{1}{2}(80)(6)=\mathbf{2400\ cm^2}$.

(b) Drawing an altitude to the end point of the top base forms a $45°-45°-90°$ isosceles triangle. The bottom base is 5 cm longer on each side than the top. Thus one leg of the isosceles triangle = 5cm. The height is also 5 cm. Use the Pythagorean Theorem to find the length of each leg of the trapezoid to be $\sqrt{5^2+5^2}=2\sqrt{2}$ cm.

(i) Perimeter = 20 + 10 + $2(5\sqrt{2})$ = $\left(\mathbf{30+10\sqrt{2}}\right)$ **cm.**

(ii) Area = $\frac{1}{2}(10+20)(5)=\mathbf{75\,cm^2}$.

19. Compute the area P of ΔACD in two ways. Let s be the length of a side of rhombus $ABCD$ and d be the length of the diagonal $\overline{BD}$ (which is perpendicular to $\overline{AC}$). Let F be the point where $\overline{BD}$ and $\overline{AC}$ intersect.

$$P=\frac{1}{2}(s)(AE)=12s$$

$$P=\frac{1}{2}(AC)\left(\frac{d}{2}\right)=10d$$

So $d=\frac{6}{5}s$. Apply the Pythagorean Theorem in ΔAFD.

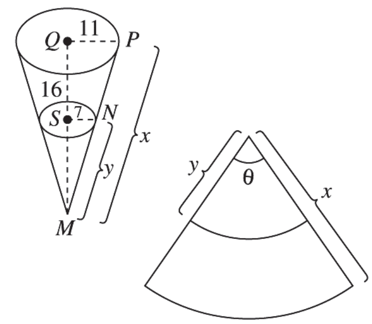
$$s^2=20^2+\left(\frac{d}{2}\right)^2\Rightarrow s^2=400+\left(\frac{3}{5}s\right)^2$$

$$\frac{16}{25}s^2=400\Rightarrow s=25$$

$$d=\frac{6}{5}\cdot 25=30$$

20. The diagonal of the square at the top of the cube is $\sqrt{1^2+1^2}=\sqrt{2}$ units. The diagonal of the cube is the hypotenuse of the right triangle created by the diagonal of any one of the faces and the edge of two adjacent faces. Thus, the diagonal of the cube has length $\sqrt{1^2+\sqrt{2}^2}=\mathbf{\sqrt{3}\ units}$.

21. The cone and the flattened region obtained by splitting the cone along a slant height are shown.



To construct the flattened ring we need to find x, y, and θ.

Because $\Delta MQP \sim \Delta MSN$, we have

$$\frac{16+MS}{MS} = \frac{11}{7} \Rightarrow 7(16+MS) = 11MS.$$

$$\Rightarrow 112 + 7MS = 11MS \Rightarrow 112 = 4MS \Rightarrow MS = 28$$

In ΔMSN, we have $28^2 + 7^2 = y^2 \Rightarrow y \approx 28.86$ cm.. In $\triangle PQM$, we have $11^2 + (16+28)^2 = x^2 \Rightarrow x \approx 45.35$ cm. To find θ, we roll the sector with radius y and central angle θ into the cone whose base is 7 cm and whose slant height is y. Notice the arc length (formula $2\pi r \cdot \frac{\theta}{360^\circ}$) and the circumference of base of the cone which is a circle with radius 7 cm are equal, hence we have

$$2\pi y \cdot \frac{\theta}{360} = 2\pi \cdot 7 \Rightarrow \theta \approx \frac{7 \cdot 360}{28.86} \approx 87^\circ 19'.$$

22. **(a)** $150 \text{ m} \cdot \frac{1 \text{ km}}{1000 \text{ m}} = 0.15 \text{ km}$

(b) $0.002 \text{ cm} \cdot \frac{10 \text{ mm}}{1 \text{ cm}} = 0.02 \text{ mm}$

(c) $1.44 \text{ yd}^2 \cdot \frac{9 \text{ ft}^2}{1 \text{ yd}^2} \cdot \frac{144 \text{ in.}^2}{1 \text{ ft}^2} = 1866.24 \text{ in.}^2$

(d) 1 in. = 2.54 cm

Assessment 14-5A: Volume, Mass and Temperature

1. **(a)** $\frac{8 \text{ m}^3}{1} \cdot \frac{(10 \text{ dm})^3}{1 \text{ m}^3} = \mathbf{8000} \text{ dm}^3$.

(b) $\frac{675000 \text{ m}^3}{1} \cdot \frac{(0.001 \text{ km})^3}{1 \text{ m}^3} = \mathbf{0.000675 \text{ km}^3}$.

(c) $\frac{7000 \text{ mm}^3}{1} \cdot \frac{(0.1 \text{ cm})^3}{1 \text{ mm}^3} = \mathbf{7} \text{ cm}^3$.

(d) $\frac{400 \text{ in.}^3}{1} \cdot \frac{1 \text{ yd}^3}{(36 \text{ in.})^3} \approx \mathbf{0.00857 \text{ yd}^3}$.

(e) $\frac{0.2 \text{ ft}^3}{1} \cdot \frac{(12 \text{ in.})^3}{1 \text{ ft}^3} = \mathbf{345.6 \text{ in.}^3}$.

2. $\frac{60 \text{ min}}{1 \text{ hr}} \cdot \frac{24 \text{ hr}}{1 \text{ day}} \cdot \frac{30 \text{ days}}{1 \text{ mo}} \cdot \frac{15 \text{ drops}}{1 \text{ min}} \cdot \frac{1 \text{ mL}}{20 \text{ drops}} \cdot \frac{1 \text{ L}}{1000 \text{ mL}} = \mathbf{32.4 \text{ L}}$ per month water wasted.

3. $V_{Great\ Pyramid} = \frac{1}{3}(771^2)(486) = 96{,}299{,}442 \text{ ft}^3$. $V_{each\ apartment} = (35)(20)(8) = 5600 \text{ ft}^3$. $\frac{96{,}299{,}442 \text{ ft}^3}{5600 \text{ ft}^3} \approx 17{,}196.36$, or the equivalent volume of about **17,197 apartments**.

4. **(a)** $V_{sphere} = \frac{4}{3}\pi r^3$ so volume of hemispherical portion $= \frac{1}{2}\left(\frac{4}{3}\right)\pi(4^3) = \left(\frac{128}{3}\right)\pi \text{ cm}^3$. $V_{cone} = \frac{1}{3}\pi r^2 h$ so volume of conical portion $= \frac{1}{3}\pi(4^2)(8) = \left(\frac{128}{3}\right)\pi \text{ cm}^3$.

Total volume $= 2\left(\frac{128}{3}\pi\right) = \left(\mathbf{\frac{256}{3}}\right)\boldsymbol{\pi} \textbf{ cm}^3$.

(b) The volume of a prism is the area of its base times its height. $V_{right\ triangular\ prism} = Bh$, where B is the area of the triangular base and h is height.

$B = \frac{1}{2}bh_{triangle} = \frac{1}{2}(6)(6) = 18$; $h_{prism} = 12$.

$V = (18)(12) = \mathbf{216 \text{ cm}^3}$.

(c) $V_{right\ circular\ cone} = \frac{1}{3}\pi r^2 h$, where $r = 3$ and $h = 5$. $V = \frac{1}{3}\pi(3^2)(5) = \mathbf{15\pi \text{ cm}^3}$.

(d) $V_{sphere} = \frac{4}{3}\pi r^3$, where $r = 10$. $V = \frac{4}{3}\pi(10^3) = \mathbf{\frac{4000}{3}\pi \text{ cm}^3}$.

(e) Volume of hemispherical portion $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$ $\frac{1}{2}\left(\frac{4}{3}\right)\pi(10^3) = \frac{2000}{3}\pi \text{ ft}^3$. Volume of cylindrical portion $= \pi r^2 h = \pi(10^2)(60) = 6000\pi \text{ ft}^3$. $V_{total} = \frac{2000}{3}\pi + 6000\pi = \mathbf{\frac{20{,}000}{3}\pi \text{ ft}^3}$.

5. $\text{cm}^3 \times 0.001 = \text{dm}^3$; $\text{dm}^3 = \text{L}$; $\text{L} \times 1000 = \text{mL}$ (i.e., $\text{cm}^3 = \text{mL}$ and $\text{dm}^3 = \text{L}$).

	(a)	(b)	(c)	(d)	(e)	(f)
cm^3	**2000**	500	**1500**	**5000**	750	4800
dm^3	2	**0.5**	**1.5**	**5**	**0.750**	**4.8**
L	**2**	**0.5**	1.5	**5**	**0.750**	**4.8**
mL	**2000**	**500**	**1500**	5000	**750**	**4800**

6. **(a)** A paper cup holds about **200.0 mL** (i.e., the average paper cup holds about a fifth of a liter).

(b) A regular soft drink bottle holds about **0.320 L** (i.e., the 12-ounce size).

(c) A quart milk container holds about **1.0 L** (a quart and a liter are roughly the same amount).

(d) A teaspoonful of cough syrup is about **5.00 mL** (or 5 cc).

7. $V_t = 4^3 = 64; V_2 = 6^3 = 216.$
$V_1 : V_2 = 64 : 216 = 8 : 27.$ (If the side lengths of the two cubes have the ratio $m : n,$ their volumes will have the ratio $m^3 : n^3$.)

8. Volume is **multiplied by a factor of 8**. Volume of a sphere is proportional to the cube of the radius, so if the radius is doubled then volume is multiplied by a factor of $2^3 = 8.$

9. For each right rectangular prism, $V = \ell wh \Rightarrow h = \frac{V}{\ell w}$. Note that in (b) and (d), units must be matched:

	(a)	(b)	(c)	(d)
Length	20 cm	10 cm	2 dm	15 cm
Width	10 cm	2 dm	1 dm	2 dm
Height	10 cm	3 dm	**2 dm**	**2.5 dm** or **25 cm**
Volume (cm^3)	**2000**	**6000**	**4000**	**7500**
Volume (dm^3)	**2**	**6**	**4**	7.5
Volume (L)	**2**	**6**	4	**7.5**

10. Volume of a sphere is proportional to the cube of the radius. A sphere with 4 times the radius of another has $4^3 = 64$ times its volume, or a ratio of **64 : 1**.

11. $V = \ell wh = (50)(25)(2) = 2500 \text{ m}^3 =$ **2,500,000 L**. ($1 \text{ m}^3 = 1000$ L.)

12. The radius of the straw is $2 \text{ mm} = 0.2 \text{ cm}.$
$V = \pi r^2 h = \pi(0.2^2)(25) = \pi \text{ cm}^3 = \boldsymbol{\pi}$ **mL**.

13. **(a)** Volume is **multiplied by $2^3 = 8$**.

(b) Volume would be **multiplied by $3^3 = 27$**.

(c) Volume will be **multiplied by n^3**.

14. Volume of any pyramid $= \frac{1}{3}$ (area of base) (height).

$$V_{Great\ Pyramid} = \frac{1}{3}\left[\left(\frac{940}{4}\right)^2\right](148) = \mathbf{2,724,433.\overline{3}\ m^3};$$

$$V_{Transamerica\ Bldg} = \frac{1}{3}\left[\left(\frac{140}{4}\right)^2\right](260) = 106,167 \text{ m}^3.$$

The Great Pyramid has a greater volume.
$\frac{2,724,433.\overline{3} \text{ m}^3}{106,167 \text{ m}^3} \approx$ **25.66 times** as much.

15. A cross-section of the cone filled to half its height shows two similar $30° - 60° - 90°$ triangles. The radius at a height of 4 cm is 2 cm. Each dimension of the cup is halved, thus $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \mathbf{\frac{1}{8}}$ **the volume** of the full cup when it is filled to half its height.

16. $100\% + 30\% = 130\% = 1.3. (1.3^3) = 2.197,$ the multiple of volume increase, or **119.7%** increase.

17. $V_{can} = \pi r^2 h = \pi(3.5^2)(2 \cdot 3.5 \cdot 3) = 257.25\pi \text{ cm}^3.$

$V_{tennis\ balls} = 3\left(\frac{4}{3}\pi r^3\right) = 3\left(\frac{4}{3}\right)\pi(3.5^3) = 171.5\pi \text{ cm}^3.$

$V_{air} = (257.25 - 171.5)\pi = 85.75\pi \text{ cm}^3.$

$\frac{85.75\pi \text{ cm}^3}{257.25 \text{ cm}^3} = \frac{1}{3} = \mathbf{33\frac{1}{3}\%}$ **air**.

18. Let r be the radius of each of the cans and h be the height of the box and the cans. The dimensions of the base of the box are $6r$ by $4r$, thus

$V_{box} = (6r)(4r)(h) = 24r^2h; V_{6\ cans} = 6\pi r^2 h.$

$V_{wasted} = 24r^2h - 6\pi r^2 h = 6r^2h(4 - \pi).$

$\frac{6r^2h(4-\pi)}{24r^2h} = \frac{4-\pi}{4} \approx 0.215,$ or **about 21.5% wasted**.

19. $V_{prism} = AB \cdot BC \cdot AP. V_{pyramid} = \frac{1}{3}(AB \cdot BC \cdot AX) = \frac{1}{3}(AB \cdot BC \cdot 3AP) = AB \cdot BC \cdot AP,$ or **equal volume**.

20. **(a)** Answers may vary. One design would be a square base with sides 5 m and height 12 m. Then $V = \frac{1}{3}Bh = \frac{1}{3}(5^2)(12) = 100 \text{ m}^3$.

(b) **Infinitely many**. $V_{pyramid} = \frac{1}{3}a^2h$, where a is the length of a side of the square base. $300 = a^2h$ is an equation having an infinite number of solutions.

21. **(a)** **Kilograms or metric tons**. A car weighs in the thousands of pounds or tons.

(b) **Kilograms**. An adult human weighs from about 100 to 250 pounds.

(c) **Grams**. Orange juice concentrate is normally weighed in ounces.

(d) **Metric tons**. An adult African elephant can weigh as much as $7\frac{1}{2}$ tons.

22. **(a)** Staple: A fraction of an ounce, or about 340 **milligrams**.

(b) Professional football player: About 250 pounds on the average, or about 110 **kilograms**.

(c) Vitamin tablet: A fraction of an ounce, or about 1100 **milligrams**.

23. In each metric case below, when converting from smaller to larger units move the decimal point to the left. When converting from larger to smaller units move the decimal point to the right.

(a) $15{,}000\,g = \mathbf{15}$ kg.

(b) 0.036 kg = **36 g**.

(c) 4320 mg = **4.320** g.

(d) 0.03 t = **30** kg.

(e) $\frac{25 \text{ oz}}{1} \cdot \frac{1 \text{ lb}}{16 \text{ oz}} = \mathbf{1\frac{9}{16}} \text{ lb} = \mathbf{1.5625} \text{ lb}$.

24. **(a)** **No,** assuming lifting by hand. 1,000,000 g = 1000 kg.

(b) **Possibly**. $\frac{1{,}000{,}000}{10} \text{ g} = 100 \text{ kg}$.

(c) **Yes**. $\frac{1{,}000{,}000}{100} \text{ g} = 10 \text{ kg}$.

(d) **Yes**. $\frac{1{,}000{,}000}{1000} \text{ g} = 1 \text{ kg}$.

(e) **Yes**. $\frac{1{,}000{,}000}{10{,}000} \text{ g} = 0.1 \text{ kg}$.

25. $V = \ell wh = (40)(20)(20) = 16{,}000 \text{ cm}^3$. 1 cm^3 of water weights 1 g. $16{,}000 \text{ cm}^3 = 16{,}000 \text{ g} =$ **16 kg**.

26. When converting from F to C: $C = \frac{5}{9}(F - 32)$.

(a) $C = \frac{5}{9}(10 - 32) = 12.\overline{2}°$, or about $^{-}\mathbf{12°C}$.

(b) $C = \frac{5}{9}(30 - 32) = {}^{-}1.\overline{1}°$, or **about** $^{-}\mathbf{1°C}$.

(c) $C = \frac{5}{9}(212 - 32) = \mathbf{100°C}$.

27. When converting from C to F: $F = \frac{9}{5}C + 32$.

(a) **Probably not**. $F = \frac{9}{5}(20) + 32 = 68°F$.

(b) **Yes**. $F = \frac{9}{5}(39) + 32 = 102.2°F$.

(c) **Yes**. $F = \frac{9}{5}(35) + 32 = 95°F$.

(d) **Hot**. $F = \frac{9}{5}(30) + 32 = 86°F$.

28. **(a)** Answers vary. Draw any segment through the intersection point of the two diagonals that has endpoints on the parallel sides. There are infinitely many ways to do this.

(b) Answers vary. Consider any plane through the intersection line that contains the point of intersection of two diagonal planes.

29. To convert from Kelvin to Celsius we subtract 273.15, so 5778 Kelvin ≈ 5505 Celsius.

30. 1 lbs = 453.592 grams so there would be about 454 $100 bills in 1 pound.

31. 1 English ton = 2000 lbs. For the alligator that is

$$\frac{1 \text{ ton}}{2000 \text{ lbs}} = \frac{x}{279 \text{ lbs.}} \Rightarrow x = \frac{279}{2000} = 0.1359 \text{ ton}$$

1 English ton = 2000 lbs. For the freshwater fish that is

$$\frac{1 \text{ ton}}{2000 \text{ lbs}} = \frac{x}{646 \text{ lbs}} \Rightarrow x = \frac{646}{2000} = 0.323 \text{ ton}$$

32. 500 sheets weigh 300 lbs and each sheet is 22" by 30". Each sheet is 660 in^2 = 4.583 ft^2 that is a total of 2291.67 ft^2 in 300 lbs or 7.6389 ft^2 per pound.

$$\frac{1 \text{ lb}}{7.6389 \text{ ft}^2}\cdot\frac{1 \text{ ft}^2}{0.092903 \text{ m}^2}\cdot\frac{453.592 \text{ g}}{1 \text{ lb}} = 639.15 \text{ g/m}^2$$

33. An 8 by 12 layout of 96 tiles will cover the floor. The grout will spread the 12 tile row by 11/8 inches, which is not enough to save any tiles; so 96 are required.

34. $\frac{1}{2}$ lbs per person = 2.5 lbs for 5 people.

$$2.5 \text{ lbs.}\cdot\frac{0.453592 \text{ kg}}{1 \text{ lbs.}} = 1.13398 \text{ kg}.$$

Assessment 14-5B

1. **(a)** $\frac{500 \text{ cm}^3}{1}\cdot\frac{1 \text{ m}^3}{(100 \text{ cm})^3} = \mathbf{0.0005\ m^3}$.

(b) $\frac{3 \text{ m}^3}{1}\cdot\frac{(100 \text{ cm})^3}{1 \text{ m}^3} = \mathbf{3{,}000{,}000\ cm^3}$.

(c) $\frac{0.002 \text{ m}^3}{1}\cdot\frac{(100 \text{ cm})^3}{1 \text{ m}^3} = \mathbf{2000\ cm^3}$.

(d) $\frac{25 \text{ yd}^3}{1}\cdot\frac{(3 \text{ ft})^3}{1 \text{ yd}^3} = \mathbf{675\ ft^3}$.

(e) $\frac{1200 \text{ in}^3}{1}\cdot\frac{1 \text{ ft}^3}{(12 \text{ in})^3} = \mathbf{0.69\overline{4}\ ft^3}$.

2. Volume of the rocks equals the volume of the 2 cm increase in the water level. (40 cm)(70 cm) (2 cm) $= 5600$ cm^3 . $\frac{5600 \text{ cm}^3}{1}\cdot\frac{1 \text{ L}}{1000 \text{ cm}^3} = \mathbf{5.6\ L}$ of rocks.

3. Volume to be heated/cooled equals the volume of the 1 ft increase in ceiling height. (2000 ft^2)(1 foot increase) $= \mathbf{2000\ ft^3}$.

4. **(a)** $V_{right\ rectangular\ prism} = \ell wh = 8\cdot 5\cdot 3 =$ $\mathbf{120\ cm^3}$.

(b) $V_{square\ pyramid} = \frac{1}{3}Bh$, where B is the area of the base $= 25$ cm^2 and h is height $= 5$ cm. $V = \frac{1}{3}(25)(5)$ cm^3 $= \frac{125}{3}$cm^3 $= \mathbf{41\frac{2}{3}\ cm^3}$.

(c) **s** where $r = 6$ and $h = 12$.

$V = \pi(6^2)(12) = \mathbf{432\pi\ cm^3}$.

(d) Volume of triangular prism portion
$= Bh = \frac{1}{2}b(h_{triangle})(h_{prism})$
$= \frac{1}{2}(30)(8)(40) = 4800$ ft^3.
Volume of rectangular prism portion
$= \ell wh = (40)(30)(15) = 18{,}000$ ft^3.
$V_{total} = 4800 + 18{,}000 = \mathbf{22{,}800\ ft^3}$.

(e) Volume of triangular prism portion
$= Bh = \frac{1}{2}(10)(6)(60) = 1800$ ft^3.
Volume of trapezoidal prism portion
$= Bh = \frac{1}{2}(50 + 10)(8)(60) = 14{,}400$ ft^3.
Volume of rectangular prism portion
$= \ell wh = (60)(50)(20) = 60{,}000$ ft^3.
$V_{total} = 1800 + 14{,}400 + 60{,}000 =$ $\mathbf{76{,}200\ ft^3}$.

5. cm^3 $\times 0.001 =$ dm^3; dm^3 $=$ L; L $\times 1000 =$ mL (i.e., cm^3 $=$ mL and dm^3 $=$ L).

	(a)	(b)	(c)	(d)	(e)	(f)
cm^3	**6000**	200	**1200**	**3000**	202	6500
dm^3	6	**0.2**	**1.2**	**3**	**0.202**	**6.5**
L	**6**	**0.2**	**1.2**	3	**0.202**	**6.5**
mL	**6000**	**200**	1200	**3000**	**202**	**6500**

6. **(a)** The block has volume 2000 cm^3, which is 2 L or **2000 mL**.

(b) 5600 cm^3 $=$ **5.6 L**.

(c) 20,000 L is more like what it takes to fill a swimming pool. A wading pool is likely filled with **200 L**.

(d) **6 mL** is more reasonable.

7. **(a)** Volume is proportional to s^3, where s is the length of a side. Thus the ratio of volumes would be $\left(\frac{2}{5}\right)^3$, or **8:125**.

(b) The cones are similar, so the radii also share the ratio $a:b$. The ratio of the volumes is

$$\frac{\frac{1}{3}\pi r^2\cdot \text{base}}{\frac{1}{3}\pi\left(\frac{b}{a}r\right)^2\cdot\left(\frac{b}{a}\cdot\text{base}\right)}$$, or $\boldsymbol{a^3{:}b^3}$.

8. Volume is **increased by a factor of 27**. Volume of a sphere is proportional to the cube of the radius, so if the radius is tripled then volume is increased by a factor of $3^3 = 27$.

9. For each right rectangular prism, $V = \ell wh \Rightarrow h = \frac{V}{\ell w}$. Note that in (b) and (d), units must be matched:

	(a)	(b)	(c)	(d)
Length	5 cm	8 cm	2 dm	15 cm
Width	10 cm	6 dm	1dm	2 dm
Height	20 cm	4 dm	**50 cm**	**40 cm**
Volume (cm^3)	**1000**	**19,200**	**10,000**	**12,000**
Volume (dm^3)	**1**	**19.2**	10	12
Volume (L)	**1**	**19.2**	10	**12**

10. Circumference of a circle $= 2\pi r \Rightarrow r = \frac{C}{2\pi}$.
Radius of larger melon $= \frac{60}{2\pi}$; radius of smaller melon $= \frac{50}{2\pi}$; ratio of radii $= \frac{\left(\frac{60}{2\pi}\right)}{\left(\frac{50}{2\pi}\right)} = 1.2$.
The volume of a sphere is proportional to the cube of the radius, so $V_{larger\ melon} = 1.2^3 = 1.728$ times $V_{smaller\ melon}$. The volume of the larger melon is 1.728 times the volume of the smaller melon at 1.5 times the price, so the **larger melon is a better buy**.

11. Partition the pool into two shapes, where the volume of each is B (area of base) $\cdot$ h (height): A right rectangular prism measuring 25 m by 10 m by 2 m and a right triangular prism with base legs 2 m by 10 m and height 10 m.
$V_{rectangular\ prisim} = (25)(10)(92) = 500\ m^3$;
$V_{triangular\ prism} = \frac{1}{2}(2)(10)(10) = 100\ m^3$.
$V_{pool} = V_{rectangular\ prism} + V_{triangular\ prism}$
$= 500 + 100 = \mathbf{600\ m^3}$.

12. $V = \pi r^2 h = \pi(6.5^2)(6) = 253.5\pi\ m^3 =$ $\mathbf{253{,}500\pi}$ **L**. ($1\ m^3 =$ **1000 L**.)

13. Convert all measurement to mm. Radius of the inner circle $= 20$ mm and height $= 20$ mm.
$V_{outer\ cylinder} = \pi r^2 h = \pi(22^2)(20) = 9680\pi\ mm^3$.
$V_{inner\ cylinder} = \pi(20^2)(20) = 8000\pi\ mm^3$.
$V_{ring} = (9680 - 8000)\pi = \mathbf{1680\pi\ mm^3}$.

14. **No**. The volume of a pyramid is $\frac{1}{3}Bh$; the volume of a box is Bh. The pyramid provides only $\frac{1}{3}$ the popcorn for $\frac{1}{2}$ the price.

15. $1\ L = 1000\ cm^3$ and $V = \pi r^2 h$, thus $1000 = \pi(12^2)\ h \Rightarrow h = \frac{1000}{144\pi}$, or **about 2.2 cm**.

16. $V_{first\ freezer} = (1.5)(1.5)(5) = 11.25\ ft^3$; $\frac{11.25\ ft^3}{\$350} \approx 0.032\ ft^3$ per dollar. $V_{second\ freezer} = (2)(2)(4) = 16\ ft^3$; $\frac{16\ ft^3}{\$400} = 0.04\ ft^3$ per dollar.
The **2 ft × 2 ft × 4 ft freezer** is a better buy.

17. $V_{sphere} = \frac{4}{3}\pi(10^3) = \frac{4000}{3}\pi\ cm^3$. Volume in a right circular cylinder is $\frac{4000}{3}\pi = \pi(10^2)h$, so $h = \frac{4000}{300} = \frac{40}{3}$ cm.
Then water height $= \left(20 + \frac{40}{3}\right) = \mathbf{33\frac{1}{3}\ cm}$.

18. Let s represent the length of a side of the box. The volume of the box is s^3. The volume of the cylinder is $\pi\left(\frac{s}{2}\right)^2 \cdot s = \frac{\pi}{4}s^3$. Thus, the cylinder takes up about $\frac{\pi}{4}$ or **78.5%**.

19. Volume is proportional to the cube of the radius, thus radius is proportional to the cube root of volume. If volume is halved, radius is decreased by a factor of $\sqrt[3]{0.5} \approx 0.794$, or **about 0.8 times the original**.

20. When $x = 20$ cm the length of each side of the box $= 200 - 2x = 160$ cm.
The height $= x = 20$ cm.
$V = (160)(160)(20) = \mathbf{512{,}000\ cm^3}$ or $\mathbf{0.512\ m^3}$.

21. **(a)** **Grams**. Mustard is normally weighed in ounces.
(b) **Grams**. Peanuts are normally weighed in ounces.
(c) **Metric tons**. An Army main battle tank weighs about 60 tons.
(d) **Kilograms**. Most cats weigh about 5-10 pounds.

22. **(a)** Dime: A fraction of an ounce, or about 2 **grams**.
(b) Add a fraction of an ounce, or 4 **grams**.
(c) Hair: A very small fraction of an ounce, or 2 **milligrams**.

23. **(a)** 8000 kg $=$ **8** t.
(b) 72 g $=$ **0.072** kg.
(c) 5.750 kg $=$ **5750** g.
(d) $\frac{2.6\text{ lb}}{1} \cdot \frac{16\text{ oz}}{1\text{ lb}} = \mathbf{41.6}$ oz.
(e) $\frac{3.8\text{ lb}}{1} \cdot \frac{16\text{ oz}}{1\text{ lb}} = \mathbf{60.8}$ oz.

24. **(a)** 1000 paper clips is only 1 kg. **Yes**, easy to 1ift.
(b) 100,000 paper clips is only 100 kg $\approx$ 220 lbs. **Yes**, possible.

25. **(a)** $\frac{1\text{ ha}}{1} \cdot \frac{10{,}000\text{ m}^2}{1\text{ ha}} \cdot \frac{10{,}000\text{ cm}^2}{1\text{ m}^2} \cdot \frac{2\text{ cm}}{1} \cdot \frac{1\text{ L}}{1000\text{ cm}^3} =$ **200,000 L** of rainfall.
(b) 1 L of water weighs 1 kg. 200,000 L $=$ 200,000 kg $=$ **200 t** of water.

26. When converting from F to C: $C = \frac{5}{9}(F - 32)$.
(a) $C = \frac{5}{9}(0 - 32) = {}^{-}17.\overline{7}°$, or **about** $\mathbf{{}^{-}18°C}$.
(b) $C = \frac{5}{9}(100 - 32) = 37.\overline{7}°$, or **about** $\mathbf{38°C}$.
(c) $C = \frac{5}{9}({}^{-}40 - 32) = \mathbf{{}^{-}40°C}$. This is the only temperature at which the Celsius and Fahrenheit measurements are the same.

27. When converting from C to F: $F = \frac{9}{5}C + 32$.
(a) **No**. $F = \frac{9}{5}(26) + 32 \approx 79°F$.
(b) **No**. $F = \frac{9}{5}(40) + 32 = 104°F$.
(c) **Chilly**. $F = \frac{9}{5}(16) + 32 \approx 61°F$.

28. **(a)** Answers vary. Draw any segment through the intersection point of the two diagonals that has endpoints on the parallel sides. There are infinitely many ways to do this.
(b) Answers vary. Use the plane determined by two parallel diagonals in opposite faces.

29. To convert from Kelvin to Celsius we subtract 273.15, so 3200 Kelvin ≈ 2927 Celsius.

30. 1 lbs = 453.592 grams so there would be about 454 \$1000 bills in 1 pound.

31. For the alligator that is
$$279\text{ lbs} \cdot \frac{1\text{ kg}}{2.20462\text{ lbs}} = 126.55\text{ kg}$$
For the freshwater fish that is
$$646\text{ lbs} \cdot \frac{1\text{ kg}}{2.20462\text{ lbs}} = 293.02\text{ kg}$$

32. 500 sheets weigh 140 lbs and each sheet is 22” by 30”. Each sheet is 660 $\text{in}^2 = 4.583\ \text{ft}^2$ that is a total of 2291.67 ft^2 in 140 lbs or 16.3691 ft^2 per pound.
$$\frac{1\text{ lbs}}{16.3691\text{ ft}^2} \cdot \frac{1\text{ ft}^2}{0.092903\text{ m}^2} \cdot \frac{453.592\text{ g}}{1\text{ lbs}} = 298.27\text{ g / m}^2$$

33. An 6 by 8 layout of 48 tiles will cover the floor. The grout will spread the 12 tile row by less than an inches, which is not enough to save any tiles; so 48 are required.

34. 0.25 kg per person = 2.5 kg for 5 people.
$$1.25\text{ kg} \cdot \frac{1\text{ lbs}}{0.453592\text{ kg}} = 2.75578\text{ lbs}$$

Mathematical Connections 14-5: Review Problems

19. **(a)** (*i*) Perimeter $= \frac{1}{2}(2\pi \cdot 6) + 2\left(\sqrt{6^2+8^2}\right)$
$= \mathbf{(6\pi + 20)\ cm}$.

(*ii*) Area $= \frac{1}{2}(\pi \cdot 6^2) + \frac{1}{2}(12)(8)$
$= \mathbf{(18\pi + 48)\ cm^2}$.

(b) (*i*) Perimeter $= \frac{1}{2}(2\pi \cdot 20) + 2\left[\frac{1}{2}(2\pi \cdot 10)\right]$
$= \mathbf{40\pi\ cm}$.

(*ii*) Area $= \frac{1}{2}(\pi \cdot 20^2) - 2\left[\frac{1}{2}(\pi \cdot 10^2)\right]$
$= \mathbf{100\pi\ cm^2}$.

20. **(a)** 350 mm $=$ **35** cm (1 cm $=$ 10 mm).

(b) $1600\ \text{cm}^2 = \mathbf{0.16}\ \text{m}^2$ ($1\ \text{m}^2 = 10{,}000\ \text{cm}^2$).

(c) $0.4\ \text{m}^2 = \mathbf{400{,}000}\ \text{mm}^2$ ($1\ \text{m}^2 =$ $1{,}000{,}000\ \text{mm}^2$).

(d) $5.2\ \text{cm}^2 = \mathbf{0.00052\ m^2}$
($1\ \text{cm}^2 = 0.0001\ \text{m}^2$).

21. **(a)** **Yes**. $1^2 + (\sqrt{2})^2 = (\sqrt{3})^2$.

(b) **No.** Sides are: $\sqrt{4^2+1^2} = \sqrt{17}$;
$\sqrt{3^2+1^2} = \sqrt{10}$; and $\sqrt{3^2+4^2} = 5$.
$(\sqrt{17})^2 + (\sqrt{10})^2 \neq 5^2$.

22. **(a)** Radius of base $= \sqrt{50^2 - 40^2} = 30$.
$SA = \pi(30^2) + \pi(30)(50) = 2400\pi\ \text{cm}^2$.

(b) Hypotenuse of triangle $= \sqrt{33^2 + 65^2}$
$= \sqrt{5314}$.
$SA = 2\left[\frac{1}{2}(65)(33)\right]$
$+ (33 + 65 + \sqrt{5314})(40)$
$= (6065 + 40\sqrt{5314})\ cm^2$.

23. First find the diagonal of the base using Pythagorean Theorem:
$6^2 + 6^2 = d^2 \Rightarrow d = \sqrt{72} = 6\sqrt{2}$, then find the non-face diagonal using Pythagorean Theorem again: $\left(6\sqrt{2}\right)^2 + 6^2 = d^2 \Rightarrow d = \sqrt{72+36} = 6\sqrt{3}$.

24. From exercise 23: $6\sqrt{3} > 6\sqrt{2}$. The non-face diagonal is the hypotenuse of a right triangle containing the diagonal of the face as a leg. Since the hypotenuse is the longest side in a right triangle, the non-face diagonal has to be larger than the face diagonal.

Chapter 14 Review

1. **(a)** $50\ \text{ft} \cdot \frac{1\ \text{yd}}{3\ \text{ft}} = \mathbf{16\frac{2}{3}\ yd}$.

(b) $947\ \text{yd} \cdot \frac{\text{mi}}{1760\ \text{yd}} = \mathbf{\frac{947}{1760}\ mi} \approx 0.538\ \text{mi}$:

(c) $0.75\ \text{mi} \cdot \frac{5280\ \text{ft}}{1\ \text{mi}} = \mathbf{3960\ ft}$.

(d) $349\ \text{in} \cdot \frac{1\ \text{yd}}{36\ \text{in}} = \mathbf{9\frac{25}{36}\ yd} \approx 9.694\ \text{yd}$.

(e) $5\ \text{km} \cdot \frac{1000\ \text{m}}{1\ \text{km}} = \mathbf{5000\ m}$.

(f) $165\ \text{cm} \cdot \frac{\text{m}}{100\ \text{cm}} = \mathbf{1.65\ m}$.

(g) $52\ \text{cm} \cdot \frac{10\ \text{mm}}{\text{cm}} = \mathbf{520\ mm}$.

(h) $125\ \text{m} \cdot \frac{1\ \text{km}}{1000\ \text{m}} = \mathbf{0.125\ km}$.

2. **(a)** The triangle inequality tells us that $r + q > p$. If $p - q > r$, then $p - q + q > r + q > p$. This would say that $p > p$, which is false.

(b) If $r = p - q$, then $r + q = p$, which would form a line segment not a triangle.

3. By the Pythagorean theorem, the other side of the rectangle is given by $\sqrt{130^2 - 120^2} = 50$ cm. Thus, the perimeter is $2(50\ \text{cm}) + 2(120\ \text{cm}) =$ **340 cm** .

4. $C = 2\pi r \Rightarrow 3\ \text{m} = 2\pi r \Rightarrow r = \mathbf{\frac{3}{2\pi}}$ m.

5. If a circle has radius 6 cm, then its circumference is $2\pi r = 2\pi(6\ \text{cm}) = 12\pi$ cm. Sarah is not correct.

6. From Pick's theorem (Assessment 14-2B, problem 7), $A = I + \frac{1}{2}B - 1$, where $I =$ the number of dots inside the polygon and $B =$ the number of dots on the polygon boundary. Alternatively, area could be determined through the rectangle method.

(a) $A = 7 + \frac{1}{2}(5) - 1 = \mathbf{8\frac{1}{2}\ cm^2}$.

(b) $A = 5 + \frac{1}{2}(5) - 1 = \mathbf{6\frac{1}{2}\ cm^2}$.

(c) $A = 2 + \frac{1}{2}(12) - 1 = \mathbf{7\,cm^2}$.

7. Rearranging as shown yields a rectangle with width $\frac{h}{2}$ and length $A'B' = b_1 + b_2$. Then $A = \ell w = \frac{1}{2}(b_1 + b_2)$, which must be the area of the initial trapezoid.

8. Area of $\Delta ABC < \Delta ABD = \Delta ABE < \Delta ABF$. All triangles have the same base, so area is proportional only to height. Ordering by height only, ΔABD and ΔABE are equal in area.

9. **(a)** $A = \frac{1}{2}ap$, where $a = \sqrt{6^2 - 3^2} = 3\sqrt{3}$ and $p = 6 \cdot 6 = 36$. $A = \frac{1}{2}(3\sqrt{3})(36) = \mathbf{54\sqrt{3}\ cm^2}$.

(b) $A = \pi r^2 = \pi(6^2) = \mathbf{36\pi\ cm^2}$.

10. **(a)** $A = \pi(4^2) - \pi(2^2) = \mathbf{12\pi\ cm^2}$.

(b) $A_{semicircle} = \frac{1}{2}\pi(3^2) = 4.5\pi\ \text{cm}^2$.

$A_{triangle} = \frac{1}{2}(6)(4) = 12\ \text{cm}^2$.

$A_{shaded\ region} = \mathbf{(4.5\pi + 12)\ cm^2}$.

(c) $A = (6)(4) = \mathbf{24\ cm^2}$.

(d) $A = \frac{40^\circ}{360^\circ}\pi(6^2) = \mathbf{4\pi\ cm^2}$.

(e) $A = \frac{1}{2}(2)(3) + \frac{1}{2}(3)(3) + (3)(15) + (3)(4) = 3 + 4.5 + 45 + 12 = \mathbf{64.5\ cm^2}$.

(f) $A = (8)(18) + \frac{1}{2}(18 + 5)(3) = \mathbf{178.5\ cm^2}$.

11. Distance from home to second base is the diagonal of a right triangle with legs 90 ft each. $d = \sqrt{90^2 + 90^2} = 90\sqrt{2}$, or **about 127.3 ft** $=$ 127 ft 3.6 in., or about 127 ft 4 in.

12. $AC = \sqrt{1^2 + 2^2} = \sqrt{5}$. $AD = \sqrt{1^2 + (\sqrt{5})^2} = \sqrt{6}$. Each succeeding segment thus adds 1 to the number under the radical; $AG = \sqrt{9} = \mathbf{3}$.

13. **(a)** **Yes**. $5^2 + 12^2 = 13^2$.

(b) **No**. $40 + 60 < 104$, thus the measures cannot represent any triangle.

14. $A_{top/bottom} = 2\left[40^2 - 4(5^2)\right] = 3000\ \text{cm}^2$.

$A_{sides} = 4(40)(15) = 2400\ \text{cm}^2$.

$A_{total} = 3000 + 2400 = \mathbf{5400\ cm^2}$.

15. In each of the following, part (*i*) is surface area and part (*ii*) is volume.

(a) **(*i*)** $SA = B + \frac{1}{2}p\ell$, where $\ell = \sqrt{6^2 + 4^2} = 2\sqrt{13}$. $SA = (8^2) + \frac{1}{2}(32)(2\sqrt{13}) = \mathbf{32(2 + \sqrt{13})\ cm^2}$.

(*ii*) $V = \frac{1}{3}Bh = \frac{1}{3}(8^2)(6) = \mathbf{128\ cm^3}$.

(b) **(*i*)** $SA = \pi r^2 + \pi r\ell$, where $\ell = \sqrt{8^2 + 6^2} = 10$. $SA = \pi(6^2) + \pi(6)(10) = \mathbf{96\pi\,cm^2}$.

(*ii*) $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6^2)(8) = \mathbf{96\pi\ cm^3}$.

(c) **(*i*)** $SA = 4\pi r^2 = 4\pi(5^2) = \mathbf{100\pi\ m^2}$.

(*ii*) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(5^3) = \mathbf{\frac{500}{3}\pi\ m^3}$.

(d) **(*i*)** $SA = 2\pi r^2 + 2\pi rh = 2\pi(3^2) + 2\pi(3)(6) = \mathbf{54\pi\ cm^2}$.

(*ii*) $V = \pi r^2 h = \pi(3^2)(6) = \mathbf{54\pi\ cm^3}$.

(e) **(*i*)** $SA = sB + ph = (2)(4)(10) + (28)(8) = \mathbf{304\ m^2}$.

(*ii*) $V = \ell wh = (10)(4)(8) = \mathbf{320\ m^3}$.

16. Lateral surface area $= \pi r\ell$, where $\ell = \sqrt{12^2 + 5^2} = 13$. Area $= \pi(5)(13) = \mathbf{65\pi\ m^2}$.

17. $V = \pi r^2 h$. If both r and h are doubled volume is $\pi(2r)^2(2h) = 8\pi r^2 h$, or eight times the original volume. The graph really shows an eight-fold growth, rather than the doubled sales.

18. Sum the areas of the four triangles.
$A = 2\left(\frac{1}{2} \cdot 5 \cdot 12\right) + 2\left(\frac{1}{2} \cdot 12 \cdot \sqrt{20^2 - 12^2}\right) =$
$60 + 192 = \mathbf{252\ cm^2}$.

19. Change the diagonal measurement to 130 cm.

(a) The length of the other side of the rectangle $= \sqrt{130^2 - 120^2} = 50$ cm. Perimeter $= 2\ell + 2w = 2(120) + 2(50) = \mathbf{340\ cm}$.

(b) $A = \ell w = (120)(50) = \mathbf{6000\ cm^2}$.

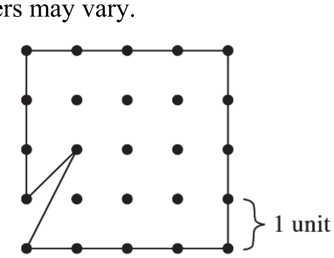
20. $\mathbf{2\sqrt{2}\ m^2}\left[h = \sqrt{3^2 - 1^2} = 2\sqrt{2} \Rightarrow A = \frac{1}{2}bh = \frac{1}{2}(2)(2\sqrt{2}) = 2\sqrt{2}\ \text{m}^2\right]$.

21. Change the area of printed matter to $2500\ \text{cm}^2$.
$h_{printed\ matter} = 74 - 24 = 50$ cm;
$w_{printed\ matter} = \frac{2500\ \text{cm}^2}{50\ \text{cm}} = 50$ cm.
$w_{poster} = 50 + 12 = \mathbf{62\ cm}$.

22. $V_{cylinder} = \pi r^2 h = 10\pi r^2$ and $V_{cone} = \frac{1}{3}\pi r^2 h$.
$10\pi r^2 = \frac{1}{3}\pi r^2 h \Rightarrow 10 = \frac{1}{3}h$. $h = \mathbf{30\ cm}$.

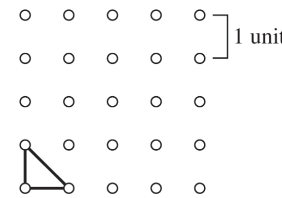
23. Answers may vary.

(a)



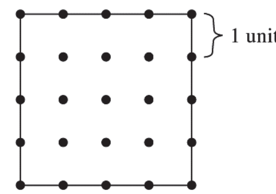
Perimeter $= 15 + \sqrt{2} + \sqrt{5}$ units. Any polygon having sides interior to the edges would have a perimeter greater than 16.

(b)



Perimeter $= 1 + 1 + \sqrt{2} = 2 + \sqrt{2}$ units. No other polygon would have a lesser perimeter.

(c)



$A = 4 \cdot 4 = 16$ units^2. Any other polygon would have space outside the measured area, and would thus have less.

24. Diagonal $= \sqrt{\left(8\frac{1}{2}\right)^2 + 11^2} = \frac{\sqrt{773}}{2} \approx$ **13.9 inches**.

25. **(a)** Given that the equation of a circle with center (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$: $h = 3$ and $k = {}^-4 \Rightarrow (x - 3)^2 + (y - {}^-4)^2 = 5^2$, or $\mathbf{(x - 3)^2 + (y + 4)^2 = 25}$.

(b) If $(x - 5)^2 + (y - {}^-3)^2 = 6^2$, then the center of the circle is $\mathbf{(5, {}^-3)}$ and the radius is **6**.

26. **(a)** **Metric tons**, which corresponds to thousands of pounds, or tons.

(b) $1\ \text{cm} \cdot 1\ \text{cm} \cdot 1\ \text{cm} = \mathbf{1\ cm^3}$.

(c) $1\ \text{cm}^3$ of water weighs **1 gram**.

(d) $\mathbf{1\ L = 1\ dm^3}$, so the two have the same volume.

(e) $\frac{1\ \text{L gas}}{12\ \text{km}} = \frac{x\ \text{L gas}}{300\ \text{km}} \Rightarrow 12x = 300$. $x = \mathbf{25\ L}$.

(f) **2000** a (1 ha = 100 a).

(g) **51,800** cm^3 ($1\ \text{L} = 1000\ \text{cm}^3$)

(h) **10,000,000** m^2 ($1\ \text{km}^2 = 1{,}000{,}000\ \text{m}^2$).

(i) **50,000** mL (1 L = 1000 mL).

(j) **5.830** L (1000 mL = 1 L).

(k) **25,000** dm^3 ($1\ \text{m}^3 = 1000\ \text{dm}^3$).

(l) **75,000** mL (1 $dm^3 = 1000$ mL).

(m) **52.813** kg (1000 g $=$ 1 kg).

(n) **4.8** t (1000 kg $=$ 1 t).

27. **(a)** $V_{tank} = \ell wh = (2)(1)(3) = 6 \text{ m}^3 =$ $6{,}000{,}000 \text{ cm}^3$. $6{,}000{,}000 \text{ cm}^3$ of water $=$ 6,000,000 g $=$ **6000 kg** of water.

(b) $V_{sphere} = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(30)^3 = 36{,}000\pi \text{ cm}^3$, or about 113,097 cm^3. To find the water's height increase from the sphere (converting length and width of the tank to cm): $V =$ $113{,}097 = (200)(100)h$. $h = \frac{113{,}097}{20{,}000}$, or about 5.65 cm rise from the sphere's volume. The tank was half full (i.e., $h = 1.5$ m), so with a rise of 5.65 cm $=$ 0.0565 m the new water height is **1.5565 m**.

28. **(a)** **80 L**, or about 21 gallons.

(b) **82 kg**, or about 181 pounds.

(c) **978 g** $=$ 0.978 kg, or about 2 pounds.

(d) **5 g**, or about 0.2 ounces.

(e) **4 kg**, or about 8.8 pounds.

(f) **1.5 metric tons** $=$ 1500 kg, or about 3300 pounds.

(g) **180 mL** $=$ 0.180 L.

29. **(a)** **unlikely**.
$F = \frac{9}{5}C + 32 = \frac{9}{5}(15) + 32 = 59°F$.

(b) **Likely**. $26°C = 78.8°F$ and $21°C = 69.8°F$.

(c) **Unlikely**. $^-5°C = 23°F$; i.e., below water's freezing point.

(d) **Unlikely**. $120°C = 248°F$, which isn't possible unless the teakettle is pressurized.

(e) **Unlikely**. Water will not freeze until its temperature drops below $0°C$.

30. **(a)** **2000 g** (1 $dm^3 = 1000 \text{ cm}^3$, or 1000 g of water).

(b) **1000 g** (1 L $= 1000 \text{ cm}^3$, or 1000 g of water).

(c) **3 g** (1 $cm^3 = 1$ g).

(d) **0.0042 kg** (1 mL of water $=$ 1 g $=$ 0.001 kg).

(e) $\mathbf{0.0002 \text{ m}^3}$ (1 L $= 1000 \text{ cm}^3 = 0.001 \text{ m}^3$).

31. The volume of a sphere is $\frac{4}{3}\pi r^3$. The ratio of the capacities is $\frac{4}{3}\pi(4^3) : \frac{4}{3}\pi(6^3) \Rightarrow \frac{4}{3}\pi 64 : \frac{4}{3}\pi 216$ $\Rightarrow 64 : 216 \Rightarrow 8 : 27$.

32. If the scale factor of two similar figures is k, then the ratio of their volumes is k^3 (Theorem 14-12). If the volumes of two cylinders have a ratio 2:3 then the scale factor is $\sqrt[3]{k}$ and their radii have a ratio of $\sqrt[3]{2} : \sqrt[3]{3}$.