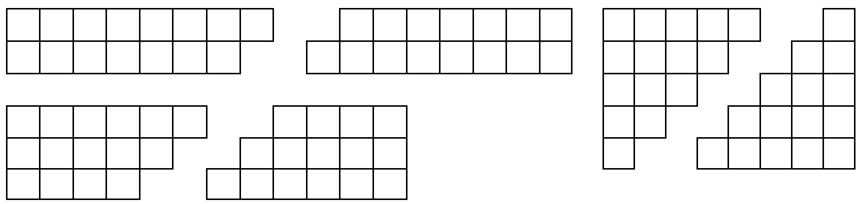
Chapter 1
An Introduction to Problem Solving

Activity 1

Box 1: UNDERSTAND THE PROBLEM

1. Answers will vary.
2. Sample Responses: Shapes are congruent if they have exactly the same shape and size. Shapes are congruent if they can be made to match exactly by placing one on top of the other.
3. a.



   b. 3 ways
4. Answers will vary.
5. Answers will vary.

Box 2: DEVISE A PLAN

1. Squares could be moved around to try to construct the staircases or they could be sketched on graph paper.
2. Simplify the problem, look for a pattern, and make a table
3. Answers will vary.

(Continued on next page.)

Activity 1 Continued

Box 4: LOOK BACK

1. To form two congruent staircases, the number of squares must be even and you must be able to form a rectangle in which one dimension is odd and the other even using the squares.

3. a. 3 ways

   b. Since the two staircases must fit together to form a rectangle with area 120, find the dimensions of all rectangles with integral dimensions and an area of 120. The possible dimensions are 1×120, 2×60, 3×40, 4×30, 5×24, 6×20, 8×15, 10×12.

   To form a staircase, both dimensions must be greater than 1, and one dimension must be even and the other odd. This leaves 3 possibilities, a 3×40 rectangle, a 5×24 rectangle, and an 8×15 rectangle.

   The 3×40 rectangle can be divided into two 3-step staircases. The bottom step is 21 squares long, the middle step is 20 steps long, and the top step is 19 squares long.

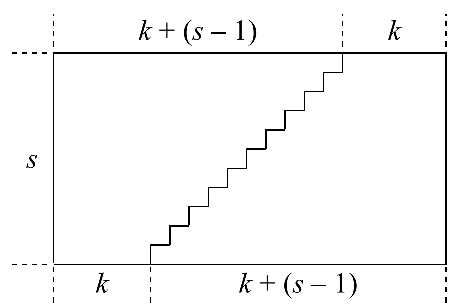
   The 5×24 rectangle can be divided into two 5-step staircases. The bottom step is 14 squares long, the next step is 13 squares long, the middle step is 12 steps long, the fourth step is 11 squares long and the top step is 10 squares long.

   The 8×15 rectangle can be divided into two 8-step staircases. The steps are 11, 10, 9, 8, 7, 6, 5, and 4 steps long.

4. To determine whether n squares can be used to construct two congruent staircases, find the factor pairs of n. There is a pair of staircases for each factor pair in which both factors are greater than 1 and one factor is odd and the other even.

   (Since the two staircases must be congruent, each must contain the same number of squares. So if n squares can be arranged into two congruent staircases, n must be even.

   Suppose a staircase has s steps and that the top step contains k squares. There are $s - 1$ steps below the top step and each step down contains one more square than the step above it, so the bottom step contains $s - 1$ more squares than the top step. That is, the bottom step contains $k + (s - 1)$ squares. When two of the staircases are put together to form a rectangle, the height of the rectangle is s and the length is $k + k + (s - 1) = 2k + (s - 1)$.



   If s is odd, $s - 1$ is even and $2k + (s - 1)$ is even since the sum of two even numbers is even.

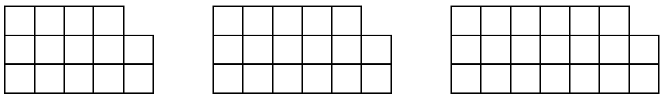
   If s is even, $s - 1$ is odd and $2k + (s - 1)$ is odd since the sum of an odd and an even number is odd.

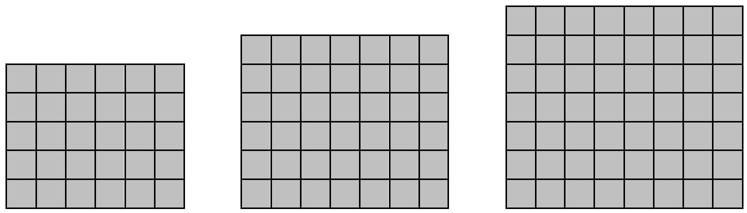
   So if n squares can be arranged into two congruent staircases, you must be able to make a rectangle with the squares in which one dimension is odd and the other dimension is even.

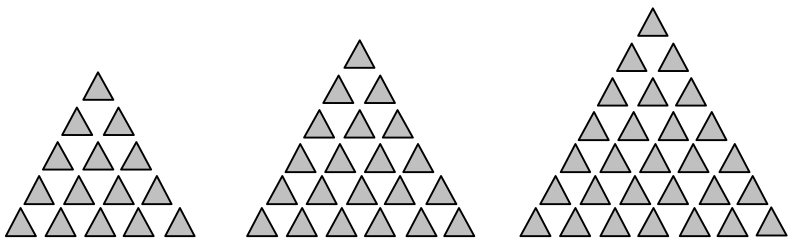
   Suppose one dimension of a rectangle is odd and the other is even. Orient the rectangle so the height is odd and the length is even. Starting at the center of the middle row of squares, form steps going up to the right and steps going down to the left. Since the number of rows above the middle row and the number of rows below it are equal, this process will result in two congruent staircases.)

Activity 2

Box 1

1. 

2. 

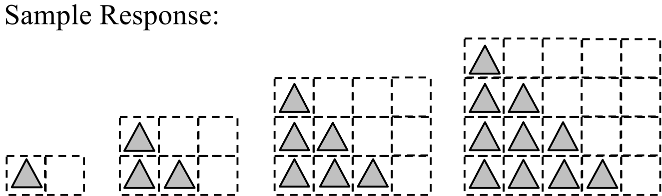
3. 

4.	6,	3,	4,	5,	6,	7
5.	14,	17,	20,	23,	26,	29
6.	25,	18,	11,	4,	–3,	–10
7.	15,	21,	28,	36,	45,	55
8.	28,	39,	52,	67,	84,	103
9.	32,	64,	128,	256,	512,	1024
10.	9,	3,	1,	$\frac{1}{3}$,	$\frac{1}{9}$,	$\frac{1}{27}$
11.	34,	55,	89,	144,	233,	377

12. Each term of the sequence in Exercise 5 is the number of squares in the corresponding term of the sequence in Exercise 1. Similarly, each term of the sequence in Exercise 7 is the number of triangles in the corresponding term of the sequence in Exercise 3.

13. a. The number of triangles in each term of the sequence in Exercise 3 is half the number of squares in the corresponding term of the sequence in Exercise 2.

    b. Sample Response:



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Activity 2 Continued

Box 2

1.	7,	12,	17,	22,	27,	32,	37,	42,	47
2.	2,	5,	8,	11,	14,	17,	20,	23,	26

Each term of the sequence in Exercise 1 on page 5 is a rectangular array 3 squares tall and *n* squares long (where *n* is the term number) with the top right-hand square removed. Thus the number of squares used to form each term is 1 less than 3 times *n*.

3.	44,	41,	38,	35,	32,	29,	26,	23,	20
4.	7,	13,	23,	37,	55,	77,	103,	133,	167
5.	2,	6,	12,	20,	30,	42,	56,	72,	90

Each term of the sequence in Exercise 2 on page 5 is a rectangular array of squares. The height of the rectangle is equal to the term number and its length is 1 more than the term number. Since the area of the rectangle is the length times the width, the number of squares in the array is *the term number* times *the term number plus 1*. That is, *the term number* times *the next term number*.

6. In Exercises 1–3, the difference between successive terms is a constant. In Exercise 1 the difference is always 5, in Exercise 2 it is always 3, and in Exercise 3 it is –3. In each case, the difference is the constant by which the term number was multiplied.

   The differences between successive terms in Exercises 4 and 5 are not constant. Note that, in these cases, the rule for generating the sequence does not involve multiplying the term number by a constant.

7. Answers will vary. One possibility is to look for a pattern in the differences between successive terms and to continue that pattern to find the next five terms in the sequence. For example, in Exercise 5 the differences are the sequence of even numbers 4, 6, 8, 10, 12, 14, 16, 18… Thus, the next five terms would be 90 + **20** = 110, 110 + **22** = 132, 132 + **24** = 156, 156 + **26** = 182, and 182 + **28** = 210.

8. 2, 4, 8, 16, 30, **52**, **84**,**128**, 186, **260**

9. The first four terms of the sequences are the same, but from there on they are different. So no matter how many of the initial terms of a sequence you may know, there may be more than one way to extend the sequence.

Activity 3

2.

Number of People	2	4	6	8	10	$2N$
Number of Pairs	1	2	3	4	5	N
Min Number of Moves	3	8	15	24	35	$N^2 + 2N$
Sequence of Moves	RLR	RLLRRL LR	RLLRRR LLLRRR LLR	RLLRRR LLLLRR RRLLLL RRRLLR	RLLRRR LLLLRR RRRLLL LLRRRR RLLLLR RRLLR	1R, 2L, 3R, ... , (N – 1)L, NR, NL, NR, (N – 1)L, ... , 3R, 2L, 1R if N is odd 1R, 2L, 3R, ... , (N – 1)R, NL, NR, NL, (N – 1)R, ... , 3R, 2L, 1R if N is even

4. 12 people
5. 255 moves
6.

Number of People	1	3	5	7	9	N
Min Number of Moves	1	5	11	19	29	$(N^2 + 4N - 1)/4$
Sequence of Moves	R	RLRRL	RLLRRR LLRRL	RLLRRR LLLRRR RLLLRR L	RLLRRR LLLLRR RRRLLL LRRRRL LLRRL	

The Legend of the Tower of Brahma

1.

Number of Disks	1	2	3	4	**5**	n
Number of Moves	1	3	7	15	**31**	$2^n - 1$

2. a. $2^{10} - 1$ = 17 min 3 sec
   b. $2^{30} - 1 \approx$ 34 years
   c. $2^{50} - 1 \approx$ 36 million years
   d. $2^{64} - 1 \approx$ 585 billion years

Activity 4

Box 1

1. 1580
2. Sample Response: From clues b and d, I knew the number had to be a multiple of 10. These numbers are easy to pick out since the must have a 0 in the units place.

Box 2

1. 153 (or 1 if it isn't a multi-digit number)
2. Sample Response: Since $7^3 = 343$, from the last two clues I knew the digits in the number had to be less than 7. From the second clue, I knew the digits could only be 1, 3, or 5.

Box 3

25 cards or 85 cards

Activity 5

2. a. 1 Subtracting 1 leaves 4 on the display.
   If Player A subtracts 1, that leaves 3 on the display and B wins by subtracting 3.
   If A subtracts 2, that leaves 2 on the display and B wins by subtracting 2.
   If A subtracts 3, it leaves 1 on the display and B wins by subtracting 1.

   b. To be certain to win, you want to be the first player. At the end of your turn you want to leave 16, 12, 8, or 4 on the display.

3. a. In this game you want to be the second player. At the end of your turn you want to leave 20, 15, 10, or 5 on the display.

   b. In this game you want to be the second player. At the end of your turn you want to leave 40, 30, 20, or 10 on the display.

Activity 6

Box 1

1. The results for Steps 1-5 will vary depending on the number selected by each person.
2. Sample Response: 10 because that was the answer I got for Persons 1 through 5.
3. Step 1: n

   Step 2: $3n$

   Step 3: $3n + 30$

   Step 4: $(3n + 30) \div 3 = n + 10$

   Step 5: $n + 10 - n$

   The result is 10.

Box 3

1. Answers will vary.
2. The display on the calculator shows the month and day of a person's birthday. For example, 1021 for October 21.
3. Step 1 m

   Step 2: $5m$

   Step 3: $5m + 20$

   Step 4: $20m + 80$

   Step 5: $20m + 73$

   Step 6: $100m + 365$

   Step 7: $100m + 365 + d$

   Step 8: $100m + \mathrm{d}$

   In Step 8, multiplying the month, m, by 100 moves the month to the hundreds place in the final answer. If m is a single digit number, that digit will be in the hundreds place of the answer. If m is a two-digit number, its tens digit will be in the thousands place of the answer and its units digit will be in the hundreds place. When the day, d, is added, the digits of d will appear in the tens and units places of the final answer.

Activity 7

Box 1

Term Number	1	2	3	4	5	6	7	8
Term	4	11	18	25	32	39	46	53

Difference 7 7 7 7 7 7 7

Constant difference = 7

Term Number		*Constant Difference*			*What Was Done?*		*To Get*	
1	′	7	→	7	–3	=	4	First Term
2	′	7	→	14	–3	=	11	Second Term
3	′	7	→	21	–3	=	18	Third Term
10	′	7	→	70	–3	=	67	Tenth Term
50	′	7	→	350	–3	=	347	Fiftieth Term

Each term is 3 less than 7 times the term number,

Rule: nth term = $7n - 3$

Box 2

					Rule	25th Term	100th Term
1.	25,	29,	33,	37	$4T + 5$	105	405
2.	30,	37,	44,	51	$7T - 5$	170	695
3.	5,	7,	9,	11	$2T - 5$	45	195
4.	90,	88,	86,	84	$-2T + 100$	50	–100
5.	65,	62,	59,	56	$-3T + 80$	5	–220

Activity 8

Box 1

1.

Beginning of month	Number of newborn pairs	Number of 1-month-old pairs	Number of pairs 2 months old or older	Total number of pairs
1	1	0	0	1
2	0	1	0	1
3	1	0	1	2
4	1	1	1	3
5	2	1	2	5
6	3	2	3	8
7	5	3	5	13

2. a. The number of 1-month old pairs in month N ($N \geq 2$) is equal to the number of newborn pairs in month $N - 1$.

   b. The number of pairs 2 months old or older in month N ($N \geq 2$) is equal to the sum of the number of 1-month old pairs and the number of pairs 2-months old or older in month $N - 1$.

   c. From the second month on, the number of newborn pairs each month is equal to the number of pairs 2 months old or older.

Box 2: CREATING A SPREADSHEET

2. a. 1 was added to the 1 in cell A2, so a 2 appears in cell A3.

   b. The formula "= A3 + 1" is copied into cell A4, the formula "= A4 + 1" is copied into cell A5, and so on. Thus for each cell A4 through A14, 1 is added to the number in the preceding cell. The result is that the numbers 3 through 13 appear in cells A4 through A14 respectively.

   c. To find the number of pairs after one year, you must extend the table to the beginning of the 13th month.

3. a. The content of cell B2 was copied into cell C3, so a 1 appears in cell C3.

7. 233 pairs

8. From the third term on, each term equals the sum of the two preceding terms.

Month	Newborn	1 Month Old	2 Months Old	Total Pairs
1	1	0	0	1
2	0	1	0	1
3	1	0	1	2
4	1	1	1	3
5	2	1	2	5
6	3	2	3	8
7	5	3	5	13
8	8	5	8	21
9	13	8	13	34
10	21	13	21	55
11	34	21	34	89
12	55	34	55	144
13	89	55	89	233

(Continued on next page.)

Activity 8 Continued

Box 3: LOOKING FOR PATTERNS

Fibonacci #s	Sums	Fib. # – 1	Squares	Sum of Squares	Products	Quotients
1	1	0	1	1	1	1
1	2	0	1	2	2	2
2	4	1	4	6	6	1.5
3	7	2	9	15	15	1.666666667
5	12	4	25	40	40	1.6
8	20	7	64	104	104	1.625
13	33	12	169	273	273	1.615384615
21	54	20	441	714	714	1.619047619
34	88	33	1156	1870	1870	1.617647059
55	143	54	3025	4895	4895	1.618181818
89	232	88	7921	12816	12816	1.617977528
144	376	143	20736	33552	33552	1.618055556
233	609	232	54289	87841	87841	1.618025751
377	986	376	142129	229970	229970	1.618037135
610	1596	609	372100	602070	602070	1.618032787
987	2583	986	974169	1576239	1576239	1.618034448
1597	4180	1596	2550409	4126648	4126648	1.618033813
2584	6764	2583	6677056	10803704	10803704	1.618034056
4181	10945	4180	17480761	28284465	28284465	1.618033963
6765	17710	6764	45765225	74049690	0	0

1. c. The first sum is 1 less than the third Fibonacci number, the second sum is 1 less than the fourth Fibonacci number, and so on—the nth sum is 1 less than Fibonacci number $n + 2$.

2. c. The numbers in the two columns are the same.

   d. The sum of the squares of the first n Fibonacci numbers equals the product of Fibonacci number n and Fibonacci number $n + 1$.

3. The quotients approach 1.61803.

Activity 9

1. Answers will vary.
2. See table. The number of layers doubles.
3.

Folds	0	1	2	3	4	5	6	7	8
No. Layers (Std. Form)	1	2	4	8	16	32	64	128	256
No. Layers (Fact. Form)	1	2	$2 \cdot 2$	$2 \cdot 2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
No. Layers (Exp. Form)	2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8
Approximate Height (cm)	0.008	0.016	0.031	0.063	0.125	0.25	0. 5	1.0	2

4. Answers will vary. The paper can be folded 5-6 times.
5. 1 fold, 2^1; 0 folds 2^0
6. See table.
7. After 14 folds, the stack is 128 cm tall and after 15 folds it is 256 cm tall, so depending on the student's height, their answer will be 14 or 15 folds.
8. b. The height of the stack after 30 folds would be about 8.4 km. This is higher than the Willis Tower (0.442 km) and less than the altitude of an orbiting satellite (altitudes range from about 150 km to 1000 km at the point nearest to Earth) and the distance to the moon (about 385,000 km).

Activity 10

1.

Square	1	2	3	4	5	6	7	8	...	64
Grains on Square	1 (2^0)	2 (2^1)	4 (2^2)	8 (2^3)	16 (2^4)	32 (2^5)	64 (2^6)	128 (2^7)	...	2^{63}
Total	1 (2^1–1)	3 (2^2–1)	7 (2^3–1)	15 (2^4–1)	31 (2^5–1)	63 (2^6–1)	127 (2^7–1)	255 (2^8–1)	...	2^{64}–1
Total on Odd Squares	1		5		21		85			
Total on Even Squares		2		10		42		170		

a. The number of grains of rice on the nth square is 2^{n-1}, so there are 2^{63} grains of rice on the 64th square. The number of grains of rice on any square is 1 more than the total number of grains of rice on all the previous squares, so there are $2^{63} - 1$ grains on squares 1 through 63.

$$\begin{aligned}\text{Total grains of rice} &= \text{number on square 64} + \text{number on squares 1 through 63}\\ &= 2^{63} + 2^{63} - 1\\ &= 2 \cdot 2^{63} - 1\\ &= 2^{64} - 1 \approx 18{,}446{,}744{,}070{,}000{,}000{,}000\end{aligned}$$

b. The number of grains of rice on the even squares is twice the number of grains on the odd squares.

The answers to Exercises 2 and 3 are based on the following data: 1000 grains of rice weigh 0.0154 lb and have a volume of 1 in^3.

2. a. $1.85 \cdot 10^{19}$ grains $\cdot \dfrac{0.0154 \text{ lb}}{1000 \text{ grains}} \approx 2.85 \cdot 10^{14}$ lb or $1.425 \cdot 10^{11}$ tons

b. $1.85 \cdot 10^{19}$ grains ÷ 1000 grains/in^3 = $1.85 \cdot 10^{16}$ in^3
1 bushel ≈ 2150 in^3
$1.85 \cdot 10^{16}$ in^3 ÷ 2150 in^3/bushel ≈ $8.6 \cdot 10^{12}$ bushels (8.6 trillion bushels)

c. Answers will vary. A 100 story building in the shape of a rectangular prism with a square base 20 mi on a side would hold the rice.

100 stories · 10 ft/story · (20 mi · 5280 ft/mi · 12 in/ft)^2 ≈ $1.9 \cdot 10^{16}$ in^3

d. In 2012, the retail price of long grain, white rice was approximately \$0.70 per pound. The rice would have a value of about \$200 trillion ($2.85 \cdot 10^{14}$ lb · \$0.70 per lb ≈ \$$2.0 \cdot 10^{14}$).

3. a. In 2010, the United States produced about $1.2 \cdot 10^7$ tons of rice. At this rate, it would take about 11,875 years ($1.425 \cdot 10^{11}$ tons ÷ $1.2 \cdot 10^7$ tons per year = $1.1875 \cdot 10^4$ years) for the U.S. to produce enough rice to meet the inventor's request.

b. In 2010, the world production of rice was $7.4 \cdot 10^8$ tons. At this rate, it would take all of the rice produced in the world for about 193 years ($1.425 \cdot 10^{11}$ tons ÷ $7.4 \cdot 10^8$ tons per year ≈ $0.193 \cdot 10^3$) years to meet the inventor's request.

c. See Parts a and b.