INSTRUCTOR'S SOLUTIONS MANUAL

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A PROBLEM SOLVING APPROACH TO MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

TWELFTH EDITION

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Publishing as Pearson, 501 Boylston Street, Boston, MA 02116.

ISBN-13: 978-0-321-99057-0
ISBN-10: 0-321-99057-9

www.pearsonhighered.com

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Contents

CHAPTER 1

AN INTRODUCTION TO PROBLEM SOLVING

Assessment 1-1A: Mathematics and Problem Solving

1. **(a)** List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 2 & + & \cdots & + & 98 & + & 99 \\ 99 & + & 98 & + & \cdots & + & 2 & + & 1 \\ \hline 100 & + & 100 & + & \cdots & + & 100 & + & 100 \end{array}$$

There are 99 sums of 100. Thus the total can be found by computing $\frac{99 \cdot 100}{2} = \mathbf{4950}$. (Another way of looking at this problem is to realize there are $\frac{99}{2} = 49.5$ pairs of sums, each of 100; thus $49.5 \cdot 100 = 4950$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1001-1}{2} + 1 = 501$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 1 & + & 3 & + & \cdots & + & 999 & + & 1001 \\ 1001 & + & 999 & + & \cdots & + & 3 & + & 1 \\ \hline 1002 & + & 1002 & + & \cdots & + & 1002 & + & 1002 \end{array}$$

There are 501 sums of 1002. Thus the total can be found by computing $\frac{501 \cdot 1002}{2} = \mathbf{251{,}001}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{300-3}{3} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{rcrcccrcr} 3 & + & 6 & + & \cdots & + & 297 & + & 300 \\ 300 & + & 297 & + & \cdots & + & 6 & + & 3 \\ \hline 303 & + & 303 & + & \cdots & + & 303 & + & 303 \end{array}$$

There are 100 sums of 303. Thus the total can be found by computing $\frac{100 \cdot 303}{2} = 15{,}150$.

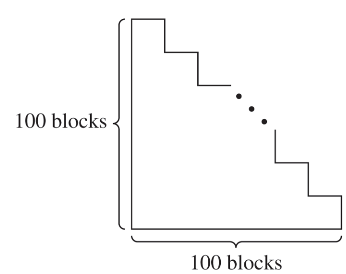
(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{400-4}{4} + 1 = 100$ terms.

List the numbers:

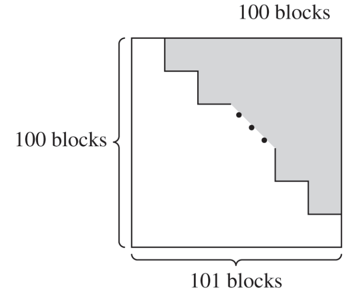
$$\begin{array}{rcrcccrcr} 4 & + & 8 & + & \cdots & + & 396 & + & 400 \\ 400 & + & 396 & + & \cdots & + & 8 & + & 4 \\ \hline 404 & + & 404 & + & \cdots & + & 404 & + & 404 \end{array}$$

There are 100 sums of 404. Thus the total can be found by computing $\frac{100 \cdot 404}{2} = 20{,}200$.

2. **(a)**



(b)



When the stack in (a) and a stack of the same size is placed differently next to the original stack in (a), a rectangle containing 100 (101) blocks is created. Since each block is represented twice, the desired sum is $100(101)/2 = \mathbf{5050}$.

While the above represents a specific example, the same thinking can be used for any natural number n to arrive at a formula $n(n+1)/2$.

3. There are $\frac{147-36}{1} + 1 = 112$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 36 & + & 37 & + & \cdots & + & 146 & + & 147 \\ 147 & + & 146 & + & \cdots & + & 37 & + & 36 \\ \hline 183 & + & 183 & + & \cdots & + & 183 & + & 183 \end{array}$$

There are 112 sums of 183. Thus the total can be found by computing $\frac{112 \cdot 183}{2} = \mathbf{10,248}$.

4. **(a)** Make a table as follows; there are 9 rows so there are 9 different ways.

6-cookie packages	2-cookie packages	single-cookie packages
1	2	0
1	1	2
1	0	4
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

(b) Make a table as follows; there are 12 rows so there are **12 different ways**.

6-cookie packages	2-cookie packages	single-cookie packages
2	0	0
1	3	0
1	2	2
1	1	4
1	0	6
0	6	0
0	5	2
0	4	4
0	3	6
0	2	8
0	1	10
0	0	12

5. If each layer of boxes has 7 more than the previous layer we can add powers of 7:

$7^0 = 1$ (red box)

$7^1 = 7$ (blue boxes)

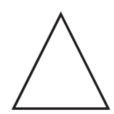
$7^2 = 49$ (black boxes)

$7^3 = 343$ (yellow boxes)

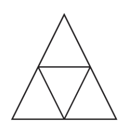
$7^4 = 2401$ (gold boxes)

$1 + 7 + 49 + 343 + 2401 = 2801$ boxes altogether.

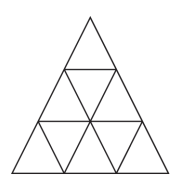
6. Using strategies from Poyla's problem solving list identify subgoals (solve simpler problems) and make diagrams to solve the original problem.



1 triangle; name this the "unit" triangle.

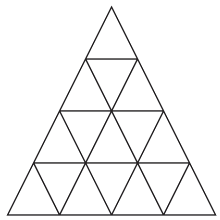


This triangle is made of 4 unit triangles. Counting the large triangle there are 5 triangles



Unit triangles	4 unit triangles	9 unit triangles
9	3	1

13 total triangles



Unit triangles	4 unit triangles	9 unit triangles	16 unit triangles
16	7	3	1

There are 27 triangles in the original figure.

7. Observe that $E = (1 + 1) + (3 + 1) + \cdots + (97 + 1) = O + 49$. Thus, E is 49 more than O..

Alternative strategy:

$$O + E = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 97 + 98$$
$$= \frac{98(99)}{2} = 49(99)$$
$$E = 2(1 + 2 + 3 + 4 + \cdots + 49) = 2\left(\frac{49(50)}{2}\right) = 49(50)$$
$$O = O + E - E = 49(99) - 49(50) = 49(49).$$

So O is 49 less than E.

8. Bubba is last; Cory must be between Alababa and Dandy; Dandy is faster than Cory. Listing from fastest to slowest, the finishing order is then Dandy, Cory, Alababa, and Bubba.

9. Make a table.

\$20 bills	\$10 bills	\$5 bills
2	1	0
2	0	2
1	3	0
1	2	2
1	1	4
1	0	6
0	5	0
0	4	2
0	3	4
0	2	6
0	1	8
0	0	10

There are twelve rows so there are twelve different ways. 12

10. The diagonal from the left, top corner to the right, bottom corner sums to $17 + 22 + 27 = 66$.

The first row sums to $17 + a + 7 = 24 + a$. So $a = 66 - 24 = \mathbf{42}$. The last column sums to $7 + b + 27 = 34 + b$. So $b = 66 - 34 = \mathbf{32}$. The first column sums to $17 + 12 + c = 29 + c$. So $c = 66 - 29 = \mathbf{37}$. The second column sums to $42 + 22 + d = 64 + d$. So $d = 66 - 64 = \mathbf{2}$.

11. Debbie and Amy began reading on the same day, since 72 pages for Debbie ÷ 9 pages per day = 8 days. Thus Amy is on 6 pages per day × 8 days = **page 4**8.

12. The last three digits must sum to 20, so the second to last digit must be $20 - (7 + 4) = 9$. Since the sum of the 11th , 12th, and 13th digits is also 20, the 11th digit is $20 - (7 + 9) = 4$.

A		7								4	7	9	4

We can continue in this fashion until we find that A is 9, or we can observe the repeating pattern from back to front, 4, 9, 7, 4, 9, 7, … and discover that A is 9.

13. Choose the box labeled Oranges and Apples (Box B). Retrieve a fruit from Box B. Since Box B is mislabeled, Box B should be labeled as having the fruit you retrieved. For example, if you retrieved an apple, then Box B should be labeled Apples. Since Box A is mislabeled, the Oranges and Apples label should be placed on Box A. These leave only one possibility for Box C; it should be labeled Oranges. If an orange was retrieved from Box B, then Box C would be labeled Oranges and Apples and Box A should be labeled Apples.

14. The electrician made \$1315 for 4 days at \$50 per hour. She spent \$15 per day on gasoline so 4 • \$15 = \$60 on gasoline. The total is then \$1315 + \$60 = \$ 1375. At \$50 per hour, she worked $\frac{1375}{50} = 27.5$ hours.

15. Working backward: Top – 6 rungs – 7 rungs + 5 rungs – 3 rungs = top – 11 rungs, which is located at the middle.

From the middle rung travel up 11 to the top or down 11 to the bottom. Along with the starting rung, then, there are 11 + 11 + 1 = **23 rungs.**

Assessment 1-1B

1. (a) List the numbers:

$$\begin{array}{ccccccccc} 1 & + & 2 & + & \cdots & + & 48 & + & 49 \\ 49 & + & 48 & + & \cdots & + & 2 & + & 1 \\ \hline 50 & + & 50 & + & \cdots & + & 50 & + & 50 \end{array}$$

There are 49 sums of 50. Thus the total can be found by computing $\frac{49 \cdot 50}{2} = \mathbf{1225}$. (Another way of looking at this problem is to realize there are $\frac{49}{2} = 24.5$ pairs of sums, each of 50; thus $24.5 \cdot 50 = 1225$.)

(b) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{2009-1}{2} + 1 = 1005$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 1 & + & 3 & + & \cdots & + & 2007 & + & 2009 \\ 2009 & + & 2007 & + & \cdots & + & 3 & + & 1 \\ \hline 2010 & + & 2010 & + & \cdots & + & 2010 & + & 2010 \end{array}$$

There are 1005 sums of 2010. Thus the total can be found by computing $\frac{1005 \cdot 2010}{2} = \mathbf{1{,}010{,}025}$.

(c) The number of terms in any sequence of numbers may be found by subtracting the first term from the last, dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{600-6}{6} + 1 = 100$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 6 & + & 12 & + & \cdots & + & 594 & + & 600 \\ 600 & + & 594 & + & \cdots & + & 12 & + & 6 \\ \hline 606 & + & 606 & + & \cdots & + & 606 & + & 606 \end{array}$$

There are 100 sums of 606. Thus the total can be found by computing $\frac{606 \cdot 100}{2} = 30{,}300$.

(d) The number of terms in any sequence of numbers may be found by subtracting the first term from the last (or the last from the first if the first is greater than the last), dividing the result by the common difference between terms, and then adding 1 (because both ends must be accounted for). Thus $\frac{1000-5}{5} + 1 = 200$ terms.

List the numbers:

$$\begin{array}{ccccccccc} 1000 & + & 995 & + & \cdots & + & 10 & + & 5 \\ 5 & + & 10 & + & \cdots & + & 995 & + & 1000 \\ \hline 1005 & + & 1005 & + & \cdots & + & 1005 & + & 1005 \end{array}$$

There are 200 sums of 1005. Thus the total can be found by computing $\frac{1005 \cdot 200}{2} = 100{,}500$.

2. (a) The diagram illustrates how the numbers can be paired to form 50 sums of 101. The sum of the first 100 natural numbers is $50(101) = \mathbf{5050}$.

(b) A diagram similar to the one in 2a would illustrate how the numbers can be paired to form 100 sums of 202. Since the last term is odd, the middle term, 101, is left unpaired. So, the sum of the first 201 natural numbers is 100 • 202 + 101 = 20,301.

3. There are $\frac{203-58}{1} + 1 = 146$ terms. terms

List the numbers:

$$\begin{array}{ccccccccc} 58 & + & 59 & + & \cdots & + & 202 & + & 203 \\ 203 & + & 202 & + & \cdots & + & 59 & + & 58 \\ \hline 261 & + & 261 & + & \cdots & + & 261 & + & 261 \end{array}$$

There are 146 sums of 261. Thus the total can be found by computing $\frac{146 \cdot 261}{2} = \mathbf{19{,}053}$.

4. There are many answers to this problem. A systematic list is a good approach. Using only two numbers and addition, 5 rows give 5 different ways.

One number	Eleven minus the number
1	10
2	9
3	8
4	7
5	6

We can view 6 + 5 as a different way than 5 + 6 and continue in this manner to find 10 different ways. Or, we can find 7 more ways using three numbers as follows.

1	1	9
2	1	8
3	1	7
4	1	6
5	1	5
1	2	8

5. After two socks are drawn, the two socks match or they don't. If they match, we are done. If they don't match, the next sock drawn will match one of the two socks already drawn.

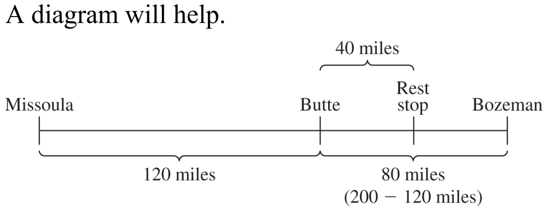
6. There are 13 squares of one unit each; 4 squares of four units each; and one square 9 units; for a total of 18 squares.

7.

$$\begin{aligned} P &= 1 + 3 + 5 + 7 + \ldots + 99 \\ Q &= \qquad\quad 5 + 7 + \ldots + 99 + 101 \\ Q - P &= (5 + \ldots + 99 + 101) - \\ &\quad (1 + 3 + 5 + \ldots + 99) \\ &= (101) - (1 + 3) \\ &= 97 \end{aligned}$$

Q is larger than P by 97.

8. A diagram will help.



The next step is 120 miles + 40 miles = **160 miles** from Missoula.

9. **(a)** Marc must have five pennies to make an even \$1.00. The minimum number of coins would have as many quarters as possible, or three quarters. The remaining 20¢ must consist of at least one dime and one nickel; the only possibility is one dime and two nickels. The minimum is 5 pennies, 2 nickels, 1 dime, and 3 quarters, or 11 coins.

(b) The maximum number of coins is achieved by having as many pennies as possible. It is a requirement to have one quarter, one dime, and one nickel = 40¢, so there may then be 60 pennies for a total of 63 coins.

10. Adding the numbers gives 99. This means that each row, diagonal, and column must add to $99 \div 3 = 33$. Write 33 as a sum of the numbers in all possible ways:

$19 + 11 + 3$
$19 + 9 + 5$
$17 + 13 + 3$
$17 + 11 + 5$
$17 + 9 + 7$
$15 + 13 + 5$
$15 + 11 + 7$
$13 + 11 + 9$

Summarizing the pattern:

Number	Nr. sums with number
3	2
5	3
7	2
9	3
11	4
13	3
15	2
17	3
19	2

Thus 11 must be in the center of the square and 5, 9, 13, and 17 must be in the corners. One solution would be:

17	7	9
3	11	19
13	15	5

11. Answers may vary; two solutions might be to:

(a) Put four marbles on each tray of the balance scale. Take the heavier four and weigh two on each tray. Take the heavier two and weigh one on each tray; the heavier marble will be evident on this third weighing.

(b) This alternative shows the heavier marble can be found more efficiently, two steps rather than three. Put three marbles in each tray of the balance scale.

(***i***) If the two trays are the same weight, the heavier marble is one of the remaining two. Weigh them to find the heavier.

(***ii***) If one side is heavier, take two of the three marbles and weigh them. If they are the same weight, the remaining marble is the heavier. If not, the heavier will be evident on this second weighing.

12. **(a)** There are:

1 partridge $\times$ 12 days $=$ 12 gifts;
2 doves $\times$ 11 days $=$ 22 gifts;
3 hens $\times$ 10 days $=$ 30 gifts;
4 birds $\times$ 9 days $=$ 36 gifts;
5 rings $\times$ 8 days $=$ 40 gifts;
6 geese $\times$ 7 days $=$ 42 gifts;
7 swans $\times$ 6 days $=$ 42 gifts;
8 maids $\times$ 5 days $=$ 40 gifts;

9 ladies $\times$ 4 days $=$ 36 gifts;
10 lords $\times$ 3 days $=$ 30 gifts;
11 pipers $\times$ 2 days $=$ 22 gifts; and
12 drummers $\times$ 1 day $=$ 12 gifts.

So the gifts given the most by your true love was 42 geese and 42 swans.

(b) $12 + 22 + \cdots + 22 + 12 =$ **364** gifts total.

13. **(a)** There must be 1 or 3 quarters for an amount ending in 5. Then dimes can add to \$1.15 plus 4 pennies to realize \$1.19. Thus:

Quarters	Dimes	Pennies	Total
3	4	4	\$1.19
1	9	4	\$1.19

and in neither case can change for \$1.00 be made.

(b) Two or zero quarters would allow an amount ending in 0. Then more combinations of dimes or pennies could add to \$1.00.

14. If the price of 15 sandwiches equals the price of 20 salads, each sandwich will buy $\frac{20}{15} = \frac{4}{3}$ salads. Thus 3 sandwiches $= 3\left(\frac{4}{3}\right) =$ **4 salads**.

15. Use a variable and a table

12 AM	5 AM	9 AM	12 PM
T	T − 15	2(T − 15)	2(T − 15) + 10

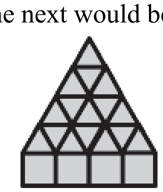
So,
$$\begin{aligned} 2(T-15)+10 &= 32 \\ 2T-30+10 &= 32 \\ 2T-20 &= 32 \\ 2T &= 52 \\ T &= \textbf{26 degrees}. \end{aligned}$$

Assessment 1-2A: Explorations with Patterns

1. **(a)** Each figure in the sequence adds one box each to the top and bottom rows. The next would be:

(b) Each figure in the sequence adds one upright and one inverted triangle. The next would be:

(c) Each figure in the sequence adds one box to the base and one row to the overall triangle. The next would be:



2. **(a)** Terms that continue a pattern are 17, 21, 25, , … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are 220, 270, 320, … . This is arithmetic because each successive term is obtained from the previous term by addition of 50.

(c) Terms that continue a pattern are 27, 81, 243, … . This is geometric because each successive term is obtained from the previous term by multiplying by 3.

(d) Terms that continue a pattern are $10^9, 10^{11}, 10^{13}, \ldots$. This is geometric because each successive term is obtained from the previous term by multiplying by 10^2.

(e) Terms that continue a pattern are $193 + 10 \cdot 2^{30}, 193 + 11 \cdot 2^{30}, 193 + 12 \cdot 2^{30}, \ldots$. This is arithmetic because each successive term is obtained from the previous term by addition of 230.

3. In these problems, let a_n represent the nth term in a sequence, a_1 represent the first term, d represent the common difference between terms in an arithmetic sequence, and r represent the common ratio between terms in a geometric sequence. In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 1$ and $d = 4$:

(*i*) $a_{100} = 1 + (100 - 1) \cdot 4$
$= 1 + 99 \cdot 4 = 397.$

(*ii*) $a_n = 1 + (n - 1) \cdot 4$
$= 1 + 4n - 4 = 4n - 3.$

(b) Arithmetic sequence: $a_1 = 70$ and $d = 50$:

(*i*) $a_{100} = 70 + (100 - 1) \cdot 50$
$= 70 + 99 \cdot 50 = 5020.$

(*ii*) $a_n = 70 + (n - 1) \cdot 50$
$= 70 + 50n - 50$ or $50n + 20.$

(c) Geometric sequence: $a_1 = 1$ and $r = 3$:

(*i*) $a_{100} = 1 \cdot 3^{100-1} = 3^{99}.$

(*ii*) $a_n = 1 \cdot 3^{n-1} = 3^{n-1}.$

(d) Geometric sequence: $a_1 = 10$ and $r = 10^2$:

(*i*) $a_{100} = 10 \cdot (10^2)^{(100-1)} = 10 \cdot (10^2)^{99}$
$= 10 \cdot 10^{198} = 10^{199}.$

(*ii*) $a_n = 10 \cdot (10^2)^{(n-1)}$
$= 10 \cdot 10^{(2n-2)} = 10^{2n-1}.$

(e) Arithmetic sequence:
$a_1 = 193 + 7 \cdot 2^{30}$ and $d = 2^{30}$:

(*i*) $a_{100} = 193 + 7 \cdot 2^{30} + (100 - 1) \cdot 2^{30}$
$= 193 + 7 \cdot 2^{30} + 99 \cdot 2^{30}$
$= 193 + 106 \cdot 2^{30}.$

(*ii*) $a_n = 193 + 7 \cdot 2^{30} + (n - 1) \cdot 2^{30}$
$= 193 + (n + 6) \cdot 2^{30}.$

4. 2, 7, 12, … . Each term is the 5th number on a clock face (clockwise) from, the preceding term.

5. **(a)** Make a table.

Number of term	Term
1	$1 \cdot 1 \cdot 1 = 1$
2	$2 \cdot 2 \cdot 2 = 8$
3	$3 \cdot 3 \cdot 3 = 27$
4	$4 \cdot 4 \cdot 4 = 64$
5	$5 \cdot 5 \cdot 5 = 125$
6	$6 \cdot 6 \cdot 6 = 216$
7	$7 \cdot 7 \cdot 7 = 343$
8	$8 \cdot 8 \cdot 8 = 512$
9	$9 \cdot 9 \cdot 9 = 729$
10	$10 \cdot 10 \cdot 10 = 1000$
11	$11 \cdot 11 \cdot 11 = 1331$

The 11^{th} term 1331 is the least 4-digit number greater than 1000.

(b) The 9^{th} term 729 is the greatest 3-digit number in this pattern.

(c) The greatest number less than 10^4 is $21 \cdot 21 \cdot 21 = 9261$.

(d) The cell A14 corresponds to the 14th term, which is $14 \cdot 14 \cdot 14 = 2744$.

6. (a) The number of matchstick squares in each windmill form an arithmetic sequence with $a_1 = 5$ and $d = 4$. The number of matchstick squares required to build the 10th windmill is thus $5 + (10 - 1) \cdot 4 = 5 + 9 \cdot 4 = \mathbf{41}$ **squares**.

(b) The ***n***th windmill would require $5 + (n - 1) \cdot 4 = 5 + 4n - 4 = \mathbf{4n + 1}$ squares.

(c) There are 16 matchsticks in the original windmill. Each additional windmill adds 12 matchsticks.

This is an arithmetic sequence with $a_1 = 16$ and $d = 12$, so $a_n = 16 + (n - 1) \cdot 12 = \mathbf{12n + 4}$ matchsticks.

7. (a) Each cube adds four squares to the preceding figure; or 6, 10, 14, … . This is an arithmetic sequence with $a_1 = 6$ and $d = 4$. Thus $a_{15} = 6 + (15 - 1) \cdot 4 = 62\,squares$ to be painted in the 10th figure.

(b) This is an arithmetic sequence with $a_1 = 6$ and $d = 6$. The ***n***th term is thus: $a_n = 6 + (n - 1) \cdot 4 = \mathbf{4n + 2}$.

8. Since the first year begins with 700 students, after the first year there would be 760, after the second there would be 820, … , and after the twelfth year the number of students would be the 13^{th} term in the sequence.

This then is an arithmetic sequence with $a_1 = 700$ and $d = 60$, so the 13^{th} term (current enrollment + twelve more years) is:

$700 + (13 - 1) \cdot 60 = 1420\,students.$

9. Using the general expression for the nth term of an arithmetic sequence with $a_1 = 24{,}000$ and $a_9 = 31{,}680$ yields:

$$31680 = 24000 + (9 - 1)d$$
$$31680 = 24000 + 8d \Rightarrow d = 960,$$

the amount by which Joe's income increased each year.

To find the year in which his income was \$45120:

$$45120 = 24000 + (n - 1) \cdot 960$$
$$45120 = 23040 + 960n$$
$$\Rightarrow n = 23.$$

Joe's income was \$45,120 in his **23rd year**.

10. (a) If the first difference of the sequence increases by 2 for each term, then the five first differences between the first six terms of the original sequence are 2, 4, 6, 8, 10. If the first term of the original sequence is 3, then the first six terms are 3, 5, 9, 15, 23, 33.

(b) If the first term is a, then $a + (a + 2) = 10 \Rightarrow a = 4$. Thus the first six terms of the original sequence are 4, 6, 10, 16, 24, 34.

(c) If the fifth term is 35, then:

Sixth term $35 + 10 = 45$

Fourth term $35 - 8 = 27$

Third term $27 - 6 = 21$

Second term $21 - 4 = 17$

First term $17 - 2 = 15$.

Thus the sequence is 15, 17, 21, 27, 35, 45.

11. (a) Look for the differences:

5	6	14	32	64	115	191
1	8	18	32	51	76	
7	10	14	19	25		
3	4	5	6			
1	1	1				

The third difference row is an arithmetic sequence with fixed difference of 1. Thus the 6th term in the second difference row $25 + 7 = 32$; the 7^{th} term in the first difference row is $76 + 32 = 108$; and the 8th term in the original sequence is $191 + 108 = 299$. Using the same reasoning, the next three terms in the original sequence will be 299, 447, 644.

(b) Look for the differences:

0	2	6	12	20	30	42
2	4	6	8	10	12	
2	2	2	2	2		

The first difference row is an arithmetic sequence with fixed difference 2. Thus the 7th term in the first difference row is $12 + 2 = 14$; the 8th term in the original sequence is $42 + 14 = 56$. Using the same reasoning, the next three terms in the original sequence are 56, 72, 90.

12. **(a)** Using the general expression for the nth term of an arithmetic sequence with $a_1 = 51, a_n = 251$, and $d = 1$ yields:

$251 = 51 + (n - 1) \cdot 1 \Rightarrow$
$251 = 50 + n \Rightarrow n = 201.$

There are 201 terms in the sequence.

(b) Using the general expression for the ***n***th term of a geometric sequence with $a_1 = 1$, $a_n = 2^{60}$, and $r = 2$ yields:

$2^{60} = 1(2)^{n-1} = 2^{n-1}$
$\Rightarrow n - 1 = 60 \Rightarrow n = 61.$

There are 61 terms in the sequence.

(c) Using the general expression for the ***n***th term of an arithmetic sequence with $a_1 = 10$, $a_n = 2000$, and $d = 10$ yields:

$2000 = 10 + (n - 1) \cdot 10 \Rightarrow$
$2000 = 10n \Rightarrow n = 200.$

There are 200 terms in the sequence.

(d) Using the general expression for the n^{th} term of a geometric sequence with $a_1 = 1$, $a_n = 1024$, and $r = 2$ yields:

$1024 = 1(2)^{n-1} \Rightarrow$
$2^{10} = 2^{n-1} \Rightarrow n - 1 = 10$
$\Rightarrow n = 11.$

There are 11 terms in the sequence.

13. **(a)**
First term: $(1)^2 + 2 = \mathbf{3}$;
Second term: $(2)^2 + 2 = \mathbf{6}$;
Third term: $(3)^2 + 2 = \mathbf{11}$;
Fourth term: $(4)^2 + 2 = \mathbf{18}$; and
Fifth term: $(5)^2 + 2 = \mathbf{27}$.

(b)
First term: $5(1) + 1 = 6$;
Second term: $5(2) + 1 = 11$;
Third term: $5(3) + 1 = 16$;
Fourth term: $5(4) + 1 = 21$; and
Fifth term: $5(5) + 1 = 26$.

(c)
First term: $10^{(1)} - 1 = \mathbf{9}$;
Second term: $10^{(2)} - 1 = \mathbf{99}$;
Third term: $10^{(3)} - 1 = \mathbf{999}$;
Fourth term: $10^{(4)} - 1 = \mathbf{9999}$; and
Fifth term: $10^{(5)} - 1 = \mathbf{99999}$.

(d)
First term: $3(1) - 2 = 1$**;**
Second term: $3(2) - 2 = 4$**;**
Third term: $3(3) - 2 = 7$**;**
Fourth term: $3(4) - 2 = 10$**;** and
Fifth term: $3(5) - 2 = 13$**.**

14. Answers may vary; examples are:

(a) If $n = 5$, then $\frac{5+5}{5} = 2 \neq 5 + 1 = 6$.

(b) If $n = 2$, then $(2 + 4)^2 = 36 \neq 2^2 + 16 = 20$.

15. **(a)** There are 1, 5, 11, 19, 29 tiles in the five figures. Each figure adds 2n tiles to the preceding figure, thus a_6 the 6th term has $29 + 12 =$ **41 tiles**.

(b) $n^2 = 1, 4, 9, 16, 25, \ldots$. Adding $(n - 1)$ to n^2 yields $1, 5, 11, 19, 29, \ldots$, which is the proper sequence. Thus the n*th* term has $n^2 + (n - 1)$.

(c) If $n^2 + (n - 1) = 1259$;

Then $n^2 + n - 1260 = 0$. This implies $(n - 35)(n + 36) = 0$, so $n = 35$.

There are 1259 tiles in the 35th figure.

16. The nth term of the arithmetic sequence is $200 + n(200)$. The sequence can also be generated by adding 200 to the previous term. The n^{th} term of the geometric sequence is 2^n. The sequence can also be generated by multiplying the previous term by 2. Make a table.

Number of the term	Arithmetic term	Geometric term
7	1600	128
8	1800	256
9	2000	512
10	2200	1024
11	2400	2048
12	2600	4096

With the 12th term, the geometric sequence is greater.

17. **(a)** Start with one piece of paper. Cutting it into five pieces gives us 5. Taking each of the pieces and cutting it into five pieces again gives $5 \cdot 5 = 25$ pieces. Continuing this process gives a geometric sequence: 1, 5, 25, 125, … . After the 5^{th} cut there are $5^5 = 3125$ pieces of paper.

(b) The number of pieces after the nth cut would be 5^n.

18. **(a)** For an arithmetic sequence there is a common difference between the terms. Between 39 and 69 there are three differences so we can find the common difference by subtracting 39 from 69 and dividing the answer by three: $69 - 39 = 30$ and $30 \div 3 = 10$. The common difference is 10 and we can find the missing terms: 39 – 10 = 29 and 39 + 10 = 49 and 49 + 10 = 59.

(b) For an arithmetic sequence there is a common difference between the terms. Between 200 and 800 there are three differences so we can find the common difference by subtracting 200 from 800 and dividing the answer by three: $800 - 200 = 600$ and $600 \div 3 = 200$. The common difference is 200 and we can find the missing terms: 200 – 200 = 0 and 200 + 200 = 400 and 400 + 200 = 600.

(c) For a geometric sequence there is a common ration between the terms. Between 5^4 and 5^{10} there are three common ratios used so we can find the common ratio by dividing 5^{10} by 5^4 and then taking the cube root:

$5^{10} \div 5^4 = 5^6$ and $(5^6)^{(\frac{1}{3})} = 5^2$. The common ratio is 52 and we can find the missing terms: $5^4 \div 5^2 = 5^2, 5^4 \cdot 5^2 = 5^6, 5^6 \cdot 5^2 = 5^8$.

19. **(a)** Let's call the missing terms a, b, c, d, and f, then the sequence becomes:

$a, b, 1, 1, c, d, e, f$

$b + 1 = 1 \to b = 0$

$a + b = 1 \to a + 0 = 1 \to a = 1$

$1 + 1 = c \to c = 2$

$1 + c = d \to 1 + 2 = d \to d = 3$

$c + d = e \to 2 + 3 = e \to e = 5$

$d + e = f \to 3 + 5 = f \to = 8.$

The missing terms are 1, 0, 2, 3, 5, and 8.

(b) Let's call the missing terms a, b, c, and d, then the sequence becomes:

$a, b, c, 10, 13, d, 36, 59$

$c + 10 = 13 \to c = 3$

$b + c = 10 \to b + 3 = 10 \to b = 7$

$a + b = c \to a + 7 = 3 \to = {}^{-}4$

$10 + 13 = d \to d = 23$

The missing terms are -4, 7, 3, and 23.

(c) If a Fibonacci-type sequence is a sequence in which the first two terms are arbitrary and in which every term starting from the third is the sum of the previous two terms, then we can add 0 and 2 to get the third term and continue the pattern:

$0 + 2 = 2$

$2 + 2 = 4$

$2 + 4 = 6$

$4 + 6 = 10$

$6 + 10 = 16$

$10 + 16 = 26$

The missing terms are 2, 4, 6, 10, 16, and 26.

20. Starting with 1 and 1 the Fibonacci sequence would be 1,1,2,3,5,8,13,21,34,55, … .

(a) 1 + 1 + 2 = 4 which is one less than the fifth term.

(b) 1 + 1 + 2 + 3 = 7 which is one less than the sixth term.

(c) 1 + 1 + 2 + 3 + 5 = 12 which is one less than the seventh term.

(d) The sum of the first n terms is one less than the (n + 2) term. OR $F1 + F2 + F3 + \ldots + F_n = F_n + 2 - 1$.

(e) The sum of the first n-2 terms of the Fibonacci sequence is one less than the nth term.

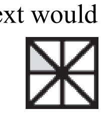
21. **(a)**

Year 1	$80 + .05(80) = 84$
Year 2	$84 + .05(84) = 88.2$
Year 3	$88.2 + .05(88.2) = 92.61$
Year 4	$92.61 + .05(92.61) = 97.2405$
Year 5	$97.2405 + .05(97.2405) = 102.102525$
	$\approx \$102.10.$

(b) This is a geometric sequence with $a1 = 80$ and $r = 1.05$, so the price after n years is $80 \cdot 1.05^n$.

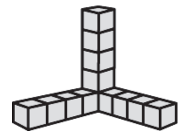
Assessment 1-2B

1. **(a)** In a clockwise direction, the shaded area moves to a new position separated from the original by one open space, then two open spaces, then by three, etc. The separation in each successive step increases by one unit; next would be:



(b) Each figure in the sequence adds one row of boxes to the base. Next would be:

(c) Each figure in the sequence adds one box to the top and each leg of the figure. Next would be:



2. **(a)** Terms that continue a pattern are 18, 22, 26, … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 4.

(b) Terms that continue a pattern are 39, 52, 65, … . This is an arithmetic sequence because each successive term is obtained from the previous term by addition of 13.

(c) Terms that continue a pattern are 256, 1024, 4096, … . This is a geometric sequence because each successive term is obtained from the previous term by multiplying by 4.

(d) Terms that continue a pattern are $2^{14}, 2^{18}, 2^{22}$, … . This is a geometric sequence because each successive term is obtained from the previous term by multiplying by 24.

(e) Terms that continue a pattern are $100 + 4 \cdot 2^{50}, 100 + 6 \cdot 2^{50}, 100 + 8 \cdot 2^{50}, \ldots.$ This is a arithmetic sequence because each successive term is obtained from the previous term by adding by $2 \cdot 2^{50}$.

3. In these problems, a_n represents the nth term in a sequence, a_1 represents the first term, d represent the common difference between terms in an arithmetic sequence, and r represents the common ratio between terms in a geometric sequence.

In an arithmetic sequence, $a_n = a_1 + (n-1)d$; in a geometric sequence, $a_n = a_1 r^{n-1}$. Thus:

(a) Arithmetic sequence: $a_1 = 2$ and $d = 4$.

(*i*) $a_{100} = 2 + (100-1) \cdot 4 = 398.$

(*ii*) $a_n = 2 + (n-1) \cdot 4$
$= 2 + 4n - 4 = 4n - 2.$

(b) Arithmetic sequence: $a_1 = 0$ **and** $d = 13$.

(*i*) $a_{100} = 0 + (100 - 1) \cdot 13 = 1287$.

(*ii*) $a_n = 0 + (n - 1) \cdot 13$
$= 13n - 13$.

(c) Geometric sequence: $a_1 = 4$ and $r = 4$.

(*i*) $a_{100} = 4 \cdot 4^{99} = 4^{100}$.

(*ii*) $a_n = 4 \cdot 4^{n-1} = 4^n$.

(d) Geometric sequence: $a_1 = 2^2$ and $r = 2^4$.

(*i*) $a_{100} = 2^2 \cdot (2^4)^{99} = 2^2 \cdot 2^{396} = 2^{398}$.

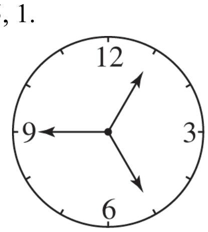
(*ii*) $a_n = 2^2 \cdot (2^4)^{(n-1)}$
$= 2^2 \cdot 2^{4n-4} = 2^{4n-2}$.

(e) Arithmetic sequence:
$a_1 = 100 + 4 \cdot 2^{50}$ and $d = 2^{51}$:

(*i*) $a_{100} = 100 + 4 \cdot 2^{50} + (100 - 1) \cdot 2^{51}$
$= 100 + 2 \cdot 2^{51} + 99 \cdot 2^{51}$
$= 100 + 101 \cdot 2^{51}$.

(*ii*) $a_n = 100 + 4 \cdot 2^{50} + (n - 1) \cdot 2^{51}$
$= 100 + 2 \cdot 2^{51} + (n - 1) \cdot 2^{51}$
$= 100 + (n + 1) \cdot 2^{51}$.

4. The hands must move 8 hours to move from 1 to 9 on the clock face. To move from 9 to 5, the hand must move 8 hours also. To move from 5 to 1, the hand must move another 8 hours. If we add 8 hours to 1 o'clock, we will land on the 9. This pattern will continue, so the next three terms are 9, 5, 1.



5. (a) Answers may vary:

(*i*) **The sum of the first n odd numbers is n^2;** e.g., $1 + 3 + 5 + 7 = 4^2$.

(*ii*) **Square the average of the first and last terms;** e.g.,

$$1 + 3 + 5 + 7 = \left(\frac{1+7}{2}\right)^2 = 4^2.$$

(b) There are $\frac{35-1}{2} + 1 = 18$ terms in this sequence.

(*i*) $1 + 3 + 5 + 7 + \cdots + 35 = 18^2$
$= \mathbf{324}$.

(*ii*) $\left(\frac{1+35}{2}\right)^2 = 18^2 = \mathbf{324}$.

6. (a) Note that 5 toothpicks are added to form each succeeding hexagon. This is an arithmetic sequence $a_1 = 6$ and $d = 5$, so $a_{10} = 6 + (10 - 1) \cdot 5 = 6 + 9 \cdot 5 = \mathbf{51}$ toothpicks.

(b) n hexagons would require $6 + (n - 1) \cdot 5 = 6 + 5n - 5 = \mathbf{5n + 1}$ toothpicks.

7. (a) Looking at the third figure, there are $5 + 3 + 1 = 9$ triangles. The fourth figure would than have $7 + 5 + 3 + 1 = 16$ triangles. An alternative to simply adding $7, 5, 3,$ and 1 together is to note that $7 + 1 = 8$ and $5 + 3 = 8$. There are $\frac{4}{2} = 2$ of these sums, and $2 \cdot 8 = 16$. Then the 100th figure would have $100 + 99 = 199$ triangles in the base, $99 + 98 = 197$ triangles in the second row, and so on until the 100th row where there would be 1 triangle. $199 + 1 = 200$; $197 + 3 = 200$; etc. and so the sum of each pair is 200 and there are $\frac{100}{2} = 50$ of these pairs. $50 \cdot 200 = 10{,}000$, or 10,000 triangles in the 100th figure.

(b) The number of triangles in the ***n***th figure is $\frac{n}{2}$(number of triangles in base + 1). The number of triangles in the base is $n + (n - 1)$, or $2n - 1$. $(2n - 1) + 1 = 2n$. Then $\frac{n}{2}(2n) = n^2$, or ***n***2 triangles in the ***n***th figure.

8. This is a geometric sequence with $a_1 = \frac{15,360}{2}$ and $r = \frac{1}{2}$. The *n*th term of a geometric sequence is $a_n = a_1 r^{n-1}$; thus the 10th term would be $15360\left(\frac{1}{2}\right)^{10} =$ **15 liters**.

Note the progression of terms in the following table:

After Day	Amount of Water Remaining
1	$15,360 \cdot \frac{1}{2} = 7680$ liters
2	$7680 \cdot \frac{1}{2} = 3840$ liters
⋮	⋮
9	$60 \cdot \frac{1}{2} = 30$ liters
10	$30 \cdot \frac{1}{2} = 15$ liters

9. This is an arithmetic sequence with $a_1 = 8\frac{1}{6}$ (i.e., 8 a.m. plus 10 minutes, or $\frac{10}{60}$ of an hour) and $d = \frac{5}{6}$ (or $\frac{50}{60}$ of an hour). Thus

$a_8 = 8\frac{1}{6} + (8 - 1) \cdot \frac{5}{6} = 14$, or **2:00 p.m**.

(14 is 2:00 p.m. on a 24-hour clock.)

10. (a) If the first difference of the sequence increases by 3 for each term, then five first differences between the first six terms of the original sequence are 3, 6, 9, 12, 15. If the first term of the original sequence is 3, then the first six terms are 3, 6, 12, 21, 33, 48.

(b) If the first term is a, then $a + (a + 3) = 7 \Rightarrow a = 2$. Thus the first six terms of the original sequence are 2, 5, 11, 20, 32, 47.

(c) If the fifth term (a_5) is 34, then:

$a_6 = 34 + 15 = 49$
$a_4 = 34 - 12 = 22$
$a_3 = 22 - 9 = 13$
$a_2 = 13 - 6 = 7$
$a_1 = 7 - 3 = 4$.

Thus the sequence is 4, 7, 13, 22, 34, 49.

11. (a) Look for the differences:

3 8 15 24 35 48
5 7 9 11 13

The first difference row is an arithmetic sequence with fixed difference of 2. Thus the 6th term in the first difference row is $13 + 2 = 15$; the 7th term in the original sequence is $48 + 15 = 63$. Using the same reasoning, the next three terms in the original sequence are 63, 80, 99.

(b) Look for the differences:

1 7 18 37 67 111
6 11 19 30 44
5 8 11 14

The second difference row is an arithmetic sequence with fixed difference of 3. Thus the 5th term in the second difference row is $14 + 3 = 17$; the 6th term in the original sequence is $111 + 61 = 172$. Using the same reasoning, the next three terms in the original sequence are 172, 253, 357.

12. (a) The ***nth*** term for this geometric sequence is 3^{n-1}. Thus $3^{99} = 3^{n-1}$.

So $99 = n - 1$, and $n = 100$.

There are 100 terms in the sequence.

(b) The $\boldsymbol{n^{th}}$ term for this arithmetic sequence is $9 + (n - 1) \cdot 4$. Thus $353 = 9 + (n - 1) \cdot 4$. Solving for ***n***, $n = 87$. There are 87 terms in the sequence.

(c) The $\boldsymbol{n^{th}}$ term for this arithmetic sequence is $38 + (n - 1) \cdot 1$. Thus $238 = 38 + (n - 1) \cdot 1$. Solving for ***n***, $n = 201$. There are 201 terms in the sequence.

13. **(a)** First term: $5(1) - 1 = 4$
Second term: $5(2) - 1 = 9$
Third term: $5(3) - 1 = 14$
Fourth term: $5(4) - 1 = 19$
Fifth term: $5(5) - 1 = 24$

(b) First term: $6(1) - 2 = \mathbf{4}$
Second term: $6(2) - 2 = \mathbf{10}$
Third term: $6(3) - 2 = \mathbf{16}$
Fourth term: $6(4) - 2 = \mathbf{22}$
Fifth term: $6(5) - 2 = \mathbf{28}$

(c) First term: $5 \cdot 1 + 1 = \mathbf{6}$
Second term: $5 \cdot 2 + 1 = \mathbf{11}$
Third term: $5 \cdot 3 + 1 = \mathbf{16}$
Fourth term: $5 \cdot 4 + 1 = \mathbf{21}$
Fifth term: $5 \cdot 5 + 1 = \mathbf{26}$

(d) First term: $1^2 - 1 = \mathbf{0}$
Second term: $2^2 - 1 = \mathbf{3}$
Third term: $3^2 - 1 = \mathbf{8}$
Fourth term: $4^2 - 1 = \mathbf{15}$
Fifth term: $5^2 - 1 = \mathbf{24}$

14. Answers may vary; examples are:

(a) If $n = 6$, then $\frac{3+6}{3} = 3 \neq 6$.

(b) If $n = 4$, then $(4 - 2)^2 = 4 \neq 4^2 - 2^2 = 12$.

15. **(a)** The first figure has 2 tiles, the second has 5 tiles, the third has 8 tiles, … . This is an arithmetic sequence where the ***n**^th^* term is $2 + (n - 1) \cdot 3$.

Thus the 7^{th} term has $2 + (7 - 1) \cdot 3 = 20$ tiles.

(b) The nth term is $2 + (n - 1) \cdot 3 = 2 + 3n - 3 = \mathbf{3n - 1}$.

(c) The question can be written as: Is there an n such that $3n - 1 = 449$. Since $3n - 1 = 449 \Rightarrow 3n = 450 \Rightarrow n = 150$, the answer is yes, the 150th figure.

16. The n^{th} term of the arithmetic sequence is $-100 + n(300)$. The sequence can also be generated by adding 300 to the previous term. The n^{th} term of the geometric sequence is 3^{n-1}. The sequence can also be generated by multiplying the previous term by 3. Make a table

Number of the term	Arithmetic term	Geometric term
7	2000	729
8	2300	2187
9	2600	6561

With the **9$^{\text{th}}$ term**, the geometric sequence is greater.

17. Use a table of Fibonacci numbers to find the pattern, F_n is the nth Fibonacci number:

Generation	Male	Female	Number in Generation	Total
1	1	0	1	1
2	0	1	1	2
3	1	1	2	4
4	1	2	3	7
5	2	3	5	12
6	3	5	8	20
⋮	⋮	⋮	⋮	⋮
n	F_{n-2}	F_{n-1}	F_n	$F_{n+2}-1$

The sum of the first n Fibonacci numbers is $F_{n+2} - 1$. $F_{12} = 144$, so there are **143 bees** in all 10 generations.

18. **(a)** For an arithmetic sequence there is a common difference between the terms. Between 49 and 64 there are three differences so we can find the common difference by subtracting 49 from 64 and dividing the answer by three: $64 - 49 = 15$ and $15 \div 3 = 5$. The common difference is 5 and we can find the missing terms: 49 – 5 = 44 and 49 + 5 = 54 and 54 + 5 = 59.

(b) For a geometric sequence there is a common ratio between the terms. Between 1 and 625 there are four common ratios used so we can find the common ratio by dividing 625 by 1 and then taking the fourth root:

$625 \div 1 = 625 \text{ and } 625^{(\frac{1}{4})} = 5.$ The common ratio is 5 and we can find the missing terms: $1 \cdot 5 = 5, 5 \cdot 5 = 25, 25 \cdot 5 = 125.$

(c) For a geometric sequence there is a common ratio between the terms. Between 310 and 319 there are three common ratios used so we can find the common ratio by dividing 319 by 310 and then taking the cube root:

$3^{19} \div 3^{10} = 3^9 \text{ and } (3^9)^{(\frac{1}{3})} = 3^3.$ The common ratio is 33 and we can find the missing terms: $3^{10} \div 3^3 = 3^7, 3^{10} \cdot 3^3 = 3^{13}, 3^{13} \cdot 3^3 = 3^{16}.$

(d) For an arithmetic sequence there is a common difference between the terms. Between ***a*** and 5***a*** there are four differences so we can find the common difference by subtracting ***a*** from 5***a*** and dividing the answer by four: $5a - a = 4a \text{ and } 4a \div 4 = a.$ The common difference is ***a*** and we can find the missing terms: ***a*** + ***a*** = 2***a***, : 2***a*** + ***a*** = 3***a***, : 3***a*** + ***a*** = 4***a***.

19. **(a)** Let's call the missing terms *x* and *y,* then the sequence becomes 1, *x*, *y*, 7, 11 and if it is a Fibonacci-type sequence then:

$1 + x = y$

$x + y = 7$

$y + 7 = 11 \rightarrow y = 11 - 7 = 4$

$\text{and } x + y = 7 \rightarrow x + 4 = 7 \rightarrow x = 3.$

The missing terms are 3 and 4.

(b) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, 2, *y*, 4, *z* and if it is a Fibonacci-type sequence then:

$x + 2 = y$

$2 + y = 4 \rightarrow y = 2$

$y + 4 = z \rightarrow 2 + 4 = z \rightarrow z = 6$

$\text{and } x + 2 = 2 \rightarrow x = 0.$

The missing terms are 0, 2, and 6.

(c) Let's call the missing terms *x*, *y*. and *z*, then the sequence becomes *x*, *y*, 3, 4, *z* and if it is a Fibonacci-type sequence then:

$x + y = 3$

$y + 3 = 4 \rightarrow y = 4 - 3 = 1$

$3 + 4 = 7$

$\text{and } x + y = 3 \rightarrow x + 1 = 3 \rightarrow x = 2.$

The missing terms are 2, 1, and 7.

20. Starting with 1 and 1 the Fibonacci sequence would be 1,1,2,3,5,8,13,21,34,55, … .

(a) $F_1 + F_3 = F_4 \rightarrow 1 + 2 = 3$

(b) $F_1 + F_3 + F_5 = F_6 \rightarrow 1 + 2 + 5 = 8.$

(c) $F_1 + F_3 + F_5 + F_7 = F_8 \rightarrow 1 + 2 + 5 + 13 = 21.$

(d) $F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k}.$

Mathematical Connections 1-2: Review Problems

17. Order the teams from 1 to 10, and consider a simpler problem of counting how many games are played if each team plays each other once. The first team plays nine teams. The second team also plays nine teams, but one of these games has already been counted. The third team also plays 9 teams, but two of these games were counted in the previous two summands. Continuing in this manner, the total is 10 + 9 + 8 + … + 3 + 2 + 1 = 9(10) / 2=45 games. Double this amount to obtain 90 games must be played for each team to play each other twice.

18. 7 ways. Make a table:

Quarters	Dimes	Nickels
1	1	1
1	0	3
0	4	0
0	3	2
0	2	4
0	1	6
0	0	8

19. If the problem is interpreted to stated that at least one 12-person tent is used, then there are 10 ways. This can be seen by the table below, which illustrates the ways 2,3,5, and -6 person tents can be combined accommodate 14 people.

6-Person	5-Person	3-Person	2-Person
2	0	0	1
1	1	1	0
1	0	2	1
1	0	0	4
0	2	0	2
0	1	3	0
0	1	1	3
0	0	4	1
0	0	2	4
0	0	0	7

Chapter 1 Review

1. Make a plan. Every 7 days (every week) the day will change from Sunday to Sunday. 365 days per year $\div 7$ days per week ≈ 52 weeks per year $+\frac{1}{7}$ weeks per year. Thus the day of the week will change from Sunday to Sunday 52 times and then change from Sunday to Monday. July 4 will be a Monday.

2. $\$5.90 \div 2 = \$2.95$ more on one of the items. That is $\$20 + \$2.95 = \$22.95$ for the more expensive item and $\$20 - \$2.95 = \$17.05$ for the less expensive item. Check that both items add up to \$40: $\$22.95 + \$17.05 = \$40$.

3. The information in the rhyme is that the scholar is a "ten o'clock scholar" who "used to come at ten o'clock." The question is "what makes you come sooner." If we assume that ten o'clock means 10 AM, then the rhyme makes no sense because the scholar came later than usual. If we assume that ten o'clock means 10 PM, the question makes sense but is not answered.

4. **(a)** 15, 21, 28 Neither. The successive differences of terms increases by one; e.g., 10 + 5, 15 + 6, … .

(b) 32, 27, 22. Arithmetic Subtract 5 from each term to obtain the subsequent term.

(c) 400, 200, 100. Geometric Each term is half the previous term.

(d) 21, 34, 55. Neither Each term is the sum of the previous two terms—this is the Fibonacci sequence.

(e) 17, 20, 23. Arithmetic Add 3 to each term to obtain the subsequent term.

(f) 256, 1024, 4096. Geometric Multiply each term by 4 to obtain the subsequent term.

(g) 16, 20, 24. Arithmetic Add 4 to each term to obtain the subsequent term.

(h) 125, 216, 343. Neither Each term is the 3rd power of the counting numbers = 13, 23, 33, … .

5. **(a)** The successive differences are 3. Each term is 3 more than the previous term. This suggests that it is an arithmetic sequence of the form $3n + ?$. Since the first term is 5, $3(1) + ? = 5$. The n^{th} term would be $\mathbf{3n + 2}$.

(b) Each term given is 3 times the previous term. This suggests that the sequence is geometric. $\boldsymbol{n^{th}}$ term will be $3^{\boldsymbol{n}}$.

(c) We are reminded of the sequence 1; 8; 27; 64; … , which is given by n^3. Since the terms in the original sequence are one less than the terms given by n^3, $\mathbf{n^3 - 1}$ is a possible n^{th} term.

6. **(a)** $3(1) - 2 = \mathbf{1}$;
$3(2) - 2 = \mathbf{4}$;
$3(3) - 2 = \mathbf{7}$;
$3(4) - 2 = \mathbf{10}$; and
$3(5) - 2 = \mathbf{13}$.

(b) $1^2 + 1 = \mathbf{2}$;
$2^2 + 2 = \mathbf{6}$;
$3^2 + 3 = \mathbf{12}$;
$4^2 + 4 = \mathbf{20}$; and
$5^2 + 5 = \mathbf{30}$.

(c) $4(1) - 1 = \mathbf{3}$;
$4(2) - 1 = \mathbf{7}$;
$4(3) - 1 = \mathbf{11}$;
$4(4) - 1 = \mathbf{15}$; and
$4(5) - 1 = \mathbf{19}$.

7. **(a)** $a_1 = 2, d = 2, a_n = 200$.

So $200 = 2 + (n-1) \cdot 2 \Rightarrow n = 100$.

Sum is $\frac{100(2+200)}{2} = \mathbf{10,100}$.

(b) $a_1 = 51,\ d = 1, a_n = 151$.

So $151 = 51 + (n-1) \cdot 1 \Rightarrow n = 101$.

Sum is $\frac{101 \cdot (51+151)}{2} = \mathbf{10,201}$.

8. **(a)** $5 + 3 = 8$, which is not odd.

(b) 15 is odd; and it does not end in a 1 or a 3.

(c) The sum of any two even numbers is always even. An even number is one divisible by 2, so any even number can be represented by $2 + 2 + 2 + \cdots$. Regardless of how many twos are added, the result is always a multiple of 2, or an even number.

9. All rows, columns, and diagonals must add to 34; i.e., the sum of the digits in row 1. Complete rows or columns with one number missing, then two, etc. to work through the square:

16	3	2	13
5	10	**11**	**8**
9	**6**	7	12
4	**15**	14	**1**

10. The ten middle tables will hold two each and the two end tables will hold three each, totaling 26 people.

11. **(a)** $\square + 2^{60} = 2^{61}$

$\square = 2^{61} - 2^{60}$

$\square = 2(2^{60}) - 2^{60}$

$\square = 2^{60}(2-1)$

$\square = 2^{60}$

(b) $\square^2 = 625$

$\sqrt{\square^2} = \sqrt{625}$

$\square = 25$

12. $100 \div 5 = 20$ plus $1 = \mathbf{21}$ posts. 1 must be added because both end posts must be counted.

13. $1 \text{ mile} = 5280$ feet.

$5280 \div 6$ feet $= 880$ turns per mile.

880×50000 miles $=$ **44,000,000 turns**.

14. There are 9 students between 7 and 17 (8 through 16). There must be 9 between them in both directions, since they are direct opposites.
$9 + 9 + 2 =$ **20 students**.

15. Let l be a large box, m be a medium box, and s be a small box:

$3l + (3l \times 2m \text{ each}) + [(3 \times 2)m \times 5s \text{ each}]$

$3l + 6m + 30s =$ **39 total boxes**.

16. Look for the differences:

$$\begin{array}{ccccccccc} 5 & & 15 & & 37 & & 77 & & 141 \\ & 10 & & 22 & & 40 & & 64 & \\ & & 12 & & 18 & & 24 & & \end{array}$$

The second difference row is an arithmetic sequence with $d = 6$. Thus the 4th term in the second difference row is $24 + 6 = 30$; the5th term in the first difference row is $64 + 30 = 94$; and the 6th term in the original sequence is $141 + 94 = \mathbf{235}$.

17. Extend the pattern of doubling the number of ants each day. This is a geometric sequence with $a_1 = 1500,\ a_n = 100{,}000$, and $r = 2$.

$$100{,}000 = 1500 \cdot 2^{n-1} \Rightarrow$$
$$66\tfrac{2}{3} = 2^{n-1}.$$

Since $2^{7-1} < 66\frac{2}{3}$ and $2^{8-1} > 66\frac{2}{3}$, the ant farm will fill sometime between **the 7th and 8th day**.

18. The best strategy would be one of guessing and checking:

(***i***) Ten 3's + two 5's = 40… close but too low.

(***ii***) Nine 3's + three 5's = 42… still too low.

(***iii***) Eight 3's + Four 5's = 44.

They must have answered four 5-point questions.

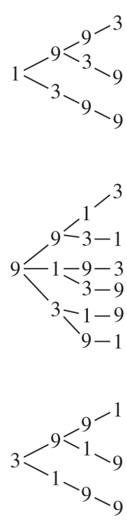
19. **Yes**. Let ℓ = length of the longest piece,
m = length of the middle-sized piece, and
s = lenth of the shortest piece.

Then $\ell = 3m$ and $s = m - 10$.

So $\ell + m + s = 90 \Rightarrow$
$3m + m + (m - 10) = 90 \Rightarrow$
$5m = 100$.

Thus $m = \mathbf{20\,cm}$;
$\ell = 3m = \mathbf{60\,cm}$; and
$s = m - 10 = 10$ **cm**.

20. Make a diagram that demonstrates all the ways four-digit numbers can be formed from left (thousands place) to right (ones place). **12 four-digit** numbers can be formed.



21. If n is 1 larger than m (i.e., n-m = 1, or equivalently, $n = m+1$), there would be 2 terms: m, and n (since $n = m + 1$). The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 2$.

If n is 2 larger than m (i.e., n-m = 2, or equivalently, $n = m+2$) there would be 3 terms: m, $m + 1$, and n, since $n = m + 2$). The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 3$.

If n is 3 larger than m there would be 4 terms: m, $m + 1$, $m + 2$, and n, since $n = m + 3$. The number of terms can be found by subtracting m from n and adding 1: $n - m + 1 = 4$.

Therefore, given that n is larger than m there would be $n - m + 1$ terms.

22. Answer may vary. Fill the 4-cup container with water and pour the water into the 7-cup container. Fill the 4-cup container again and pour water into the 7-cup container until it is full. Four minus three (1) cups of water will remain in the 4-cup container. Empty the 7-cup container and pour the contents of the 4-cup container into the 7- cup container. The 7-cup container now holds 1 cup of water. Refill the 4-cup container and pour it into the 7-cup container. The 7-cup container now contains exactly 5 cups of water.

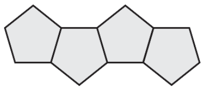
23. A possible pattern is to increase each rectangle by one row of dots and one column of dots to obtain the next term in the sequence. Make a table.

Number of the term	Row of dots	Column of dots	Term (row × column)
1	1	2	2
2	2	3	6
3	3	4	12
4	4	5	20
5	5	6	30
6	6	7	42
7	7	8	56
⋮			
100	100	101	10100
⋮			
n	n	$n + 1$	$n(n + 1)$

We also observe that the number of the term corresponds to the number of rows in the arrays and that the number of columns in the array is the number of the term plus one. Thus, the next three terms are **30, 42, and 56**. The 100th term is 10100 and the n^{th} term is $n(n + 1)$.

24. A possible pattern is that each successive figure is constructed by adjoining another pentagon to the previous figure

(a)



(b) Observe that the perimeter of the first figure is 5 and that when a new pentagon is adjoined 4 new sides are added and one side (where the new pentagon is adjoined) is lost. Make a table.

Number of terms	1-unit sides (perimeter)
1	5
2	$5 - 1 + 4 = 8$
3	$8 - 1 + 4 = 11$
4	$11 - 1 + 4 = 14$

(c) and **(d)** Looking at the terms in the sequence and noting that the difference of terms is three, we suspect that the sequence is arithmetic and conjecture that the n^{th} term is $3n+2$. However, we need to be sure. Looking at the 4th term (part a) we observe that the pentagons on the end Contribute 4 sides to the perimeter and the "middle" pentagons contribute 3 sides. Thus, in the n^{th} figure there will be 2 "end" pentagons that contribute 4 1-unit sides and $n - 2$ "middle" pentagons that contribute $3(n - 2)$ 1 unit sides. The total will be $\mathbf{3(n - 2) + 2(4)} = 3n + 2$ units. Thus the 100th term is $3(100) + 2 = \mathbf{30}2$.

25. **(a)** The circled terms will constitute an arithmetic sequence because the common difference will be twice the difference in the original series.

(b) The new sequence will be a geometric sequence because the ratio will be the square of the ratio of the original series.

26. If the sequence is $a_1, a_2, a_3, a_4, \ldots$ to find the first term we substitute $n = 1$ and get $1^2 - 1$ or 0, so $a_1=0$. For $n = 2$, we get 2^2-2 or 2. Thus $a_1 + a_2=2$; hence $a_2 = 2$. For $n = 3$, $a_1 + a_2 + a_3 = 3^2 - 3 = 6$. Substituting for a_1 and a_2, we get $a_3 = 6 - 2 = 4$. For $n = 4$, $a_1 + a_2 + a_3 + a_4 = 4^2 - 4 = 12$. Substituting for a_1, a_2, and a_3, we get $0 + 2 + 4 + a_4 = 12$. Hence $a_4 = 6$.

27. **(a)** Let's call the missing terms a, and b then the sequence becomes:

$$13, a, b, 27$$
$$13 + a = b \rightarrow a = b - 13$$
$$a + b = 27 \rightarrow b - 13 + b = 27$$
$$\rightarrow 2b = 40$$
$$\rightarrow b = 20$$
$$a = b - 13 \rightarrow a = 7$$

So 7 and 20 are the missing terms.

(b) Let's call the missing terms a, and b then the sequence becomes:

$$137, a, b, 163$$
$$137 + a = b \rightarrow a = b - 137$$
$$a + b = 163 \rightarrow b - 137 + b = 163$$
$$\rightarrow 2b = 300$$
$$\rightarrow b = 150$$
$$a = b - 137 \rightarrow a = 13$$

So 13 and 150 are the missing terms.

(c) Let's call the missing terms x, and y, then the sequence becomes:

$$b, x, y, a$$
$$b + x = y \rightarrow x = y - b$$
$$x + y = a \rightarrow y - b + y = a$$
$$\rightarrow 2y = a + b$$
$$\rightarrow y = \frac{a+b}{2}$$
$$x = \frac{a+b}{2} - b \rightarrow x = \frac{a+b}{2} - \frac{2b}{2}$$
$$\rightarrow x = \frac{a-b}{2}$$

So the missing terms are $\frac{a-b}{2}$ and $\frac{a+b}{2}$.

CHAPTER 2

INNTRODUCTION TO LOGIC AND SETS

Assessment 2-1A: Reasoning and Logic: An Introduction

1. **(a)** **False statement**. A statement is a sentence that is either true or false, but not both.

(b) **False statement.**

(c) **Not a statement.**

(d) **True statement.**

2. **(a)** **There exists at least one** natural number n such that $n + 8 = 11.$

(b) **There exists at least one** natural number n such that $n^2 = 4.$

(c) For **all** natural numbers n, $n + 3 = 3 + n.$

(d) For **all** natural numbers n, $5n + 4n = 9n.$

3. **(a)** For **all** natural numbers n, $n + 8 = 11.$

(b) For **all** natural numbers n, $n^2 = 4.$

(c) There is **no** natural number x such that $x + 3 = 3 + x.$

(d) There is **no** natural number x such that $5x + 4x = 9x.$

4. **(a)** The book **does not have** 500 pages.

(b) $3 \cdot 5 \neq 15.$

(c) **Some** dogs **do not have** four legs.

(d) **No** rectangles are squares.

(e) **All** rectangles are squares.

(f) **Some** dogs have fleas.

5. **(a)** If $n = 4$, or $n = 5$, then $n < 6$ *and* $n > 3$, so the statement is **true**, since it can be shown to work for some natural numbers n..

(b) If $n = 10$, then $n > 0$; *or* if $n = 1$, then $n < 5$, so the statement is **true**.

6. **(a)**

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

(b)

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

(c) **Yes**. The truth table entries are the same.

(d) **No**. The truth table entries are not the same.

7. **(a)**

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	
T	T	T	F	T	
T	F	F	F	F	
F	T	T	T	T	
F	F	T	T	T	

(b) Answers will vary. Here are two possible examples:1. Let p be "the Bobcats win" and q be "the Bobcats make the playoffs". Then Column 3 would read "If the Bobcats win, then the Bobcats make the playoffs". Column 5 would read "The Bobcats lose or the Bobcats make the playoffs."2. Let p be "it is summer vacation" and q be "I am at home". Then Column 3 would read "If it is summer vacation, then I am at home". Column 5 would read "It is not summer vacation or I am at home."

(c) In this problem, p is the statement "2 + 3 = 5" and q is the statement "4 + 6 = 10". That would make the statement in this problem be in the form of $\sim p \vee q$. So, it will be logically equivalent to a statement in the form of $p \to q$ or "if 2 + 3 = 5, then 4 + 6 = 10."

8. **(a)** $q \wedge r$. Both q and r are true.

(b) $r \vee \sim q$. r is true or q is not true.

(c) $\sim(q \wedge r)$. q and r are not both true.

(d) $\sim q$. q is not true.

9. **(a)** **False**. The statement is a conjunction. The two parts could be stated as such: p is the statement $2 + 3 = 5$ and q is the statement $4 + 7 = 10$. In this situation, p is true, but q is false. In order for a conjunction to be true, both p and q must be true; otherwise, the conjunction is false.

(b) **True**. This statement is false, since Barack Obama was president in 2013.

(c) **False**. The United States Supreme Court currently has nine justices.

(d) **True**. The only triangles that have three sides of the same length are equilateral triangles. In every case, an equilateral triangle will have two sides the same length as well.

(e) **False**. Isosceles triangles have two sides equal in length, but the third side is not equal to the other two.

10. **(a)** By DeMorgan's Laws, the negation of $p \wedge q$ is $\sim p \vee \sim q$. Therefore, the answer is $2 + 3 \neq 5$ or $4 + 7 \neq 10$.

(b) The president of the United States in 2013 was Barack Obama.

(c) With every seat filled, the Supreme Court of the United States does not have 12 justices.

(d) The triangle has three sides of the same length and the triangle does not have two sides of the same length.

(e) The triangle has two sides of the same length or the triangle does not have three sides of the same length.

In both (d) and (e) above, the negation of a conditional statement $p \rightarrow q$ is $p \wedge \sim q$.

11. **(a)**

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \vee q)$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	T

Since the truth values for $\sim p \vee \sim q$ are not the same as for $\sim(p \vee q)$, the statements are **not logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are not the same as for $\sim p \wedge \sim q$, the statements are **not logically equivalent**.

12. Dr. No is a male spy. He is not poor, and he is not tall.

13. If p = "it is raining" and q = "the grass is wet":

(a) $\boldsymbol{p \rightarrow q}$. **(b)** $\boldsymbol{\sim p \rightarrow q}$.

(c) $\boldsymbol{p \rightarrow \sim q}$.

(d) $\boldsymbol{p \rightarrow q}$. The hypothesis is "it is raining;" the conclusion is "the grass is wet."

(e) $\boldsymbol{\sim q \rightarrow \sim p}$. **(f)** $\boldsymbol{p \leftrightarrow q}$.

14. **(a)** **Converse**: If a triangle has no two sides of the same length, then the triangle is scalene. **Inverse**: If a triangle is not scalene, then the triangle has (at least) two sides of the same length. **Contrapositive**:If a triangle does not have two sides of the same length, then the triangle is not scalene. Note that this statement is a biconditional.

(b) **Converse:** If an angle is a right angle, then it is not acute. **Inverse**: If an angle is acute, then it is not a right angle. **Contrapositive:** If an angle is not a right angle, then it is acute. Note that the original statement and the contrapositive are not true, while the converse and inverse are true.

(c) **Converse**: If Mary is not a citizen of Cuba, then she is a U.S. citizen. **Inverse**: If Mary is not a U.S. citizen, then she is a citizen of Cuba. **Contrapositive**: If Mary is a citizen of Cuba, then she is not a U.S. citizen. Note that the original statement and the contrapositive are true, while the converse and inverse are not true

(d) **Converse**: If a number is not a natural number, then it is a whole number. **Inverse**: If a number is not a whole number, then it is a natural number. **Contrapositive**: If a number is a natural number, then it is not a whole number.

15. The statements are negations of each other.

p	q	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$	$p \to q$	$\sim(p \to q)$
T	T	F	F	T	T	F
T	F	T	T	F	F	T
F	T	T	F	T	T	F
F	F	F	F	T	T	F

16. The contrapositive is logically equivalent: "If a number is not a multiple of 4 then it is not a multiple of 8."

17. **(a)** **Valid.** This is valid by the transitivity property. "All squares are quadrilaterals" is $p \to q$ "all quadrilaterals are polygons" is $q \to r$; and "all squares are polygons" is $p \to r$.

(b) **Invalid.** We do not know what will happen to students who are not freshman. There is no statement "sophomores, juniors, and seniors do not take mathematics."

18.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

19. In the set of teachers, some have IQ's of 145 or more, and some are in Mensa. Notice that those two subsets intersect; their intersection represents the group of teachers who belong in both categories. Therefore, the argument is valid: some teachers with IQ's of 145 or more are in Mensa.

20. **(a)** Since all students in Integrated Mathematics I make A's, and some of those students are in Beta Club, then **some Beta Club students make A's**.

(b) Let $p =$ I study for the final, $q =$ I pass the final, $r =$ I pass the course, $s =$ I look for a teaching job. Then $p \to q$, if I study for the final, then I will pass the final. $q \to r$, if I pass the final, then I will pass the course. $r \to s$, if I pass the course, I will look for a teaching job. So $p \to s$, **if I study for the final I will look for a teaching job**.

(c) The first statement could be rephrased as "If a triangle is equilateral, then it is isosceles." Let $p =$ equilateral triangle and $q =$ isosceles triangle. So the first statement is $p \to q$, . The second statement is simply p ; then the conclusion should be q or **there exist triangles that are isosceles**.

21. **(a)** If a figure is a square, then it is a rectangle.

(b) If a number is an integer, then it is a rational number.

(c) If a polygon has exactly three sides, then it is a triangle.

22. **(a)** If $\sim p \vee \sim q \equiv \sim(p \wedge q)$, then
$\mathbf{3 \cdot 2 \neq 6}$ ***or*** $\mathbf{1 + 1 = 3}$.

(b) If $\sim p \wedge \sim q \equiv \sim(p \vee q)$, then you cannot pay me now and you cannot pay me later.

Assessment 2-1B

1. **(a)** **Not a statement**. A statement is a sentence that is either true or false.

(b) **Not a statement**. A statement must be either true of false; this could be either.

(c) True statement.

(d) **False statement**. $2 + 3 \neq 8$.

(e) Not a statement.

2. **(a)** For **all** natural numbers n, $n + 0 = n$.

(b) There exists **no** natural number n such that $n + 1 = n + 2$.

(c) **There exists at least one** natural number n such that $3 \cdot (n + 2) = 12$.

(d) **There exists at least one** natural number n such that $n^3 = 8$.

3. **(a)** There is **no** natural number n such that $n + 0 = n$.

(b) **There exists at least one** natural number n such that $n + 1 = n + 2$.

(c) For **all** natural numbers n, $3 \cdot (n + 2) \neq 12$.

(d) For **all** natural numbers n, $n^3 \neq 8$.

4. **(a)** Six is **greater than** or equal to 8. Another way to express this would be to say 6 **is not less than** 8.

(b) **All** cats **have** nine lives. Another way to express this would be to say that **no** cats **do not have** nine lives.

(c) There exists **a** square that is **not** a rectangle. Another way to express this would be to say **some** squares **are not** rectangles.

(d) **All** numbers **are** positive.

(e) **No** people **have** blond hair.

5. **(a)** If $n = 10$, then $n > 5$ and $n > 2$, so the statement is **true**.

(b) x could equal 5, so the statement is **false**.

6. **(a)** $p \vee q$ is false only if both p and q are false, so if p is true the statement is **true** regardless of the truth value of q.

(b) An implication is false only when p is true and q is false, so if p is false then the statement is **true** regardless of the truth value of q.

7. **(a)** $\boldsymbol{q \wedge r}$.

(b) $\boldsymbol{q \wedge \sim r}$.

(c) $\boldsymbol{\sim r \vee \sim q}$.

(d) $\boldsymbol{\sim(q \wedge r)}$.

8. **(a)** **True**. This statement is a disjunction. The two parts could be stated as such: p is the statement "4 + 6 = 10", while q is the statement "2 + 3 = 5". In this situation p is true, and q is true. In order for a disjunction to be true, either p or q (or both) have to be true; the only way a disjunction can be false is if both p and q are false. So therefore, this statement is true.

(b) **False**. If a team has more than 11 players on the field, it is a penalty and the play will not count.

(c) True.

(d) **False**. To see, sketch a drawing where three sides are the same length, but with the two angles where the sides intersect being different measures (in fact, make one a right angle, the other an obtuse angle). You should easily make a quadrilateral with a side length different from the other three.

(e) **True**. If a rectangle has four sides of the same length, then by default it must have three sides the same length. Of course, a rectangle with four equal sides is a square!.

9. **(a)** By DeMorgan's Laws, the negation of the disjunction $p \vee q$ is $\sim p \wedge \sim q$. So, the statement would be $4 + 6 \neq 10$ and $2 + 3 \neq 5$

(b) A National Football League team cannot have more than 11 players on the field while a game is in progress.

(c) The first president of the United States was not George Washington.

(d) A quadrilateral has three sides of the same length and the quadrilateral does not have four sides of the same length.

(e) A rectangle has four sides of the same length and that rectangle does not have three sides of the same length.

In both (d) and (e), the negation of the conditional statement $\boldsymbol{p \to q}$ is $\boldsymbol{p \wedge \sim q}$.

10. **(a)**

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Since the truth values for $\sim(p \vee q)$ are the same as for $\sim p \wedge \sim q$, the statements **are logically equivalent**.

(b)

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Since the truth values for $\sim(p \wedge q)$ are the same as for $\sim p \vee \sim q$, the statements are **logically equivalent**.

11. Ms. Makeover is not single. She has straight blond hair.

12. **(a)** $\boldsymbol{p \to q}$.

(b) $\boldsymbol{\sim p \to q}$.

(c) $\boldsymbol{p \to \sim q}$.

(d) q if p, or $\boldsymbol{p \to q}$.

(e) $\boldsymbol{\sim p \to \sim q}$.

(f) $\sim q \to \sim p$.

13. **(a)** Converse: **If $\boldsymbol{x^2 = 9}$, then $\boldsymbol{x = 3}$**.

Inverse: **If $\boldsymbol{x \neq 3}$, then $\boldsymbol{x^2 \neq 9}$**.

Contrapositive: **If $\boldsymbol{x^2 \neq 9}$, then $\boldsymbol{x \neq 3}$**.

(b) Converse: If classes are canceled, then it snowed.

Inverse: If it does not snow, then classes are not canceled.

Contrapositive: If classes are not canceled, then it did not snow.

14. **No**. This is the inverse; i.e., if it does not rain then Iris can either go to the movies or not without making her statement false.

15. **(a)** **Valid**. Use modus ponens: Hypatia was a woman $\to$ all women are mortal $\to$ Hypatia was mortal.

(b) **Valid**. Since *Dirty Harry* was not written by J.K. Rowling, and she wrote all the *Harry Potter* books, then *Dirty Harry* cannot be a *Harry Potter* book.

(c) **Not valid**. There exist some whole numbers that are not natural numbers. It might be easier to understand if the word seven is replaced by a variable "x". So, it reads: Some whole numbers are not natural numbers. "x" is a whole number. Conclusion: "x" is a natural number.

16. **(a)** Since all students in Integrated Mathematics I are in Kappa Mu Epsilon, and Helen is in Integrated Mathematics I, then the conclusion is that **Helen is in Kappa Mu Epsilon**.

(b) Let p = all engineers need mathematics and q = Ron needs mathematics.

Then $p \to q$, or if all engineers need mathematics then Ron needs mathematics.

p is true, but q is false, Ron does not need mathematics.

So Ron is not an engineer.

(c) Since all bicycles have tires and all tires use rubber, then the conclusion is **all bicycles use rubber.**

17. **(a)** **If** a number is a natural number, **then** it is a real number.

(b) **If** a figure is a circle, **then** it is a closed figure.

18. DeMorgan's Laws are that:

$\sim(p \wedge q)$ is the logical equivalent of $\sim p \vee \sim q$.

$\sim(p \vee q)$ is the logical equivalent of $\sim p \wedge \sim q$.

Thus:

(a) The negation is $\mathbf{3 + 5 = 9}$ **or** $\mathbf{3 \cdot 5 \neq 15}$.

(b) The negation is I am not going and she is not going.

19. Therefore, a square is a parallelogram.

Assessment 2-2A: Describing Sets

1. **(a)** Either a list or set-builder notation may be used: $\{\mathbf{a, s, e, m, n, t}\}$ or $\{x \mid x$ **is a letter in the word *assessment*$\}$**.

(b) $\{\mathbf{21, 22, 23, 24, \ldots}\}$ or $\{x \mid x$ **is a natural number and** $x > 20\}$ or $\{x \mid x \in N$ **and** $x > 20\}$.

2. **(a)** $P = \{p, q, r, s\}$.

(b) $\{\mathbf{1, 2}\} \subset \{\mathbf{1, 2, 3}\}$. The symbol $\subset$ refers to a proper subset.

(c) $\{\mathbf{0, 1}\} \not\subseteq \{\mathbf{1, 2, 3}\}$. The symbol $\subseteq$ refers to a subset.

3. **(a)** **Yes**. $\{1, 2, 3, 4, 5\} \sim \{m, n, o, p, q\}$ because both sets have the same number of elements and thus exhibit a one-to-one correspondence.

(b) **Yes**. $\{a, b, c, d, e, f, \ldots, m\} \sim \{1, 2, 3, \ldots, 13\}$ because both sets have the same number of elements.

(c) **No**. $\{x \mid x$ is a letter in the word *mathematics*$\}$ $\not\sim \{1, 2, 3, 4, \ldots, 11\}$; there are only eight unduplicated letters in the word *mathematics*.

4. **(a)** The first element of the first set can be paired with any of the six in the second set, leaving five possible pairings for the second element, four for the third, three for the fourth, two for the fifth, and one for the sixth. Thus there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{720}$ one-to-one correspondences.

(b) There are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$ possible one-to-one correspondences. The first element of the first set can be paired with any of the n elements of the second set; for each of those n ways to make the first pairing, there are $n - 1$ ways the second element of the first set can be paired with any element of the second set; which means there are $n - 2$ ways the third element of the first set can be paired with any element of the third set; and so on. The Fundamental Counting Principle states that the choices can be multiplied to find the total number of correspondences.

5. **(a)** If x must correspond to 5, then y may correspond to any of the four remaining elements of $\{1, 2, 3, 4, 5\}, z$ may correspond to any of the three remaining, etc. Then $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ **one-to-one correspondences**.

(b) There would be $1 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = \mathbf{6}$ **one-to-one correspondences**.

(c) The set $\{x, y, z\}$ could correspond to the set $\{1,3,5\}$ in $3 \cdot 2 \cdot 1 = 6$ ways. The set $\{u, v\}$ could correspond with the set $\{2,4\}$ in $2 \cdot 1 = 2$ ways. There would then be $6 \cdot 2 = \mathbf{12}$ **one-to-one correspondences**.

6. **(*i*)** $\boldsymbol{A = C}$. The order of the elements does not matter.

(*ii*) $\boldsymbol{E = H}$; they are both the null set.

(*iii*) $\boldsymbol{I = L}$. Both represent the numbers $1,3,5,7,\ldots$.

7. **(a)** Assume an arithmetic sequence with $a_1 = 201, a_n = 1100,$ and $d = 1$. Thus $1100 = 201 + (n-1) \cdot 1$; solving, $n = 900$. The cardinal number of the set is therefore **900**.

(b) Assume an arithmetic sequence with $a_1 = 1, a_n = 101,$ and $d = 2$. Thus $101 = 1 + (n-1) \cdot 2$; solving, $n = 51$. The cardinal number of the set is therefore **51**.

(c) Assume a geometric sequence with $a_1 = 1, a_n = 1024,$ and $r = 2$. Thus $1024 = 1 \cdot 2^{n-1} \Rightarrow 2^{10} = 2^{n-1} \Rightarrow n - 1 = 10 \Rightarrow n = 11$. The cardinal number of the set is therefore **11**.

(d) If $k = 1,2,3,\ldots,100$, the cardinal number of the set $\{x \mid x = k^3, k = 1,2,3,\ldots,100\} = \mathbf{100}$, since there are 100 elements in the set.

8. $\overline{A}$ represents all elements in U that are not in A, or the set of all college students with at least one grade that is not an A.

9. **(a)** A proper subset must have at least one less element than the set, so the maximum $\boldsymbol{n(B) = 7}$.

(b) Since $B \subset C$, and $n(B) = 8$ then C could have any number of elements in it, so long as it was greater than eight.

10. **(a)** The sets arc equal, so $\boldsymbol{n(D) = 5}$.

(b) Answers vary.For example, the sets are equal; the sets are also equivalent

11. **(a)** A has 5 elements, thus $2^5 = \mathbf{32}$ **subsets**.

(b) Since A is a subset of A and A is the only subset of A that is not proper, A has $2^5 - 1 =$ **31 proper subsets**.

(c) Let $B = \{b,c,d\}$. Since $B \subset A$, the subsets of B are all of the subsets of A that do not contain a and e. There are $2^3 = 8$ of these subsets. If we join (union) a and e to each of these subsets there are still **8 subsets**.

Alternative. Start with $\{a,e\}$. For each element b, c, and d there are two options: include the element or don't include the element. So there are $2 \cdot 2 \cdot 2 = 8$ ways to create subsets of A that include a and e.

12. If there are n elements in a set, 2^n subsets can be formed. This includes the set itself. So if there are 127 proper subsets, then there are 128 subsets. Since $2^7 = 128$, the set has **7 elements**.

13. In roster format, $A = \{3,6,9,12,\ldots\}, B = \{6,12,18,24,\ldots\}$, and $C = \{12,24,36,\ldots\}$. Thus, $C \subset A, C \subset B$, and $B \subset A$.

Alternatively: $12n = 6(2n) = 3(4n)$. Since $2n$ and $4n$ are natural number $\boldsymbol{C \subset A, C \subset B}$, and $\boldsymbol{B \subset A}$.

14. **(a)** $\notin$. There are no elements in the empty set.

(b) $\in$. $1024 = 2^{10}$ and $10 \in N$.

(c) $\in$. $3(1001) - 1 = 3002$ and $1001 \in N$.

(d) $\notin$. For example, $x = 3$ is not an element because for $3 = 2^n$, $n \notin N$.

15. **(a)** $\not\subseteq$. 0 is not a set so cannot be a subset of the empty set, which has only one subset, $\varnothing$.

(b) $\not\subseteq$. 1024 is an element, not a subset.

(c) $\not\subseteq$. 3002 is an element, not a subset.

(d) $\not\subseteq$. x is an element, not a subset.

16. **(a)** **Yes**. Any set is a subset of itself, so if $A = B$ then $A \subseteq B$.

(b) **No**. A could equal B; then A would be a subset but not a proper subset of B.

(c) **Yes**. Any proper subset is also a subset.

(d) **No**. Consider $A = \{1, 2\}$ and $B=\{1,2,3\}$.

17. **(a)** Let $A = \{1, 2, 3, \ldots, 100\}$ and $B = \{1, 2, 3\}$. Then $n(A) = 100$ and $n(B) = 3$. Since $B \subset A$, $\boldsymbol{n(B) = 3 < 100 = n(A)}$.

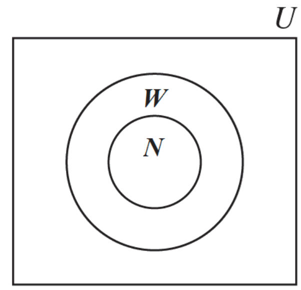
(b) $n(\emptyset) = 0$. Let $A = \{1, 2, 3\} \Rightarrow n(A) = 3$. $\emptyset \subset A$, which implies that there is at least one more element in A than in $\emptyset$. Thus $\mathbf{0 < 3}$.

18. There are seven senators to choose from and 3 will be chosen. Consider the ways to form subsets with only three members. If we pick the first member, there are 7 senators to choose from. To pick the second member, there are only 6 to choose from, since 1 member has already been chosen. For the third seat, there are 5 to choose from. This yields $7 \cdot 6 \cdot 5$. However, this calculation counts {Able, Brooke, Cox} as a different committee that {Brooke, Able, Cox}. In fact, for any 3 names, there are $3 \cdot 2 \cdot 1$ ways to arrange the names. Thus, the number of unique committees is $7 \cdot 6 \cdot 5/3 \cdot 2 \cdot 1 = 7 \cdot 5 = \mathbf{35}$.

19. Answers vary. For example, the set of all odd natural numbers and the set of all even natural numbers are two infinite sets that are equivalent but not equal.. Another possibility is the set of all natural numbers and the set of all whole numbers.

20. Each even natural number $2n$ can be paired with each odd natural number $2n - 1$ in a one-to-one correspondence.

21.



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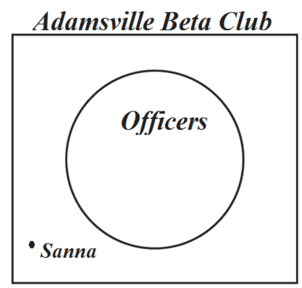


23. Answers vary. Example: All members of the Adamsville Beta Club are officers.

Assessment 2-2B

1. **(a)** Either a list or set-builder notation may be used: $\{\boldsymbol{a, l, g, e, b, r}\}$ or $\{x | x$ **is a letter in the word *algebra*}**

(b) $\mathbf{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}}$ or $\{x | x$ **is a natural number and** $x<10\}$ or $\{x | x \in N$ **and** $x<10\}$.

2. **(a)** $\boldsymbol{Q = \{q, r, s\}}$

(b) $\mathbf{\{1, 3\} = \{3, 1\}}$. The symbol $=$ refers to the sets being equal (containing the same elements).

(c) $\mathbf{\{1, 3\} \not\subset \{1, 4, 6\}}$. The symbol $\not\subset$ refers to "not a proper subset."

3. **(a)** **Yes.** $\{1, 2, 3, 4\} \sim \{w, c, y, z\}$ because both sets have the same number of elements and thus exhibit a one-to-one correspondence.

(b) **Yes**, because both sets have the same number of elements.

(c) **No**. $\{x|x$ is a letter in the word *geometry*$\}$ $\nsim \{1,2,3,4,...,8\}$; there are only seven unduplicated letters in geometry.

4. **(a)** The first element of the first set can be paired with any of the eight in the second set, leaving seven possible pairings for the second element, six for the third, five for the fourth, etc. Thus, there are $8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 =$ **40,320** one-to-one correspondences.

(b) There are $\boldsymbol{(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot3\cdot2\cdot1}$ possible one-to-one correspondences. The first element of the first set can be paired with any of the $n-1$ elements of the second set; there are $n-2$ ways the second element of the first set can be paired with any element of the second set; which means there are $n-3$ ways the third element of the first set can be paired with any element of the third set; and so on. The Fundamental Counting Principle states that the choices can be multiplied to find the total number of correspondences

5. **(a)** If b must correspond to 3, then a may correspond to any of the three remaining elements of $\{1,2,3,4\}$, and c may correspond to any of the two remaining, etc. Then $1\cdot3\cdot2\cdot1 =$ **6 one-to-one correspondences.**

(b) **T**here would be a $1\cdot1\cdot2\cdot1 =$ **2 one-to-one correspondences.**

(c) The set $\{a,c\}$ could correspond to the set $\{2,4\}$ in $2\cdot1 = 2$ ways. The set $\{b,d\}$ could correspond with the set $\{1,3\}$ in $2\cdot1 = 2$ ways. There would then be $2\cdot2 =$ **4 one-to-one correspondences.**

6. $\boldsymbol{A = C}$.

7. **(a)** Assume an arithmetic sequence with $a_1 = 19$, $a_n = 99$, and $d = 1$. Thus $99 = 19 + (n-1)\cdot 1$; solving, $n = 81$. The cardinal number of the set is therefore **81**.

(b) Assume an arithmetic sequence with $a_1 = 2$, $a_n = 1002$, and $d = 2$. Thus $1002 = 2 + (n-1)\cdot 2$; solving, $n = 501$. The cardinal number of the set is therefore **501**.

(c) Assume an arithmetic sequence with $a_1 = 1$, $a_n = 99$, and $d = 2$. Thus $99 = 1 + (n-1)\cdot 2$; solving, $n = 50$. The cardinal number of the set is therefore **50**.

(d) There are no natural numbers that have the property $x = x + 1$, so the set is empty and the cardinal number is **0**.

8. $\overline{G}$ represents all elements in U that are not in G, or the set of all women who are not alumni of Georgia State University. In set-builder notation, $\overline{G} = \{x|x$ is a woman who is not a graduate of Georgia State University$\}$.

9. **(a)** Since the empty set is a subset of any set, A could be the empty set; then the minimum number of elements in A would be **0**.

(b) **Yes**. Since A is not assumed to be a proper subset, A and B could be equal . Thus, both sets could be empty.

10. To be subsets of each other, the two sets must be **equal** and **equivalent**.

11. **(a)** Since $A = \{1,2,3,4,5,6,7,8,9\}$ has 9 elements, A has $2^9 =$ **512 subsets**.

(b) Since A is a subset of A and not proper, A has $2^9 - 1 =$ **511 subsets**.

12. If there are n elements in a set, 2^n subsets can be formed. Thus, $16 = 2^n$ and $n =$ **4 elements**.

13. In roster format, $A = \{4,7,10,13,16,...\}$, $B = \{7,13,18,...\}$, and $C = \{13,25,38,...\}$.
Thus, $C \subset A$, $C \subset B$, and $B \subset A$.
Alternative: $12n + 1 = 6(2n) + 1 = 3(4n) + 1$. Since $2n$ and $4n$ are natural numbers $\boldsymbol{C \subset A}$, $\boldsymbol{C \subset B}$, and $\boldsymbol{B \subset A}$.

14. **(a)** $\in$. The set containing the empty set has one element; the empty set.

(b) $\in$. $1022 = 2^{10} - 2$ and $10 \in N$.

(c) $\in$. $3(1001) + 1 = 3004$ and $1001 \in N$.

(d) $\in$. 17 is an element of the natural numbers.

15. **(a)** **No**. For example, if $A = \{1\}$ and $B = \{1,2\}$, then $A \subseteq B$ but $A \neq B$.

(b) **No**. $A \subset B$ implies that A must have at least one less element than B.

(c) **No**. A and B must have the same number of elements, but not necessarily be equal; for example, if $A = \{1,2\}$ and $B = \{a,b\}$.

(d) **No**. See part (c).

16. **(a)** $n(\emptyset) = 0$. Let $A = \{1,2\} \Rightarrow n(A) = 2$. $\emptyset \subset A$, which implies that there is at least one more element in A than in $\emptyset$. Thus $\mathbf{0 < 2}$.

(b) Let $A = \{1,2,3,\ldots 99\}$ and $B = \{1,2,3,\ldots, 100\}$. $n(A) = 99$ and $n(B) = 100$, but $A \subset B$ so $\boldsymbol{n(A) = 99 < 100 = n(B)}$.

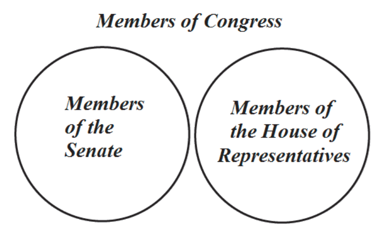
17. **(a)** There are 4 flavors from which to pick the first scoop, leaving 3 from which to pick the second, 2 from which to pick the third, and only 1 from which to pick the last. Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = \mathbf{24}$ **ways to pick the four flavors**.

(b) Each of the four scoops may be picked in four different ways, thus there are $4^4 = \mathbf{256}$ **ways to pick the four flavors**.

18. 6. Note that $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. The Fundamental Counting Principle states that the product of choices gives the number of ways that the one-to-one correspondence can happen; since there are 720 one-to-one correspondences, then there must be six elements in each set.

19. Let every natural number n be paired with the number $n - 1$ from the set of whole numbers. The pairing is a one-to-one correspondence between the set of natural numbers and the set of whole numbers.

20. The Congress of the United States consists of all members of the Senate, and all members of the House of Representatives; however, no member of the Senate is also a member of the House of Representatives, and vice-versa. Therefore, a Venn diagram consisting of two disjoint circles will work for this problem, as shown below:



21. Since both sets are equal, that means both sets contain all of the same elements, by defintion of equal sets. So, you can infer that every swimmer in the 100 meter butterfly race is from the Maryville Swim Team.

22. For voting in a primary election, some states make voters declare a party affiliation (so that you vote in that parties' primary only). No state would allow a voter to declare as both a Republican and a Democrat, so the Venn diagram would consist of two disjoint circles, as shown below: