A Modern Introduction to Differential Equations

Third Edition

Instructor's Resource Manual

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Preface

In addition to the solutions to all the exercises, I have provided teaching suggestions, comments on the text and exercises, historical tidbits, and references, including some with links to the Internet. Some of the references in this manual can form the basis for independent study reports or modeling projects. If differential equations are your forte and you're experienced with using technology in the classroom, forgive me for preaching to the choir. If, however, you are either new to teaching differential equations or do not have experience using technology in your teaching, I hope you find some help in this manual.

Since I do not emphasize mathematical modeling in the text, I offer the following supplementary resources:

CODEE, the **C**ommunity **o**f **D**ifferential **E**quations **E**ducators (originally C•ODE•E, with a slightly different translation): https://scholarship.claremont.edu/codee/. This link takes you to the home page of the *CODEE Journal*, an open access journal. As I write this *Preface*, the latest issue (Volume 12, 2019) is devoted to the theme "Linking Differential Equations to Social Justice and Environmental Concerns." If you click on Resources, you can access such goodies as a bibliography of papers on the teaching and learning of ODEs, senior theses involving differential equations, and past issues of the C•ODE•E *Newsletter* (1992–1997), which was a major inspiration for the first edition of this text. You can also download the 368 page *Differential Equations Laboratory Workbook* by Borelli, Coleman, and Boyce.

SIMIODE, the **S**ystematic **I**nitiative for **M**odeling **I**nvestigations and **O**pportunities with **D**ifferential **E**quations: https://www.simiode.org/resources. Among the categories of resources listed here are Articles and Publications, Modeling Scenarios, and Potential Scenario Ideas. This is a rich source of modeling material.

I have established a website for this book: **www.moderndifferentialequations.com**. I plan to update it regularly with corrections and supplementary material (exercises, references to recent journal articles, user comments, etc.). Your comments on the book and manual should be sent to **henry@mec.cuny.edu**, preferably with some ODE reference in the subject line.

CHAPTER

Introduction to differential equations

1

1.1 Basic terminology

If students have seen an introduction to differential equations as part of the calculus sequence, this section can be covered quickly. However, I've found that a review is always helpful. In particular, students who have recently taken two or more semesters of calculus sometimes have trouble distinguishing dependent variables from independent variables in differentiation problems and have trouble with dummy variables in integration. I want students to focus on the *form* of a differential equation, not on the particular variables used or on the derivative notation employed. Throughout the book I have deliberately mixed the Leibniz (d/dx), Newton (dot), and Lagrange (prime) notations for derivatives, although the dot notation becomes dominant in later chapters as the dynamical systems interpretation becomes more pronounced.

Parameters in equations often cause difficulty, but the student should become more comfortable with this concept as the course progresses. Students will have trouble with the general form(s) of an nth-order differential equation, especially if they are not familiar with functions of several variables. Concrete examples are necessary. Many students need time to understand the idea of a *linear* differential equation. Later discussions of linearity (in particular, the Superposition Principle) should help.

The idea of a *system* of differential equations is introduced early because of its importance in later chapters, but sometimes I postpone any classroom mention of systems until Chapter 5 or 6. If the students are reading the book (*Ha!*, you say), they'll see this on their own.

The text is dedicated to the proposition that technology is a valuable tool that can aid a student's understanding and that may be essential in solving certain problems. The November 1994 issue (Vol. 25, No. 5) of the *College Mathematics Journal* is devoted to the teaching of differential equations. However, I don't want to spend a great deal of valuable class time teaching the intricacies of the syntax of any CAS or other software I may be using. For example, in using *Maple* I've found that a few basic commands should be mastered and used over and over again, making minimal changes to accommodate different problems. I've handed out summaries of these commands and have encouraged students to use the "Help" system. Getting comfortable with the various options (numerical, graphical, and analytic) may take some time, and I have learned to avoid embarrassment in class by preparing ahead of time and saving examples on a USB drive. I hand out hard copies of certain *Maple* worksheets for the students to use as templates. Students sometimes make their own

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electronic copies of worksheets that may be on a departmental server. There are many books dealing with ODEs and various computer algebra systems. However, it's important for students to realize that computers don't have all the answers. I've found that showing problems that the CAS can't solve or can only solve incompletely is a sound pedagogical technique.

There is a wealth of ODE information on the Internet. See, for example, my article "Internet Resources for Differential Equations" in the MAA newsletter *Focus* (May/June, 2002; August/September, p. 24). (Past issues of *Focus* can be found at www.maa.org/press/periodicals/maa-focus#Past.) Some of the links may no longer be viable, but there are still some good search tips in the article.

For the history of differential equations I recommend the classic book *Ordinary Differential Equations* by E.L. Ince (New York: Dover Publications, 1956); *Analysis by Its History* by E. Hairer and G. Wanner (New York: Springer-Verlag, 1996); and *Mathematics of the 19th Century* by A.N. Kolmogorov and A.P. Yushkevich (Boston: Birkhäuser, 1998). Two Internet sources are www.eoht.info/page/History+of+differential+equations and a brief history (with references) by John E. Sasser: http://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=C56A1A79F28D5CD1E873E9AEA1610BAB?doi=10.1.1.112.3646&rep=rep1&type=pdf.

"Ten Lessons I Wish I Had Learned Before I Started Teaching Differential Equations" by Gian-Carlo Rota (https://web.williams.edu/Mathematics/lg5/Rota.pdf) is an amusing and insightful article by a master.

Finally, "The Dynamical Systems Approach to Differential Equations" by Morris W. Hirsch (*Bulletin* of the American Mathematical Society, **11**, Number 1, July 1984), although somewhat dated, is an authoritative survey of the early history of the subject, whereas "Dynamical Systems Theory: What in the World Is It?" by Mike Hochman (http://math.huji.il/~mhochman/research-expo.html) is a more modern *apologia* for this field of study.

A

1. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) linear
2. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) linear
3. (a) The independent variable is not indicated, but the dependent variable is x; (b) second-order; (c) nonlinear because of the term $\exp(-x)$—the equation cannot be written in the form (1.1.1), where y is replaced by x and x is replaced by the independent variable.
4. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because of the term $(y')^2 = y'(x) \cdot y'(x)$—the equation cannot be written in the form (1.1.1).
5. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because you get the terms $x^2(y')^2$ and $x\,y'\,y$ when you remove the parentheses.

6. (a) The independent variable is t and the dependent variable is r; (b) second-order; (c) linear

7. (a) The independent variable is x and the dependent variable is y; (b) fourth-order; (c) linear

8. (a) The independent variable is t and the dependent variable is y; (b) second-order; (c) nonlinear because of the term $-y'(y^2-1)$

9. (a) The independent variable is t and the dependent variable is x; (b) third-order; (c) linear

10. (a) The independent variable is t and the dependent variable is x; (b) seventh-order; (c) linear

11. (a) The independent variable is x and the dependent variable is y; (b) first-order; (c) nonlinear because of the term $e^{y'}$

12. (a) The independent variable is t and the dependent variable is R; (b) third-order; (c) linear

13. **a.** Nonlinear; the first equation is nonlinear because of the term $4xy = 4x(t)y(t)$.

b. Linear

c. Nonlinear; the first and second equations are nonlinear because each contains a product of dependent variables.

d. Linear

14. Using the "B" definition of the FTC in Section A.4, we have $y(x) = \int_1^x \sin t\, dt$, $y'(x) = \sin x$, $y''(x) = \cos x$, $y'''(x) = -\sin x$. Therefore, $y'''(x) + y'(x) = -\sin x + \sin x = 0$.

B

15. The terms $(a^2 - a)x\frac{dx}{dt}$ and $te^{(a-1)x}$ make the equation nonlinear. If $a^2 - a = 0$—that is, if $a = 0$ or $a = 1$—then the first troublesome term disappears. However, only the value $a = 1$ makes the second nonlinear term vanish as well. Thus, $a = 1$ is the answer.

16. **a.** $\frac{dx}{dt} = \ln(2^x) = x\ln 2 = (\ln 2)x$, a linear equation

b. $x' = \begin{cases} \frac{x^2-1}{x-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = \begin{cases} x+1 & \text{for } x = 1 \\ 2 & \text{for } x = 1 \end{cases} = x + 1$ for all x, which is linear

c. $x' = \begin{cases} \frac{x^4-1}{x^2-1} & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = \begin{cases} x^2+1 & \text{for } x \neq 1 \\ 2 & \text{for } x = 1 \end{cases} = x^2 + 1$ for all x, which is nonlinear.

C

17. Using the formula for arc length and the formula for area under a curve, our given conditions translate to $\int_a^x \sqrt{1 + \{f'(t)\}^2}\, dt = \int_a^x f(t)\, dt$. Differentiating each side, we get $\sqrt{1+\{f'(x)\}^2} = f(x)$, which implies that $1+\{f'(x)\}^2 = f^2(x)$, or $\{f'(x)\}^2 = f^2(x) - 1$.

1.2 Solutions of differential equations

If students have worked with differential equations before, much of this material can be covered quickly as a review. However, before the formal solution methods are discussed in Chapter 2, I want the students to develop some facility in guessing and verifying solutions of differential equations. *Implicit* solutions are important, and students usually need a quick review of implicit differentiation. You may also want to introduce/review the use of technology in plotting implicit functions.

The following reference works are dictionaries/encyclopedias of differential equations and their solutions:

Ordinary Differential Equations and Their Solutions by G.M. Murphy (New York: Van Nostrand, 1960)

Handbook of Exact Solutions for Ordinary Differential Equations by A.D. Polyanin and V.F. Zaitsev (Boca Raton: CRC Press, 1995)

A Compendium on Nonlinear Ordinary Differential Equations by P.L. Sachdev (New York: John Wiley & Sons, 1997)

Handbook of Differential Equations (*Third Edition*) by D. Zwillinger (New York: Academic Press, 1998)

A

1. $y = \sin x$, $y' = \cos x$, $y'' = -\sin x$; thus, $y'' + y = -\sin x + \sin x = 0$.

2. $x = -\pi\, e^{3t} + \frac{2}{3}e^{2t}$, $x' = -3\pi\, e^{3t} + \frac{4}{3}e^{2t}$, $x'' = -9\pi\, e^{3t} + \frac{8}{3}e^{2t}$; thus, $x'' - 5x' + 6x = \left(-9\pi\, e^{3t} + \frac{8}{3}e^{2t}\right) - 5\left(-3\pi\, e^{3t} + \frac{4}{3}e^{2t}\right) + 6\left(-\pi\, e^{3t} + \frac{2}{3}e^{2t}\right) = -9\pi\, e^{3t} + \frac{8}{3}e^{2t} + 15\pi\, e^{3t} - \frac{20}{3}e^{2t} - 6\pi\, e^{3t} + \frac{12}{3}e^{2t} = 0$.

3. $y = x^2$, $dy/dx = 2x$; thus, $(1/4)(dy/dx)^2 - x(dy/dx) + y = (1/4)(2x)^2 - x(2x) + x^2 = (1/4)(4x^2) - 2x^2 + x^2 = x^2 - 2x^2 + x^2 = 0$.

4. $R = t(c - \cos t)$, $dR/dt = t(\sin t) + (c - \cos t)$; thus, $t(dR/dt) - R = t(t\sin t + c - \cos t) - t(c - \cos t) = t^2\sin t + ct - t\cos t - tc + t\cos t = t^2 \sin t$.

5. $y = at^3 + bt^2 + ct + d$, $dy/dt + 3at^2 + 2bt + c$, $d^2y/dt^2 = 6at + 2b$, $d^3y/dt^3 = 6a$, $d^4y/dt^4 = 0$.

6. $r = ce^{bt} - (a/b)t - a/b^2$, $dr/dt = bce^{bt} - a/b$; thus, $at + br = at + b\left(ce^{bt} - (a/b)t - a/b^2\right) = at + bce^{bt} - at - a/b = bce^{bt} - a/b = dr/dt$.

7. $y = \ln x^2$, $y' = \left(1/x^2\right)(2x) = 2/x$; thus, $xy' - 2 = x(2/x) - 2 = 2 - 2 = 0$.

8. $y = \left(e^{ax} + e^{-ax}\right)/2a$, $y' = \left(ae^{ax} - ae^{-ax}\right)/2a = \left(e^{ax} - e^{-ax}\right)/2$, $y'' = \left(a^2e^{ax} + a^2e^{-ax}\right)/2a = \left(ae^{ax} + ae^{-ax}\right)/2$; thus,

$$a\sqrt{1+(y')^2} = a\sqrt{1+\left(e^{ax}-e^{-ax}\right)^2/4} = a\sqrt{\frac{4+\left(e^{2ax}-2+e^{-2ax}\right)}{4}}$$

$$= a\sqrt{\frac{2+\left(e^{2ax}+e^{-2ax}\right)}{4}} = a\sqrt{\frac{\left(e^{ax}+e^{-ax}\right)^2}{4}} = a\sqrt{\left(\frac{\left(e^{ax}+e^{-ax}\right)}{2}\right)^2}$$

$$= a\frac{\left(e^{ax}+e^{-ax}\right)}{2} = \left(ae^{ax}+ae^{-ax}\right)/2 = y''.$$

9. $y = \int_1^x \frac{\sin t}{t}\,dt$; By the FTC, we have $y' = \frac{\sin x}{x}$, so that $xy' - \sin x = x(\sin x/x) - \sin x = \sin x - \sin x = 0$. (See Appendix A.4, statement (B), for the FTC.)

10. $y = \int_3^x e^{-t^2}\,dt$, so that $y' = e^{-x^2}$ and $y'' = -2xe^{-x^2}$. Thus, $y'' + 2xy' = \left(-2xe^{-x^2}\right) + 2x\left(e^{-x^2}\right) = -2xe^{-x^2} + 2xe^{-x^2} = 0$. (See Appendix A.4, statement (B), for the FTC.)

11. **a.** For example, $y' = 1/c$, so that $cy' = 1$ is a possible differential equation satisfied by y.

b. Note that $y' = be^{ax}\cos bx + ae^{ax}\sin bx = be^{ax}\cos bx + ay$, so that $y' - ay = be^{ax}\cos bx$.

c. We have $y' = (A+Bt)\,e^t + Be^t = y + Be^t$, so that $y' - y = Be^t$.
Other possibilities are the equations $y'' - y = 2Be^t$ and $y'' - y' = Be^t$.

d. We have $\dot{y} = -3\,e^{-3t} + t\,y(t)$, or $\dot{y} - t\,y = -3\,e^{-3t}$, for example. Other possibilities include $\ddot{y} - t\,\dot{y} - y = 9\,e^{-3t}$.

12. Differentiating implicitly, we find that $xy' + y - y'/y = 0$, so that $xyy' + y^2 - y' = (xy-1)y' + y^2 = 0$, a first-order nonlinear equation, is a possible answer.

13. We get $y' + y'/\left(1+y^2\right) = 1 + 1/\left(1+x^2\right)$, so that $y'\left(\frac{y^2+2}{y^2+1}\right) = \frac{x^2+2}{x^2+1}$, or $y' = \left(\frac{y^2+1}{y^2+2}\right)\left(\frac{x^2+2}{x^2+1}\right)$, a first-order nonlinear equation.

14. We get $3y^2y' - 3 + 3y' = 0$, so $y' = 3/\left(3y^2+3\right)$ is a possible answer.

15. We have $x^2y' + 2xy + 4y' = 0$, or $y' = -2xy/\left(x^2+4\right)$, a linear equation.

16. Differentiating implicitly, we get $2x + 2yy' - 6 + 10y' = 0$, $2yy' + 10y' = 6 - 2x$, $2(y+5)y' = 2(3-x)$, so $y' = (3-x)/(y+5)$ *for those values of x for which $y \neq -5$.*

B

17. $y = x^2/2 + (x/2)\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$,

$$y' = x + (x/2)(1/2)\left(2x\Big/\sqrt{x^2+1}\right) + (1/2)\sqrt{x^2+1}$$

$$+ (1/2)\left(1 + x\Big/\sqrt{x^2+1}\right)\Big/\left(x+\sqrt{x^2+1}\right) = x + \sqrt{x^2+1};\ \text{thus,}$$

$$xy' + \ln(y') = \left(x^2 + x\sqrt{x^2+1}\right) + \ln\left(x+\sqrt{x^2+1}\right)$$

$$= \left(x^2 + x\sqrt{x^2+1}\right) + \ln\left(x+\sqrt{x^2+1}\right)$$

$$= \left(x^2 + x\sqrt{x^2+1}\right) + 2\ln\sqrt{x+\sqrt{x^2+1}}$$
$$= 2\left(x^2/2 + (x/2)\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}\right) = 2y.$$

18. If you differentiate a polynomial of degree n, you get a polynomial of degree $n-1$, so that the derivative can't be a constant multiple of the original function. The derivative of any basic trigonometric function is another trigonometric function that is not a constant multiple of the function you started with. Finally, if you differentiate the logarithm function to any base, you'll get a multiple of the *reciprocal* of the original function.

19. **a.** The given equation is equivalent to $(y')^2 = -1$. Since there is no real-valued function y' whose square is negative, there can be no real-valued function y satisfying the equation.

b. The only way that two absolute values can have a sum equal to zero is if each absolute value is itself zero. This says that y is identically equal to zero, so that the zero function is the only solution. The graph of this solution is the x-axis (if the independent variable is x).

20. If $x(t) \neq t$, the expression $-|x-t|$ is always negative, so that $\sqrt{-|x-t|}$ is not a real number. If $x(t) = t$, then the equation becomes $dx/dt = 0$, which contradicts the fact that dx/dt must equal to 1 in this case.

21. If $y = \pm\sqrt{c^2-x^2} = \pm\left(c^2-x^2\right)^{1/2}$, then $dy/dx = \pm\frac{1}{2}\left(c^2-x^2\right)^{-1/2}\cdot(-2x) = \mp x\left(c^2-x^2\right)^{-1/2}$ and $y\,dy/dx + x = \pm\left(c^2-x^2\right)^{1/2}\cdot \mp x\left(c^2-x^2\right)^{1/2} + x = -x + x = 0$. If $x > c$ or $x < -c$, then $c^2 - x^2 < 0$ and then the functions $y = \pm\sqrt{c^2-x^2}$ do not exist as real-valued functions. If $x = \pm c$, then each function is the zero function, which is not a solution of the differential equation.

22. **a.** If $y = \ln(|C_1 x|) + C_2$, then $y' = C_1/C_1 x = 1/x$ for all values of C_1 and C_2 (with $C_1 \neq 0$).

b. Note that $y = \ln(|C_1 x|) + C_2 = \ln(|C_1|) + \ln(|x|) + C_2 = \ln(|x|) + (\ln(|C_1|) + C_2) = \ln(|x|) + C$, where $C = \ln(|C_1|) + C_2$.

23. If $y(x) = c_1 \sin x + c_2 \cos x$, then $dy/dx + y = (c_1\cos x - c_2 \sin x) + (c_1 \sin x + c_2 \cos x) = (c_1 - c_2)\sin x + (c_2 + c_1)\cos x$. If this last expression must equal $\sin x$, then we must have $c_1 - c_2 = 1$ and $c_2 + c_1 = 0$. Adding these last equations, we find that $c_1 = 1/2$, and thus $c_2 = -1/2$. Therefore, the solution is $y(x) = (1/2)(\sin x - \cos x)$.

24. Suppose the polynomial is $y(x) = ax^2 + bx + c$, so that $y' = 2ax + b$. Then $2y' - y = 2(2ax+b) - (ax^2+bx+c) = -ax^2 + (4a-b)x + (2b-c) = 3x^2 - 13x + 7$, which implies that $a = -3$, $4a - b = -13$, and $2b - c = 7$. Thus, $a = -3$, $b = 1$, and $c = -5$, so that the solution is $y(x) = -3x^2 + x - 5$.

25. We have $y = Cx \pm \sqrt{C^2+1}$, so that $y' = C$. Then $(xy'-y)^2 - (y')^2 - 1 = (Cx - Cx \mp \sqrt{C^2+1})^2 - C^2 - 1 = \left(C^2+1\right) - C^2 - 1 = 0$. But if we assume that a function y is defined implicitly and we differentiate the relation $x^2 + y^2 = 1$ implicitly with respect to x, we get $2x + 2yy' = 0$, or $y' = -x/y$.

26. **a.** We have $y = e^x = y' = y''$. Therefore, $xy'' - (x+n)y' + ny = xy - (x+n)y + ny = 0$.

b. We have $y = \sum_{k=0}^{n} \frac{x^k}{k!}$, $y' = \sum_{k=0}^{n} \frac{k \cdot x^{k-1}}{k!} = \sum_{k=1}^{n} \frac{x^{k-1}}{(k-1)!} = y - \frac{x^n}{n!}$, and $y'' = y' - \frac{n\,x^{n-1}}{n!} = y' - \frac{x^{n-1}}{(n-1)!}$. Therefore, $xy'' - (x+n)y' + n\,y = x\left[y' - \frac{x^{n-1}}{(n-1)!}\right] - (x+n)y' + ny = xy' - \frac{x^n}{(n-1)!} - xy' - ny' + n\,y = -\frac{x^n}{(n-1)!} - ny' + ny = -\frac{x^n}{(n-1)!} - n\left[y - \frac{x^n}{n!}\right] + ny = -\frac{x^n}{(n-1)!} - ny + \frac{x^n}{(n-1)!} + ny = 0$.

C

27. We have $y(t) = \cos t + \int_0^t (t-u)y(u)\,du = \cos t + t\int_0^t y(u)\,du - \int_0^t u\,y(u)\,du$, so (using the Product Rule and the FTC) $y'(t) = -\sin t + t\,y(t) + \int_0^t y(u)\,du - t\,y(t) = -\sin t + \int_0^t y(u)\,du$. Differentiating again, we get $y''(t) = -\cos t + y(t)$, or $y'' - y = -\cos t$. In this problem it is important to distinguish between t and the "dummy variable" u.

28. To see this, we apply Leibniz's rule for differentiation under the integral sign with respect to the parameter x to get

$$y' = \int_0^{\pi} \cos(x\cos t)\cos t\,dt \tag{1}$$

$$y'' = -\int_0^{\pi} \sin(x\cos t)\cos^2 t\,dt. \tag{2}$$

Applying integration by parts to integral (1) yields

$$\begin{aligned} y' &= \int_0^{\pi} \underbrace{\cos(x\cos t)}_{u}\,\underbrace{\cos t\,dt}_{dv} \\ &= \cos(x\cos t)\sin t\,\Big|_0^{\pi} - \int_0^{\pi} -\sin(x\cos t)(-x\sin t)\sin t\,dt \\ &= -x\int_0^{\pi} \sin(x\cos t)\sin^2 t\,dt. \end{aligned} \tag{3}$$

Then, using the definition of y and the expressions (2) and (3), we see that

$$\begin{aligned} xy'' + y' + xy &= -x\int_0^{\pi} \sin(x\cos t)\cos^2 t\,dt - x\int_0^{\pi} \sin(x\cos t)\sin^2 t\,dt \\ &\quad + x\int_0^{\pi} \sin(x\cos t)\,dt \\ &= -x\int_0^{\pi} \sin(x\cos t)\cos^2 t\,dt - x\int_0^{\pi} \sin(x\cos t)(1-\cos^2 t)\,dt \end{aligned}$$

$$+ x \int_0^{\pi} \sin(x \cos t)\, dt$$
$$= -x \int_0^{\pi} \sin(x \cos t)\, dt + x \int_0^{\pi} \sin(x \cos t)\, dt = 0.$$

[**COMMENT**: Problem 81.1 in the Summer 2018 issue of the Irish Mathematical Society *Bulletin* (https://www.maths.tcd.ie/pub/ims/bull81/problems81.pdf) asked for a homogeneous linear ordinary differential equation of order two satisfied by the function y. My solution (and those of others, I am sure) yielded the differential equation given in this exercise. HOWEVER, as reported in a later issue of that journal, it can be shown (an exercise for the reader) that $y(x) \equiv 0$ (!)]

1.3 Initial-value problems and boundary-value problems

I emphasize the fact that an initial-value problem or a boundary-value problem may have *no* solution, *one* solution, or *many* solutions, anticipating the formal discussion of existence and uniqueness in Section 2.8.

A

1. $R(t) = t(c - \cos t)$, so that $R(\pi) = \pi(c - \cos\pi) = \pi(c+1) = 0$ implies that $c = -1$. Thus, the solution of the IVP is $R(t) = \pi(-1 - \cos t) = -\pi(1 + \cos t)$.
2. Since $y = at^3 + bt^2 + ct + d$, $y(0) = 1$ implies that $d = 1$; $y'(0) = 0$ implies that $c = 0$; $y''(0) = 1$ implies that $2b = 1$, or $b = 1/2$; and $y'''(0) = 6$ tells us that $6a = 6$, or $a = 1$. Thus, the solution of the IVP is $y = t^3 + (1/2)t^2 + 1$.
3. $r(t) = ce^{bt} - (a/b)t - a/b^2$, so that $r(0) = ce^0 - (a/b)(0) - a/b^2 = c - a/b^2 = 0$ implies that $c = a/b^2$. Thus, the solution of the IVP can be written as $r(t) = (a/b^2)\,e^{bt} - (a/b)t - a/b^2 = (a/b)\left(e^{bt}/b - t - 1/b\right)$.
4. We have $y = (e^{ax} + e^{-ax})/2a$ and $y' = \left(ae^{ax} - ae^{-ax}\right)/2a = \left(e^{ax} - e^{-ax}\right)/2$, so that $y(0) = \left(e^0 + e^0\right)/2a = 1/a = 2$ implies that $a = 1/2$. Noting that $y'(0) = \left(ae^0 - ae^0\right)/2 = 0$ for *all* values of a, we conclude that $y(x) = e^{x/2} + e^{-x/2}$ is the solution of the initial value problem.
5. We have $y' = -\frac{1}{4} + \frac{66}{296}\,e^{6x} + 2Ax + B\cos x - C\sin x$, $y'' = \frac{396}{296}\,e^{6x} + 2A - B\sin x - C\cos x$, $y''' = \frac{6(396)}{296}\,e^{6x} - B\cos x + C\sin x$. Substituting these derivatives in the original differential equation and simplifying, we get $(-B + 6C)\cos x + (C + 6B)\sin x - 12A = 3 - \cos x$. Equating coefficients of like functions—a technique that will come in handy in Chapter 4—we get the system $\{-B + 6C = -1, C + 6B = 0, -12A = 3\}$, which has the solution $A = -1/4$, $B = 1/37$, $C = -6/37$. [Of course, the boundary conditions yield the same result.]

6. $2 = C/\sqrt{C^2-1} \Rightarrow 4 = C^2/(C^2-1) \Rightarrow 3C^2 = 4 \Rightarrow C = \pm 2\sqrt{3}/3$.
On the other hand, $1 = 2C/\sqrt{C^2-4} \Rightarrow C^2 - 4 = 4C^2 \Rightarrow C^2 = -4/3$, so there is no such function in this family.

B

7. As was illustrated in Example 1.3.1, the velocity function is the derivative of the position function, so that we have $\frac{dx}{dt} = \frac{1}{t^2+1}$. Integrating both sides, we get $x(t) - x(0) = x(t) = \int_0^t \frac{1}{u^2+1}\, du = \arctan(t)$. (See also Eq. (1.3.1).) For $t \geq 0$, $\arctan(t) \leq \pi/2$, so that $x(t) \leq \pi/2$. (Look at the graph of the arctangent.)

8. If $y_1(x) \equiv 0$, then $\frac{dy_1}{dx} = 0 = 3(0)^{2/3}$. Also, $y_1(x_0) = 0$. If $y_2(x) = (x - x_0)^3$, then $\frac{dy_2}{dx} = 3(x-x_0)^2 = 3[(x-x_0)^3]^{2/3} = 3\, y_2^{2/3}$. Furthermore, $y_2(x_0) = 3(x_0 - x_0)^3 = 0$.

9. We calculate that $y' = e^{x^2}\left(\int_1^x e^{-t^2}\, dt\right)' + \left(e^{x^2}\right)' \int_1^x e^{-t^2}\, dt = e^{x^2}\left(e^{-x^2}\right) + 2x\, e^{x^2} \int_1^x e^{-t^2}\, dt = 1 + 2x\, e^{x^2} \int_1^x e^{-t^2}\, dt = 1 + 2x\left(e^{x^2} \int_1^x e^{-t^2}\, dt\right) = 1 + 2xy$. Also, $y(1) = e^{1^2} \int_1^1 e^{-t^2}\, dt = e(0) = 0$.

10. **a.** Integrating successively, we see that $y''' = -24\cos\left(\frac{\pi}{2}x\right) \Rightarrow y'' = -\frac{48}{\pi}\sin\left(\frac{\pi}{2}x\right) + C_1 \Rightarrow y' = \frac{96}{\pi^2}\cos\left(\frac{\pi}{2}x\right) + C_1 x + C_2 \Rightarrow y = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}x\right) + C_1\frac{x^2}{2} + C_2 x + C_3$. There are **three** parameters involved.

b. Now $y(0) = -4$ implies that $-4 = y(0) = \frac{192}{\pi^3}\sin(0) + C_1\frac{0^2}{2} + C_2(0) + C_3 = C_3$; and $y(1) = 0$ implies that $0 = y(1) = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}\right) + C_1\frac{1^2}{2} + C_2(1) + C_3 = \frac{192}{\pi^3} + \frac{C_1}{2} + C_2 + C_3$, or $\frac{192}{\pi^2} + \frac{C_1}{2} + C_2 = 4$. Also, $y'(1) = 6$ implies that $6 = y'(1) = \frac{96}{\pi^2}\cos\left(\frac{\pi}{2}\right) + C_1 + C_2 = C_1 + C_2$. Solving the last two simultaneous equations, we find that $C_1 = \left(4\pi^3 + 384\right)/\pi^3$ and $C_2 = \left(2\pi^3 - 384\right)/\pi^3$. Thus, $y = \frac{192}{\pi^3}\sin\left(\frac{\pi}{2}x\right) + \left(\frac{2\pi^3+192}{\pi^3}\right)x^2 + \left(\frac{2\pi^3-384}{\pi^3}\right)x - 4$.

11. No. An equation of order n requires an n-parameter family of solutions. Essentially, to solve a differential equation of order 4 requires 4 integrations, each of which introduces a constant of integration (parameter).

12. **a.** We have $y(x) = cx + c^2 \Rightarrow y' = c \Rightarrow (y')^2 + xy' = c^2 + x(c) = y$.

b. We see that $y(x) = -1/4x^2 = cx + c^2 \Rightarrow x^2 + 4cx + 4c^2 = 0 \Rightarrow$ (regarding this as a quadratic equation in c) $c = \left(-4x \pm \sqrt{16x^2 - 4(4x^2)}\right)/8 = -x/2$, so we can't find a constant c. It is easy to verify that $y(x) = -x^2/4$ satisfies the given differential equation.

13. The given information is that $v(0) = 0$, $v(30 \text{ seconds}) = v(30/3600 \text{ hours}) = 200$ mph, and $a(t) = C$, a constant. Now $a(t) = C \Rightarrow v(t) = \int a(t)\, dt = Ct + K$. Then $v(0) = 0 \Rightarrow K = 0 \Rightarrow v(t) = C\,t$. Therefore, $200 = v(30/3600) =$