1.2 Convergence

1. Compute each of the following limits and determine the corresponding rate of convergence.

(a) $\lim_{n\to\infty} \frac{n-1}{n^3+2}$

(b) $\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

(c) $\lim_{n\to\infty} \frac{\sin n}{n}$

(d) $\lim_{n\to\infty} \frac{3n^2-1}{7n^2+n+2}$

(a) For $n > 1$,

$$\left|\frac{n-1}{n^3+2} - 0\right| = \frac{n-1}{n^3+2} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Thus, $\frac{n-1}{n^3+2}$ converges to 0 with rate of convergence $O(1/n^2)$.

(b) Note that

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

Because

$$|(\sqrt{n+1} - \sqrt{n}) - 0| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}},$$

it follows that $\sqrt{n+1} - \sqrt{n}$ converges to 0 with rate of convergence $O(1/\sqrt{n})$.

(c) Since $-1 \le \sin n \le 1$ for all n, it follows that

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

for all n. Then, by the squeeze theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$. Moreover, because

$$\left|\frac{\sin n}{n} - 0\right| \le \frac{1}{n},$$

the rate of convergence is $O(1/n)$.

(d) For $n > 13$,

$$\left|\frac{3n^1-1}{7n^2+n+2} - \frac{3}{7}\right| = \frac{3n+13}{7(7n^2+n+2)} < \frac{4n}{49n^2} < \frac{1}{10n}.$$

Therefore, $\frac{3n^2-1}{7n^2+n+2}$ converges to $\frac{3}{7}$ with rate of convergence $O(1/n)$.

2. Compute each of the following limits and determine the corresponding rate of convergence.

(a) $\lim_{x\to 0} \frac{e^x-1}{x}$

(b) $\lim_{x\to 0} \frac{\sin x}{x}$

(c) $\lim_{x\to 0} \frac{e^x-\cos x - x}{x^2}$

(d) $\lim_{x\to 0} \frac{\cos x - 1 + x^2/2 - x^4/24}{x^6}$

(a) From Taylor's Theorem, $e^x = 1 + x + \frac{1}{2}x^2e^\xi$ for some ξ between 0 and x. Hence,

$$\frac{e^x-1}{x} = 1 + \frac{1}{2}xe^\xi.$$

Because

$$\left|\frac{e^x-1}{x} - 1\right| = \frac{1}{2}|x|e^\xi < |x|$$

for all x satisfying $|x| < \ln 2$, it follows that

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1 \text{ with rate of convergence } O(x).$$

(b) From Taylor's Theorem, $\sin x = x - \frac{x^3}{6}\cos\xi$ for some ξ between 0 and x. Then,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6}\cos\xi$$

and

$$\left|\frac{\sin x}{x} - 1\right| = \frac{1}{6}|x^2\cos\xi| \le \frac{1}{6}x^2.$$

Finally,

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \text{ with rate of convergence } O(x^2).$$

(c) From Taylor's Theorem, we have

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3e^{\xi_1}$$

and

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{6}x^3\sin\xi_2$$

for some ξ_1 and ξ_2 between 0 and x. Then

$$\frac{e^x - \cos x - x}{x^2} = 1 + \frac{x}{6}\left(e^{\xi_1} - \sin\xi_2\right).$$

For sufficiently small x, $e^{\xi_1} < 2$, so $|e^{\xi_1} - \sin\xi_2| < 2 + 1 = 3$. Thus,

$$\left|\frac{e^x - \cos x - x}{x^2} - 1\right| = \frac{|x|}{6}\left|e^{\xi_1} - \sin\xi_2\right| \le \frac{1}{2}|x|,$$

and
$$\lim_{x\to 0}\frac{e^x-\cos x-x}{x^2}=1 \text{ with rate of convergence } O(x).$$

(d) From Taylor's Theorem, we have
$$\cos x = 1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\frac{1}{8!}x^8\cos\xi$$
for some ξ between 0 and x. Hence,
$$\frac{\cos x - 1+\frac{1}{2}x^2-\frac{1}{24}x^4}{x^6}=-\frac{1}{720}+\frac{1}{8!}x^2\cos\xi,$$
and
$$\left|\frac{\cos x - 1+\frac{1}{2}x^2-\frac{1}{24}x^4}{x^6}+\frac{1}{720}\right|=\frac{1}{8!}|x^2\cos\xi|\le\frac{1}{8!}|x^2|.$$
It therefore follows that
$$\lim_{x\to 0}\frac{\cos x - 1+\frac{1}{2}x^2-\frac{1}{24}x^4}{x^6}=-\frac{1}{720}$$
with rate of convergence $O(x^2)$.

3. Numerically determine which of the following sequences approaches 1 faster, and then confirm the numerical evidence by determining the rate of convergence of each sequence.
$$\lim_{x\to 0}\frac{\sin x^2}{x^2} \quad \text{versus} \quad \lim_{x\to 0}\frac{(\sin x)^2}{x^2}.$$

The values in the following table suggest that $\frac{\sin x^2}{x^2}$ converges toward 1 more rapidly than $\frac{(\sin x)^2}{x^2}$.

x	$\frac{\sin x^2}{x^2}$	$\frac{(\sin x)^2}{x^2}$
1.000	0.84147098480790	0.70807341827357
0.100	0.99998333341667	0.99667110793792
0.010	0.99999999833333	0.99996666711111
0.001	0.99999999999983	0.99999966666671

To confirm this conclusion, note that by Taylor's Theorem,
$$\sin u = u-\frac{1}{6}u^3\cos\xi,$$
for some ξ between 0 and u. Using the substitution $u=x^2$, we find
$$\sin x^2 = x^2-\frac{1}{6}x^6\cos\xi$$

for some ξ between 0 and x^2. Consequently,

$$\left|\frac{\sin x^2}{x^2} - 1\right| = \frac{1}{6}x^4|\cos\xi| \le \frac{1}{6}x^4.$$

Starting from $f(x) = (\sin x)^2$, we find

$$f'(x) = 2\sin x\cos x = \sin 2x, \quad f''(x) = 2\cos 2x, \quad f'''(x) = -4\sin 2x,$$

and $f^{(4)}(x) = -8\cos 2x$. Therefore,

$$(\sin x)^2 = x^2 - \frac{1}{3}x^4\cos 2\xi$$

for some ξ between 0 and x, and

$$\left|\frac{(\sin x)^2}{x^2} - 1\right| = \frac{1}{3}x^2|\cos 2\xi| \le \frac{1}{3}x^2.$$

Finally,

$$\frac{\sin x^2}{x^2} = 1 + O(x^4) \qquad \text{and} \qquad \frac{(\sin x)^2}{x^2} = 1 + O(x^2).$$

4. Suppose that $0 < a < b$.

(a) Show that if $\alpha_n = \alpha + O(1/n^b)$, then $\alpha_n = \alpha + O(1/n^a)$.

(b) Show that if $f(x) = L + O(x^b)$, then $f(x) = L + O(x^a)$.

(a) Suppose $\alpha_n = \alpha + O(1/n^b)$. Then, there exists a constant λ such that for sufficiently large n, $|\alpha_n - \alpha| \le \lambda\frac{1}{n^b}$. Because $a < b$, it follows that $n^a < n^b$ and $\frac{1}{n^a} > \frac{1}{n^b}$ for all $n > 1$. Thus,

$$|\alpha_n - \alpha| \le \lambda\frac{1}{n^b} < \lambda\frac{1}{n^a},$$

and $\alpha_n = \alpha + O(1/n^a)$.

(b) Suppose $f(x) = L + O(x^b)$. Then, there exists a constant K such that for all sufficiently small x, $|f(x) - L| \le K|x|^b$. Because $a < b$, it follows that for all $|x| \le 1$, $|x|^b \le |x|^a$. Thus, for sufficiently small x,

$$|f(x) - L| \le K|x|^b \le K|x|^a,$$

and $f(x) = L + O(x^a)$.

5. Suppose that $f_1(x) = L_1 + O(x^a)$ and $f_2(x) = L_2 + O(x^b)$. Show that

$$c_1 f_1(x) + c_2 f_2(x) = c_1 L_1 + c_2 L_2 + O(x^c),$$

where $c = \min(a, b)$.

Suppose $f_1(x) = L_1 + O(x^a)$ and $f_2(x) = L_2 + O(x^b)$. Then, there exist constants K_1 and K_2 such that for all sufficiently small x, $|f_1(x) - L_1| \le K_1|x^a|$ and $|f_2(x) - L_2| \le K_2|x^b|$. Let c_1 and c_2 be any real numbers. Applying the triangle inequality, we find

$$\begin{aligned} |c_1 f_1(x) + c_2 f_2(x) - (c_1 L_1 + c_2 L_2)| &\le |c_1||f_1(x) - L_1| + |c_2||f_2(x) - L_2| \\ &\le |c_1|K_1|x^a| + |c_2|K_2|x^b|. \end{aligned}$$

Now, let $c = \min(a, b)$. Then, for $|x| < 1$,

$$|c_1|K_1|x^a| + |c_2|K_2|x^b| < |c_1|K_1|x^c| + |c_2|K_2|x^c| = (|c_1|K_1 + |c_2|K_2)|x^c|.$$

Consequently,

$$c_1 f_1(x) + c_2 f_2(x) = c_1 L_1 + c_2 L_2 + O(x^c).$$

6. The table below lists the errors of successive iterates for three different methods for approximating $\sqrt[3]{5}$. Estimate the order of convergence of each method, and explain how you arrived at your conclusions.

Method 1	Method 2	Method 3
4.0×10^{-2}	3.7×10^{-4}	4.3×10^{-3}
9.1×10^{-4}	1.2×10^{-15}	1.8×10^{-8}
4.8×10^{-7}	1.5×10^{-60}	1.4×10^{-24}

If a sequence converges of order α, then the error in each term of the sequence is roughly the error in the previous term raised to the power α. From the data for "Method 1," we see that each error is roughly the previous error squared; therefore, we estimate the order of convergence to be $\alpha = 2$. From the data for "Method 2," we see that each error is roughly the previous error raised to the fourth power; therefore, we estimate the order of convergence to be $\alpha = 4$. Finally, from the data for "Method 3," we see that each error is roughly the previous error raised to the third power; therefore, we estimate the order of convergence to be $\alpha = 3$.

7. Let $\{p_n\}$ be a sequence which converges to the limit p.

(a) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = 0,$$

what can be said about the order of convergence of $\{p_n\}$ to p?

(b) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \to \infty,$$

what can be said about the order of convergence of $\{p_n\}$ to p?

(a) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = 0,$$

then the numerator approaches zero faster than the denominator. In order to achieve a nonzero limit, we must increase the power in the denominator. Therefore, the order of convergence must be greater than α.

(b) If

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \to \infty,$$

then the denominator approaches zero faster than the numerator. In order to achieve a nonzero limit, we must decrease the power in the denominator. Therefore, the order of convergence must be less than α.

8. Suppose theory indicates that the sequence $\{p_n\}$ converges to p of order 1.5. Explain how you would numerically verify this order of convergence.

To numerically verify the order of convergence, calculate the ratio

$$\frac{|p_{n+1} - p|}{|p_n - p|^{1.5}}$$

for several successive values of n. If the order of convergence is $\alpha = 1.5$, these ratios should approach a constant, specifically the asymptotic error constant.

9. Theory indicates that the following sequence should converge to $\sqrt{3}$ of order 1.618. Does the sequence actually achieve an order of convergence of 1.618? If not, what is the actual order?

n	p_n
0	2.000000000000000
1	1.666666666666667
2	1.727272727272727
3	1.732142857142857
4	1.732050680431722
5	1.732050807565499

Because the values in the third column of the following table appear to be approaching a constant, the evidence suggests that the sequence does, in fact, converge toward $\sqrt{3}$ with order of convergence $\alpha = 1.618$.

n	p_n	$\lvert p_n - \sqrt{3}\rvert / \lvert p_{n-1} - \sqrt{3}\rvert^{1.618}$
1	2.000000000000000	
2	1.666666666666667	0.55066002953142
3	1.727272727272727	0.39429299851516
4	1.732142857142857	0.52358803162884
5	1.732050680431722	0.43100791441420
6	1.732050807565499	0.48525581579327

10. Theory indicates that the following sequence should converge to $4/3$ of order 1.618. Does the sequence actually achieve an order of convergence of 1.618? If not, what is the actual order?

n	p_n
0	1.498664098580016
1	1.497353997792205
2	1.428801977335339
3	1.401092915389552
4	1.376493676051456
5	1.361345745573130
6	1.351034482500881
7	1.344479850695066

Because the values in the third column of the following table are increasing with n, the evidence suggests that the sequence does not have order of convergence $\alpha = 1.618$, but rather that the order of convergence is less than 1.618. Because the values in the fourth column appear to be approaching a constant, these values suggest that the sequence is converging to $4/3$ with order of convergence $\alpha = 1$.

n	p_n	$\lvert p_n - 4/3\rvert/\lvert p_{n-1} - 4/3\rvert^{1.618}$	$\lvert p_n - 4/3\rvert/\lvert p_{n-1} - 4/3\rvert$
1	1.49866409858002		
2	1.49735399779221	3.01718763541581	0.99207588021590
3	1.42880197733534	1.77891367138598	0.58205253781266
4	1.40109291538955	3.03079120639280	0.70975745769255
5	1.37649367605146	3.36181849329742	0.63696294174768
6	1.36134574557313	4.52671513900300	0.64903127444432
7	1.35103448250088	5.75689539760301	0.63190377951100
8	1.34447985069507	7.61855893491390	0.62970586012393

11. Show that the convergence of the sequence generated by the formula

$$x_{n+1} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a}$$

toward $\sqrt{a}$ is third-order. What is the asymptotic error constant?

Note

$$\begin{aligned} x_{n+1} - \sqrt{a} = \frac{x_n^3 + 3x_n a}{3x_n^2 + a} - \sqrt{a} &= \frac{x_n^3 - 3x_n^2\sqrt{a} + 3x_n a - a^{3/2}}{3x_n^2 + a} \\ &= \frac{(x_n - \sqrt{a})^3}{3x_n^2 + a}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{a}|}{|x_n - \sqrt{a}|^3} = \lim_{n\to\infty} \frac{1}{3x_n^2 + a} = \frac{1}{4a}.$$

Consequently, $x_n \to \sqrt{a}$ with order of convergence $\alpha = 3$ and asymptotic error constant $\lambda = \frac{1}{4a}$.

12. Let a be a non-zero real number. For any x_0 satisfying $0 < x_0 < 2/a$, the recursive sequence defined by

$$x_{n+1} = x_n(2 - ax_n)$$

converges to $1/a$. What are the order of convergence and the asymptotic error constant?

Note

$$\begin{aligned} x_{n+1} - \frac{1}{a} = x_n(2 - ax_n) - \frac{1}{a} &= -ax_n^2 + 2x_n - \frac{1}{a} \\ &= -a\left(x_n^2 - \frac{2}{a}x_n + \frac{1}{a^2}\right) = -a\left(x_n - \frac{1}{a}\right)^2. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{|x_{n+1} - \frac{1}{a}|}{|x_n - \frac{1}{a}|^2} = \lim_{n\to\infty} a = a.$$

Consequently, $x_n \to \frac{1}{a}$ with order of convergence $\alpha = 2$ and asymptotic error constant $\lambda = a$.

13. Suppose that the sequence $\{p_n\}$ converges linearly to the limit p with asymptotic error constant λ. Further suppose that $p_{n+1} - p$, $p_n - p$ and $p_{n-1} - p$ are all of the same sign. Show that

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} \approx \lambda.$$

Suppose the sequence $\{p_n\}$ converges linearly to p with asymptotic error constant λ. Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda,$$

so, for sufficiently large n,

$$|p_{n+1} - p| \approx \lambda|p_n - p|.$$

Moreover,

$$|p_n - p| \approx \lambda|p_{n-1} - p| \quad \text{or} \quad |p_{n-1} - p| \approx \frac{1}{\lambda}|p_n - p|.$$

Because we are given that $p_{n+1} - p$, $p_n - p$ and $p_{n-1} - p$ are all of the same sign, we may drop the absolute values from the above expressions. Now,

$$\frac{p_{n+1} - p_n}{p_n - p_{n-1}} = \frac{p_{n+1} - p - (p_n - p)}{p_n - p - (p_{n-1} - p)}$$

$$\approx \frac{\lambda(p_n - p) - (p_n - p)}{p_n - p - \frac{1}{\lambda}(p_n - p)}$$
$$= \frac{\lambda - 1}{1 - \frac{1}{\lambda}} = \lambda.$$

14. A sequence $\{p_n\}$ converges *superlinearly* to p provided

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = 0.$$

Show that if $p_n \to p$ of order α for $\alpha > 1$, then $\{p_n\}$ converges superlinearly to p.

Suppose the sequence $\{p_n\}$ converges p of order $\alpha > 1$ with asymptotic error constant λ. Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} &= \lim_{n\to\infty} \frac{|p_{n+1} - p| \cdot |p_n - p|^{\alpha-1}}{|p_n - p|^\alpha} \\
&= \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} \cdot \lim_{n\to\infty} |p_n - p|^{\alpha-1} \\
&= \lambda \cdot 0 = 0.
\end{aligned}$$

Therefore, $\{p_n\}$ converges superlinearly to p.

15. Suppose that $\{p_n\}$ converges superlinearly to p (see Exercise 14). Show that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = 1.$$

Note that

$$\frac{p_{n+1} - p_n}{p_n - p} = \frac{p_{n+1} - p - (p_n - p)}{p_n - p} = \frac{p_{n+1} - p}{p_n - p} - 1.$$

Because $\{p_n\}$ converges superlinearly to p, it then follows that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|} = \left| \lim_{n\to\infty} \left(\frac{p_{n+1} - p}{p_n - p} - 1 \right) \right| = |0 - 1| = 1.$$

16. **(a)** Determine the third-degree Taylor polynomial and associated remainder term for the function $f(x) = \ln(1 - x)$. Use $x_0 = 0$.

(b) Using the results of part (a), approximate $\ln(0.25)$ and compute the theoretical error bound associated with this approximation. Compare the theoretical error bound with the actual error.

(c) Compute the following limit and determine the corresponding rate of convergence:

$$\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3}.$$

(a) Let $f(x) = \ln(1-x)$. Then

$$f'(x) = -\frac{1}{1-x},\ f''(x) = -\frac{1}{(1-x)^2},\ f'''(x) = -\frac{2}{(1-x)^3},\text{ and } f^{(4)}(x) = -\frac{6}{(1-x)^4}.$$

Moreover,

$$f(0) = \ln 1 = 0,\ f'(0) = -1,\ f''(0) = -1,\ f'''(0) = -2,\text{ and } f^{(4)}(\xi) = -\frac{6}{(1-\xi)^4}.$$

Finally,

$$\begin{aligned} \ln(1-x) &= P_3(x) + R_3(x) \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4(1-\xi)^4}, \end{aligned}$$

for some ξ between 0 and x.

(b) Using the result of part (a),

$$\ln(0.25) \approx P_3(0.75) = -0.75 - \frac{0.75^2}{2} - \frac{0.75^3}{3} = -1.171875.$$

Because $0 < \xi < 0.75$, $(1-\xi)^{-4} \le 4^4$ and

$$|\text{error}| = |R_3(0.75)| = \frac{0.75^4}{4(1-\xi)^4} \le \frac{81}{4} = 20.25.$$

The actual error is $|\ln(0.25) - P_3(0.75)| \approx 0.214419$, which is significantly less than the theoretical error bound.

(c) Once again using the result from part (a), we find

$$\frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} = -\frac{1}{3} - \frac{x}{4(1-\xi)^4}.$$

Moreover,

$$\left|\frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} + \frac{1}{3}\right| = \frac{|x|}{4|1-\xi|^4} \le |x|,$$

for all sufficiently small x. Therefore,

$$\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{1}{2}x^2}{x^3} = -\frac{1}{3},$$

with rate of convergence $O(x)$.

17. **(a)** Determine the third-degree Taylor polynomial and associated remainder term for the function $f(x) = \sqrt{1+x}$. Use $x_0 = 0$.

(b) Using the results of part (a), approximate $\sqrt{1.5}$ and compute the theoretical error bound associated with this approximation. Compare the theoretical error bound with the actual error.

(c) Compute the following limit and determine the corresponding rate of convergence:

$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}.$$

(a) Let $f(x) = \sqrt{1+x}$. Then

$$f'(x) = \frac{1}{2}(1+x)^{-1/2},\ f''(x) = -\frac{1}{4}(1+x)^{-3/2},\ f'''(x) = -\frac{3}{8}(1+x)^{-5/2},$$

and $f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$. Moreover,

$$f(0) = 1,\ f'(0) = \frac{1}{2},\ f''(0) = -\frac{1}{4},\ f'''(0) = \frac{3}{8},$$

and $f^{(4)}(\xi) = -\frac{15}{16}(1+\xi)^{-7/2}$. Finally,

$$\begin{aligned} \sqrt{1+x} &= P_3(x) + R_3(x) \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4(1+\xi)^{-7/2}, \end{aligned}$$

for some ξ between 0 and x.

(b) Using the result of part (a),

$$\sqrt{1.5} \approx P_3(0.5) = 1 + \frac{1}{2}(0.5) - \frac{1}{8}(0.5)^2 + \frac{1}{16}(0.5)^3 = 1.2265625.$$

Because $0 < \xi < 0.5$, $(1+\xi)^{-7/2} \le 1$ and

$$|\text{error}| = |R_3(0.5)| \le \frac{5}{128}(0.5)^4 \approx 2.44 \times 10^{-3}.$$

The actual error is $|\sqrt{1.5} - P_3(0.5)| \approx 1.82 \times 10^{-3}$, which is less than the theoretical error bound.

(c) Once again using the result from part (a), we find

$$\frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = -\frac{1}{8} - \frac{x}{16}(1+\xi)^{-5/2}.$$

Moreover,

$$\left| \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} + \frac{1}{8} \right| = \frac{|x|}{16}|1-\xi|^{-5/2} \le \frac{1}{16}|x|,$$

for all sufficiently small x. Therefore,

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{x^2}=-\frac{1}{8},$$

with rate of convergence $O(x)$.

In Exercises 18 - 21, verify that Taylor's theorem produces the indicated formula, where ξ is between 0 and x.

18.

$$e^x = 1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}e^{\xi}$$

Let $f(x)=e^x$. Then $f^{(n)}(x)=e^x$ and $f^{(n)}(0)=1$ for all n. Therefore, by Taylor's Theorem with $x_0=0$,

$$\begin{aligned} e^x &= \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\\ &= 1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}e^{\xi}, \end{aligned}$$

for some ξ between 0 and x.

19.

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cos\xi$$

Let $f(x)=\sin x$. Then

$$f'(x)=\cos x,\ f''(x)=-\sin x,\ \text{and } f'''(x)=-\cos x.$$

Moreover,

$$f(0)=0,\ f'(0)=1,\ f''(0)=0,\ \text{and } f'''(0)=-1.$$

As higher-order derivatives are calculated, this cycle of four values repeats indefinitely. In particular, we find

$$f^{(2n)}(0)=0 \ \text{ and } \ f^{(2n+1)}(0)=(-1)^n.$$

Therefore, by Taylor's Theorem with $x_0=0$,

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cos\xi,$$

for some ξ between 0 and x.

20.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cos \xi$$

Let $f(x) = \cos x$. Then

$$f'(x) = -\sin x, \; f''(x) = -\cos x, \text{ and } f'''(x) = \sin x.$$

Moreover,

$$f(0) = 1, \; f'(0) = 0, \; f''(0) = -1, \text{ and } f'''(0) = 0.$$

As higher-order derivatives are calculated, this cycle of four values repeats indefinitely. In particular, we find

$$f^{(2n)}(0) = (-1)^n \text{ and } f^{(2n+1)}(0) = 0.$$

Therefore, by Taylor's Theorem with $x_0 = 0$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cos \xi,$$

for some ξ between 0 and x.

21.

$$\frac{1}{1+x} = 1 - x + x^2 - + \cdots + (-1)^n x^n + \frac{(-1)^{n+1} x^{n+1}}{(1+\xi)^{n+2}}$$

Let $f(x) = \frac{1}{1+x} = (1+x)^{-1}$. Then,

$$f'(x) = -1 \cdot (1+x)^{-2}, \; f''(x) = 1 \cdot 2 \cdot (1+x)^{-3}, \; f'''(x) = -1 \cdot 2 \cdot 3 \cdot (1+x)^{-4},$$

and, in general, $f^{(n)}(x) = (-1)^n \cdot n! \cdot (1+x)^{-n-1}$. Therefore, by Taylor's Theorem with $x_0 = 0$,

$$\frac{1}{1+x} = 1 - x + x^2 - + \cdots + (-1)^n x^n + \frac{(-1)^{n+1} x^{n+1}}{(1+\xi)^{n+2}},$$

for some ξ between 0 and x.