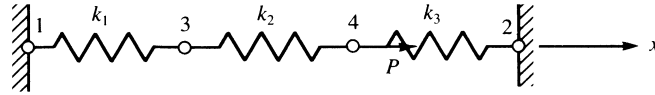


# Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so  $u_1 = 0$  and  $u_2 = 0$  and  $[K]$  becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K] \{d\} \Rightarrow [K^{-1}] \{F\} = [K^{-1}] [K] \{d\}$$

$$\Rightarrow [K^{-1}] \{F\} = \{d\}$$

Using the adjoint method to find  $[K^{-1}]$

$$C_{11} = k_2 + k_3 \qquad C_{21} = (-1)^3 (-k_2)$$

$$C_{12} = (-1)^{1+2} (-k_2) = k_2 \qquad C_{22} = k_1 + k_2$$

$$[C] = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}$$

$$\det [K] = | [K] | = (k_1 + k_2)(k_2 + k_3) - (-k_2)(-k_2)$$

$$\Rightarrow | [K] | = (k_1 + k_2)(k_2 + k_3) - k_2^2$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{(k_1 + k_2)(k_2 + k_3) - k_2^2} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow u_3 = \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow u_4 = \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

(c) In order to find the reaction forces we go back to the global matrix  $F = [K] \{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

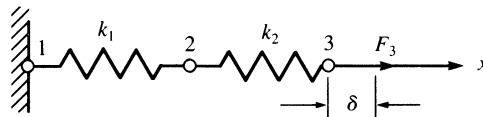
$$F_{1x} = -k_1 u_3 = -k_1 \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1 k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$F_{2x} = -k_3 u_4 = -k_3 \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3 (k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

## 2.2



$$k_1 = k_2 = k_3 = 1000 \frac{\text{lb}}{\text{in.}}$$

$$[k^{(1)}] = \begin{matrix} (1) & (2) \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{matrix} (1) \\ (2) \end{matrix} \end{matrix}; \quad [k^{(2)}] = \begin{matrix} (2) & (3) \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} & \begin{matrix} (2) \\ (3) \end{matrix} \end{matrix}$$

By the method of superposition the global stiffness matrix is constructed.

$$\begin{matrix}
 (1) & (2) & (3) \\
 [K] = \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}
 \end{matrix}$$

Node 1 is fixed  $\Rightarrow u_1 = 0$  and  $u_3 = \delta$

$$\{F\} = [K] \{d\}$$

$$\begin{cases} F_{1x}=? \\ F_{2x}=0 \\ F_{3x}=? \end{cases} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{cases} u_1=0 \\ u_2=? \\ u_3=\delta \end{cases}$$

$$\Rightarrow \begin{cases} 0 \\ F_{3x} \end{cases} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{cases} u_2 \\ \delta \end{cases} \Rightarrow \begin{cases} 0 = 2k u_2 - k \delta \\ F_{3x} = -k u_2 + k \delta \end{cases}$$

$$\Rightarrow u_2 = \frac{k\delta}{2k} = \frac{\delta}{2} = \frac{1 \text{ in.}}{2} \Rightarrow u_2 = 0.5''$$

$$F_{3x} = -k(0.5'') + k(1'')$$

$$F_{3x} = \left(-1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') + \left(1000 \frac{\text{lb}}{\text{in.}}\right)(1'')$$

$$F_{3x} = 500 \text{ lbs}$$

Internal forces

Element (1)

$$\begin{cases} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{cases} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{cases} u_1=0 \\ u_2=0.5'' \end{cases}$$

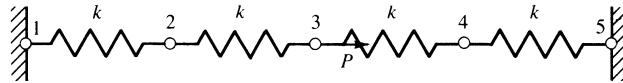
$$\Rightarrow f_{1x}^{(1)} = \left(-1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') \Rightarrow f_{1x}^{(1)} = -500 \text{ lb}$$

$$f_{2x}^{(1)} = \left(1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') \Rightarrow f_{2x}^{(1)} = 500 \text{ lb}$$

Element (2)

$$\begin{cases} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{cases} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{cases} u_2=0.5'' \\ u_3=1'' \end{cases} \Rightarrow \begin{cases} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{cases}$$

2.3



$$(a) [k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

By the method of superposition we construct the global  $[K]$  and knowing  $\{F\} = [K] \{d\}$  we have

$$\begin{cases} F_{1x}=? \\ F_{2x}=0 \\ F_{3x}=P \\ F_{4x}=0 \\ F_{5x}=? \end{cases} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{cases} u_1=0 \\ u_2 \\ u_3 \\ u_4 \\ u_5=0 \end{cases}$$

$$(b) \begin{cases} 0 \\ P \\ 0 \end{cases} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{cases} u_2 \\ u_3 \\ u_4 \end{cases} \Rightarrow \begin{cases} 0 = 2ku_2 - ku_3 \\ P = -ku_2 + 2ku_3 - ku_4 \\ 0 = -ku_3 + 2ku_4 \end{cases}$$

$$\Rightarrow u_2 = \frac{u_3}{2}; u_4 = \frac{u_3}{2}$$

Substituting in the equation in the middle

$$P = -k u_2 + 2k u_3 - k u_4$$

$$\Rightarrow P = -k \left( \frac{u_3}{2} \right) + 2k u_3 - k \left( \frac{u_3}{2} \right)$$

$$\Rightarrow P = k u_3$$

$$\Rightarrow u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k}; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation  $\{F\} = [K] \{d\}$

$$F_{1x} = -k u_2 = -k \frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k u_4 = -k \frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

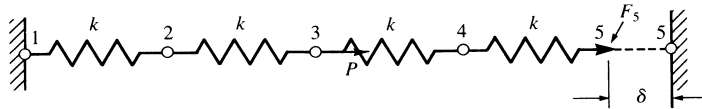
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left( -\frac{P}{2} \right) + P = 0$$

$$\Rightarrow 0 = 0$$

## 2.4



$$(a) [k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

By the method of superposition the global  $[K]$  is constructed.

Also  $\{F\} = [K] \{d\}$  and  $u_1 = 0$  and  $u_5 = \delta$

$$\begin{cases} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{cases} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{cases} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{cases}$$

$$(b) \quad 0 = 2k u_2 - k u_3 \quad (1)$$

$$0 = -k u_2 + 2k u_3 - k u_4 \quad (2)$$

$$0 = -k u_3 + 2k u_4 - k \delta \quad (3)$$

From (2)

$$u_3 = 2 u_2$$

From (3)

$$u_4 = \frac{\delta + 2 u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -k(u_2) + 2k(2u_2) - k\left(\frac{\delta + 2\delta_{2x}}{2}\right)$$

$$\Rightarrow -u_2 + 4u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2 \frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\left(\frac{\delta}{4}\right)}{2} \Rightarrow u_4 = \frac{3\delta}{4}$$

(c) Going back to the global equation

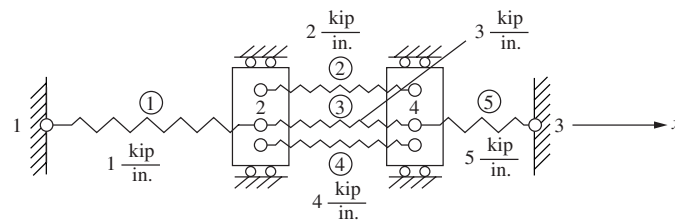
$$\{F\} = [K] \{d\}$$

$$F_{1x} = -k u_2 = -k \frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k \delta}{4}$$

$$F_{5x} = -k u_4 + k \delta = -k \left(\frac{3\delta}{4}\right) + k \delta$$

$$\Rightarrow F_{5x} = \frac{k \delta}{4}$$

## 2.5



$$[k^{(1)}] = \begin{bmatrix} d_1 & d_2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} d_2 & d_4 \\ 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} d_2 & d_4 \\ 3 & -3 \\ -3 & 3 \end{bmatrix}; \quad [k^{(4)}] = \begin{bmatrix} d_2 & d_4 \\ 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$[k^{(5)}] = \begin{bmatrix} d_4 & d_3 \\ 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Assembling global  $[K]$  using direct stiffness method

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2+3+4 & 0 & -2-3-4 \\ 0 & 0 & 5 & -5 \\ 0 & -2-3-4 & -5 & 2+3+4+5 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

**2.6** Now apply + 2 kip at node 2 in spring assemblage of P 2.5.

$$\therefore F_{2x} = 2 \text{ kip}$$

$$[K]\{d\} = \{F\}$$

$[K]$  from P 2.5

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 2 \\ F_3 \\ 0 \end{Bmatrix} \quad (\text{A})$$

where  $u_1 = 0, u_3 = 0$  as nodes 1 and 3 are fixed.

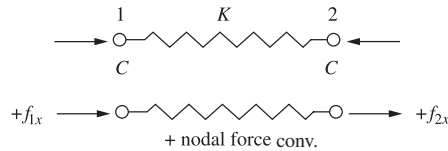
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 10 & -9 \\ -9 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 0.475 \text{ in.}, \quad u_4 = 0.305 \text{ in.}$$

**2.7**



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

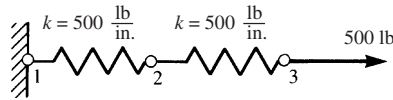
$$\therefore f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \text{ same as for tensile element}$$

2.8



$$k_1 = 500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; k_2 = 500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

So

$$[K] = 500 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\Rightarrow \begin{bmatrix} F_1 = ? \\ F_2 = 0 \\ F_3 = 1000 \end{bmatrix} = 500 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{cases} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{cases}$$

$$\Rightarrow 0 = 1000 u_2 - 500 u_3 \tag{1}$$

$$500 = -500 u_2 + 500 u_3 \tag{2}$$

From (1)

$$u_2 = \frac{500}{1000} u_3 \Rightarrow u_2 = 0.5 u_3 \tag{3}$$

Substituting (3) into (2)

$$\Rightarrow 500 = -500 (0.5 u_3) + 500 u_3$$

$$\Rightarrow 500 = 250 u_3$$

$$\Rightarrow u_3 = 2 \text{ in.}$$

$$\Rightarrow u_2 = (0.5) (2 \text{ in.}) \Rightarrow u_2 = 1 \text{ in.}$$

Element 1-2

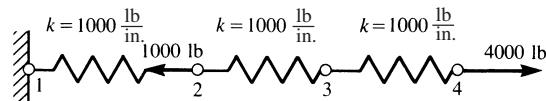
$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = 500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \text{ in.} \\ 1 \text{ in.} \end{Bmatrix} \Rightarrow \begin{cases} f_{1x}^{(1)} = -500 \text{ lb} \\ f_{2x}^{(1)} = 500 \text{ lb} \end{cases}$$

Element 2-3

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = 500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \text{ in.} \\ 2 \text{ in.} \end{Bmatrix} \Rightarrow \begin{cases} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{cases}$$

$$F_{1x} = 500 [1 \ -1 \ 0] \begin{bmatrix} 0 \\ 1 \text{ in.} \\ 2 \text{ in.} \end{bmatrix} \Rightarrow F_{1x} = -500 \text{ lb}$$

2.9



$$[k^{(1)}] = \begin{matrix} & \begin{matrix} (1) & (2) \end{matrix} \\ \begin{matrix} (1) \\ (2) \end{matrix} & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{matrix}$$

$$[k^{(2)}] = \begin{matrix} & \begin{matrix} (2) & (3) \end{matrix} \\ \begin{matrix} (2) \\ (3) \end{matrix} & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{matrix}$$

$$[k^{(3)}] = \begin{matrix} & \begin{matrix} (3) & (4) \end{matrix} \\ \begin{matrix} (3) \\ (4) \end{matrix} & \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \end{matrix}$$

$$[K] = \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix} & \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -1000 & 0 \\ 0 & -1000 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \end{matrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -1000 \\ F_{3x} = 0 \\ F_{4x} = 4000 \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -1000 & 0 \\ 0 & -1000 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{aligned} u_1 &= 0 \text{ in.} \\ u_2 &= 3 \text{ in.} \\ u_3 &= 7 \text{ in.} \\ u_4 &= 11 \text{ in.} \end{aligned}$$

Reactions

$$F_{1x} = [1000 \quad -1000 \quad 0 \quad 0] \begin{Bmatrix} u_1 = 0 \\ u_2 = 3 \\ u_3 = 7 \\ u_4 = 11 \end{Bmatrix} \Rightarrow F_{1x} = -3000 \text{ lb}$$

Element forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 3 \end{Bmatrix} \Rightarrow \begin{aligned} f_{1x}^{(1)} &= -3000 \text{ lb} \\ f_{2x}^{(1)} &= 3000 \text{ lb} \end{aligned}$$

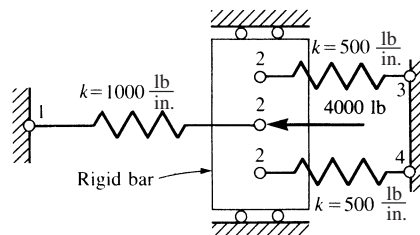
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 3 \\ 7 \end{Bmatrix} \Rightarrow \begin{aligned} f_{2x}^{(2)} &= -4000 \text{ lb} \\ f_{3x}^{(2)} &= 4000 \text{ lb} \end{aligned}$$

Element (3)

$$\begin{Bmatrix} f_{3x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 7 \\ 11 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= -4000 \text{ lb} \\ f_{4x}^{(3)} &= 4000 \text{ lb} \end{aligned}$$

## 2.10





$$[k^{(1)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = -4000 \\ F_{3x} = ? \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow u_2 = \frac{-4000}{2000} = -2 \text{ in.}$$

Reactions

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -4000 \\ 1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ -2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2000 \\ -2000 \end{Bmatrix} \text{ lb}$$

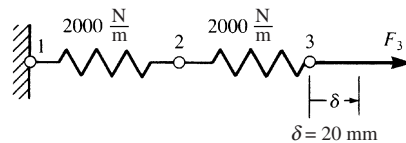
Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

Element (3)

$$\begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} -2 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} f_{2x}^{(3)} \\ f_{4x}^{(3)} \end{Bmatrix} = \begin{Bmatrix} -1000 \\ 1000 \end{Bmatrix} \text{ lb}$$

## 2.11



$$[k^{(1)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}; [k^{(2)}] = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{1x}=? \\ F_{2x}=0 \\ F_{3x}=? \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 & 0 \\ -2000 & 4000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix} \begin{Bmatrix} u_1=0 \\ u_2=? \\ u_3=0.02 \text{ m} \end{Bmatrix}$$

$$\Rightarrow u_2 = 0.01 \text{ m}$$

Reactions

$$F_{1x} = (-2000)(0.01) \Rightarrow F_{1x} = -20 \text{ N}$$

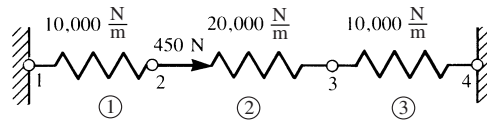
Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.01 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix} \text{ N}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.02 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -20 \\ 20 \end{Bmatrix} \text{ N}$$

## 2.12



$$[k^{(1)}] = [k^{(3)}] = 10000 \begin{Bmatrix} 1 & -1 \\ -1 & 1 \end{Bmatrix}$$

$$[k^{(2)}] = 10000 \begin{Bmatrix} 2 & -2 \\ -2 & 2 \end{Bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{1x}=? \\ F_{2x}=4500 \text{ N} \\ F_{3x}=0 \\ F_{4x}=? \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1=0 \\ u_2=? \\ u_3=? \\ u_4=0 \end{Bmatrix}$$

$$0 = -2u_2 + 3u_3 \Rightarrow u_2 = \frac{3}{2}u_3 \Rightarrow u_2 = 1.5u_3$$

$$450 \text{ N} = 30000(1.5u_3) - 20000u_3$$

$$\Rightarrow 450 \text{ N} = (25000 \frac{\text{N}}{\text{m}})u_3 \Rightarrow u_3 = 1.8 \times 10^{-2} \text{ m}$$

$$\Rightarrow u_2 = 1.5(1.8 \times 10^{-2}) \Rightarrow u_2 = 2.7 \times 10^{-2} \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 2.7 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x}^{(1)} \\ \hat{f}_{2x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -270 \text{ N} \\ 270 \text{ N} \end{Bmatrix}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 2.7 \times 10^{-2} \\ 1.8 \times 10^{-2} \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}^{(2)} = 180 \text{ N} \\ \hat{f}_{3x}^{(2)} = -180 \text{ N} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = 10000 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 1.8 \times 10^{-2} \\ 0 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{3x}^{(3)} = 180 \text{ N} \\ \hat{f}_{4x}^{(3)} = -180 \text{ N} \end{matrix}$$

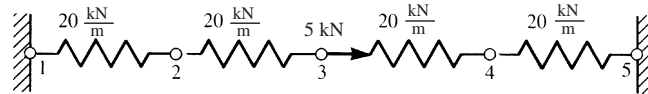
Reactions

$$\{F_{1x}\} = \left(10000 \frac{\text{N}}{\text{m}}\right) [1 \ -1] \begin{Bmatrix} 0 \\ 2.7 \times 10^{-2} \end{Bmatrix} \Rightarrow F_{1x} = -270 \text{ N}$$

$$\{F_{4x}\} = \left(10000 \frac{\text{N}}{\text{m}}\right) [-1 \ 1] \begin{Bmatrix} 1.8 \times 10^{-2} \\ 0 \end{Bmatrix}$$

$$\Rightarrow F_{4x} = -180 \text{ N}$$

### 2.13



$$[k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 10 \text{ kN} \\ F_{4x} = 0 \\ F_{5x} = ? \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = 0 \end{Bmatrix}$$

$$\left. \begin{matrix} 0 = 2u_2 - u_3 \Rightarrow u_2 = 0.5u_3 \\ 0 = -u_3 + 2u_4 \Rightarrow u_4 = 0.5u_3 \end{matrix} \right\} \Rightarrow u_2 = u_4$$

$$\Rightarrow 5 \text{ kN} = -20u_2 + 40(2u_2) - 20u_2$$

$$\Rightarrow 5 = 40u_2 \Rightarrow u_2 = 0.125 \text{ m}$$

$$\Rightarrow u_4 = 0.125 \text{ m}$$

$$\Rightarrow u_3 = 2(0.125) \Rightarrow u_3 = 0.25 \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{1x}^{(1)} = -2.5 \text{ kN} \\ \hat{f}_{2x}^{(1)} = 2.5 \text{ kN} \end{matrix}$$

Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0.25 \end{Bmatrix} \Rightarrow \begin{matrix} \hat{f}_{2x}^{(2)} = -2.5 \text{ kN} \\ \hat{f}_{3x}^{(2)} = 2.5 \text{ kN} \end{matrix}$$

Element (3)

$$\begin{Bmatrix} f_{3x} \\ f_{4x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.25 \\ 0.125 \end{Bmatrix} \Rightarrow \begin{aligned} f_{3x}^{(3)} &= 2.5 \text{ kN} \\ f_{4x}^{(3)} &= -2.5 \text{ kN} \end{aligned}$$

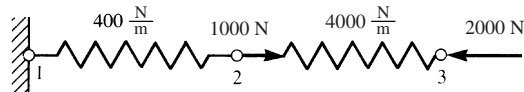
Element (4)

$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{5x} \end{Bmatrix} = 20 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow \begin{aligned} \hat{f}_{4x}^{(4)} &= 2.5 \text{ kN} \\ \hat{f}_{5x}^{(4)} &= -2.5 \text{ kN} \end{aligned}$$

$$F_{1x} = 20 [1 \quad -1] \begin{Bmatrix} 0 \\ 0.125 \end{Bmatrix} \Rightarrow F_{1x} = -2.5 \text{ kN}$$

$$F_{5x} = 20 [-1 \quad 1] \begin{Bmatrix} 0.125 \\ 0 \end{Bmatrix} \Rightarrow F_{5x} = -2.5 \text{ kN}$$

2.14



$$[k^{(1)}] = [k^{(2)}] = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = 100 \\ F_{3x} = -200 \end{Bmatrix} = 400 \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \end{Bmatrix}$$

$$100 = 800 u_2 - 400 u_3$$

$$-200 = -400 u_2 + 400 u_3$$

$$-100 = 400 u_2 \Rightarrow u_2 = -0.25 \text{ m}$$

$$100 = 800(-0.25) - 400 u_3 \Rightarrow u_3 = -0.75 \text{ m}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow \begin{aligned} \hat{f}_{1x}^{(1)} &= 100 \text{ N} \\ \hat{f}_{2x}^{(1)} &= -100 \text{ N} \end{aligned}$$

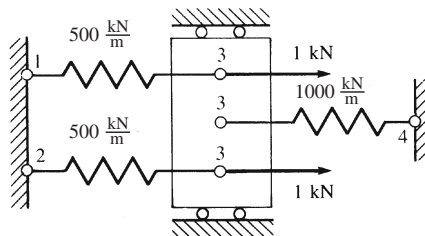
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = 400 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} -0.25 \\ -0.75 \end{Bmatrix} \Rightarrow \begin{aligned} \hat{f}_{2x}^{(2)} &= 200 \text{ N} \\ \hat{f}_{3x}^{(2)} &= -200 \text{ N} \end{aligned}$$

Reaction

$$\{F_{1x}\} = 400 [1 \quad -1] \begin{Bmatrix} 0 \\ -0.25 \end{Bmatrix} \Rightarrow F_{1x} = 100 \text{ N}$$

2.15



$$[k^{(1)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; [k^{(2)}] = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix}; [k^{(3)}] = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix}$$

$$\begin{Bmatrix} F_{1x} = ? \\ F_{2x} = ? \\ F_{3x} = 2 \text{ kN} \\ F_{4x} = ? \end{Bmatrix} = \begin{bmatrix} 500 & 0 & -500 & 0 \\ 0 & 500 & -500 & 0 \\ -500 & -500 & 2000 & -1000 \\ 0 & 0 & -1000 & 1000 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\Rightarrow u_3 = 0.001 \text{ m}$$

Reactions

$$F_{1x} = (-500)(0.001) \Rightarrow F_{1x} = -0.5 \text{ kN}$$

$$F_{2x} = (-500)(0.001) \Rightarrow F_{2x} = -0.5 \text{ kN}$$

$$F_{4x} = (-1000)(0.001) \Rightarrow F_{4x} = -1.0 \text{ kN}$$

Element (1)

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

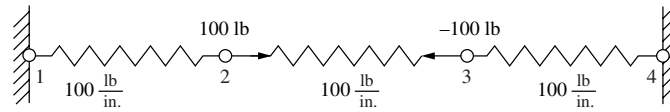
Element (2)

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 500 & -500 \\ -500 & 500 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.001 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{Bmatrix} -0.5 \text{ kN} \\ 0.5 \text{ kN} \end{Bmatrix}$$

Element (3)

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0.001 \\ 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{Bmatrix} 1 \text{ kN} \\ -1 \text{ kN} \end{Bmatrix}$$

2.16



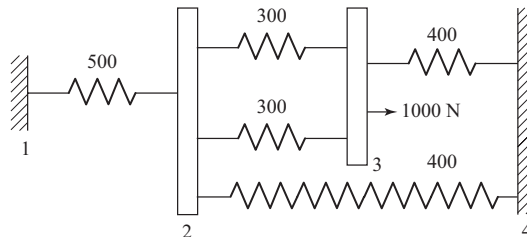
$$\begin{Bmatrix} F_{1x} \\ 100 \\ -100 \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 100 + 100 & -100 & 0 \\ 0 & -100 & 100 + 100 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} 100 \\ -100 \end{Bmatrix} = \begin{Bmatrix} 200 & -100 \\ -100 & 200 \end{Bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$$u_2 = \frac{1}{3} \text{ in.}$$

$$u_3 = -\frac{1}{3} \text{ in.}$$

2.17



$$\begin{Bmatrix} F_{1x}=? \\ 0 \\ 1000 \text{ N} \\ F_{4x}=? \end{Bmatrix} = \begin{bmatrix} 500 & -500 & 0 & 0 \\ -500 & (400+300) & -300-300 & -400 \\ 0 & (500+300) & (300+300+400) & -400 \\ 0 & -300-300 & -400 & 400+400 \end{bmatrix} \begin{Bmatrix} u_1=0 \\ u_2 \\ u_3 \\ u_4=0 \end{Bmatrix}$$

$$0 = 1500 u_2 - 600 u_3$$

$$1000 = -600 u_2 + 1000 u_3$$

$$u_3 = \frac{15 \cancel{\delta} \cancel{\delta}}{6 \cancel{\delta} \cancel{\delta}} u_2 = 2.5 u_2$$

$$1000 = -600 u_2 + 1000 (2.5 u_2)$$

$$1000 = 1900 u_2$$

$$u_2 = \frac{1000}{1900} = \frac{1}{1.9} \text{ mm} = 0.526 \text{ mm}$$

$$u_3 = 2.5 \left( \frac{1}{1.9} \right) \text{ mm} = 1.316 \text{ mm}$$

$$F_{1x} = -500 \left( \frac{1}{1.9} \right) = -263.16 \text{ N}$$

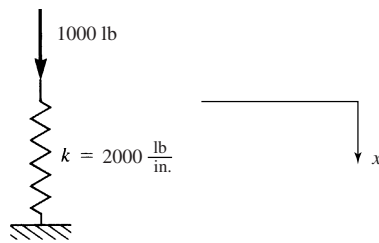
$$F_{4x} = -400 \left( \frac{1}{1.9} \right) - 400 \left( 2.5 \left( \frac{1}{1.9} \right) \right)$$

$$= -400 \left( \frac{1}{1.9} + \frac{2.5}{1.9} \right) = -736.84 \text{ N}$$

$$\Sigma F_x = -263.16 + 1000 - 736.84 = 0$$

### 2.18

(a)



As in Example 2.4

$$\pi_p = U + \Omega$$

$$U = \frac{1}{2} k x^2, \Omega = -Fx$$

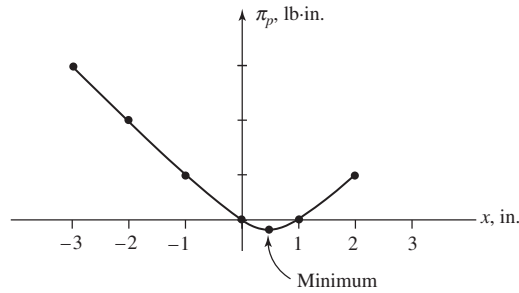
Set up table

$$\pi_p = \frac{1}{2} (2000) x^2 - 1000 x = 1000 x^2 - 1000 x$$

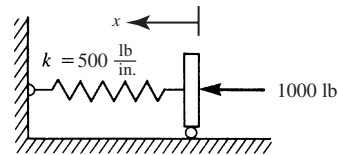
Deformation $x$ , in.	$\pi_p$ , lb·in.
-3.0	6000
-2.0	3000

-1.0	1000
0.0	0
0.5	-125
1.0	0
2.0	1000

$$\frac{\partial \pi_p}{\partial x} = 2000x - 1000 = 0 \Rightarrow x = 0.5 \text{ in. yields minimum } \pi_p \text{ as table verifies.}$$



(b)



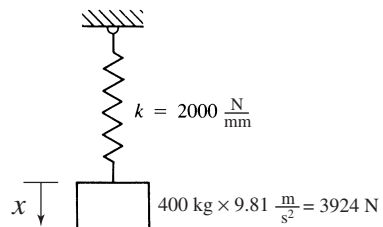
$$\pi_p = \frac{1}{2} kx^2 - F_x = 250x^2 - 1000x$$

$x$ , in.	$\pi_p$ , lb·in.
-3.0	11250
-2.0	3000
-1.0	1250
0	0
1.0	-750
2.0	-1000
3.0	-750

$$\frac{\partial \pi_p}{\partial x} = 500x - 1000 = 0$$

$$\Rightarrow x = 2.0 \text{ in. yields } \pi_p \text{ minimum}$$

(c)



$$\pi_p = \frac{1}{2} (2000) x^2 - 3924 x = 1000 x^2 - 3924 x$$

$$\frac{\partial \pi_p}{\partial x} = 2000 x - 3924 = 0$$

⇒  $x = 1.962$  mm yields  $\pi_p$  minimum

$$\pi_{p \text{ min}} = \frac{1}{2} (2000) (1.962)^2 - 3924 (1.962)$$

⇒  $\pi_{p \text{ min}} = -3849.45$  N·mm

(d) 
$$\pi_p = \frac{1}{2} (400) x^2 - 981 x$$

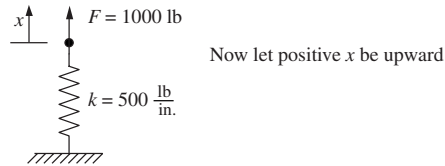
$$\frac{\partial \pi_p}{\partial x} = 400 x - 981 = 0$$

⇒  $x = 2.4525$  mm yields  $\pi_p$  minimum

$$\pi_{p \text{ min}} = \frac{1}{2} (400) (2.4525)^2 - 981 (2.4525)$$

⇒  $\pi_{p \text{ min}} = -1202.95$  N·mm

## 2.19



$$\pi_p = \frac{1}{2} kx^2 - Fx$$

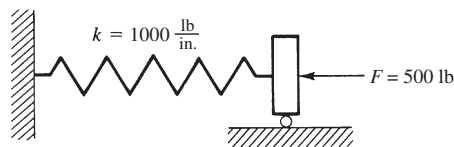
$$\pi_p = \frac{1}{2} (500) x^2 - 1000 x$$

$$\pi_p = 250 x^2 - 1000 x$$

$$\frac{\partial \pi_p}{\partial x} = 500 x - 1000 = 0$$

⇒  $x = 2.0$  in. ↑

## 2.20



$$F = k\delta^2 \quad (x = \delta)$$

$$dU = F dx$$



$$U = \int_0^x (kx^2) dx$$

$$U = \frac{kx^3}{3}$$

$$\Omega = -Fx$$

$$\pi_p = \frac{1}{3} kx^3 - 500x$$

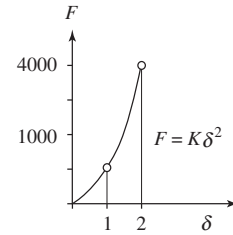
$$\frac{\partial \pi_p}{\partial x} = 0 = kx^2 - 500$$

$$0 = 1000x^2 - 500$$

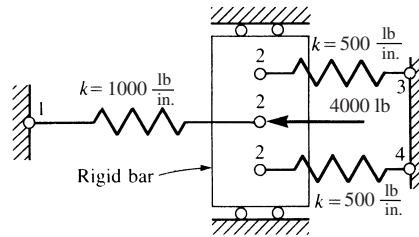
$$\Rightarrow x = 0.707 \text{ in. (equilibrium value of displacement)}$$

$$\pi_{p \text{ min}} = \frac{1}{3} (1000) (0.707)^3 - 500 (0.707)$$

$$\pi_{p \text{ min}} = -235.7 \text{ lb}\cdot\text{in.}$$



**2.21** Solve Problem 2.10 using P.E. approach



$$\pi_p = \sum_{e=1}^3 \pi_p^{(e)} = \frac{1}{2} k_1 (u_2 - u_1)^2 + \frac{1}{2} k_2 (u_3 - u_2)^2 + \frac{1}{2} k_3 (u_4 - u_2)^2$$

$$-f_{1x}^{(1)} u_1 - f_{2x}^{(1)} u_2 - f_{2x}^{(2)} u_2$$

$$-f_{3x}^{(2)} u_3 - f_{2x}^{(3)} u_2 - f_{4x}^{(3)} u_4$$

$$\frac{\partial \pi_p}{\partial u_1} = -k_1 u_2 + k_1 u_1 - f_{1x}^{(1)} = 0 \quad (1)$$

$$\begin{aligned} \frac{\partial \pi_p}{\partial u_2} &= k_1 u_2 - k_1 u_1 - k_2 u_3 + k_2 u_2 - k_3 u_4 \\ &\quad + k_3 u_2 - f_{2x}^{(1)} - f_{2x}^{(2)} - f_{2x}^{(3)} = 0 \end{aligned} \quad (2)$$

$$\frac{\partial \pi_p}{\partial u_3} = k_2 u_3 - k_2 u_2 - f_{3x}^{(2)} = 0 \quad (3)$$

$$\frac{\partial \pi_p}{\partial u_4} = k_3 u_4 - k_3 u_2 - f_{4x}^{(3)} = 0 \quad (4)$$

In matrix form (1) through (4) become

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \\ f_{3x}^{(2)} \\ f_{4x}^{(3)} \end{Bmatrix} \quad (5)$$

or using numerical values

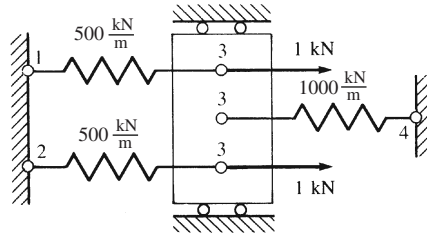
$$\begin{bmatrix} 1000 & -1000 & 0 & 0 \\ -1000 & 2000 & -500 & -500 \\ 0 & -500 & 500 & 0 \\ 0 & -500 & 0 & 500 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 = 0 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ -4000 \\ F_{3x} \\ F_{4x} \end{Bmatrix} \quad (6)$$

Solution now follows as in Problem 2.10

Solve 2<sup>nd</sup> of Equations (6) for  $u_2 = -2$  in.

For reactions and element forces, see solution to Problem 2.10

**2.22** Solve Problem 2.15 by P.E. approach



$$\begin{aligned} \pi_p &= \sum_{e=1}^3 \pi_p^{(e)} = \frac{1}{2} k_1 (u_3 - u_1)^2 + \frac{1}{2} k_2 (u_3 - u_2)^2 \\ &\quad + \frac{1}{2} k_3 (u_4 - u_3)^2 - f_{1x}^{(1)} u_1 \\ &\quad - f_{3x}^{(1)} u_3 - f_{2x}^{(2)} u_2 - f_{3x}^{(2)} u_3 \\ &\quad - f_{3x}^{(3)} u_3 - f_{3x}^{(4)} u_4 \end{aligned}$$

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = -k_2 u_3 + k_2 u_2 - f_{2x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 + k_2 u_3 - k_2 u_2 - k_3 u_4 + k_3 u_3 - f_{3x}^{(2)} - f_{3x}^{(3)} - f_{3x}^{(1)} - k_1 u_1$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_3 u_4 - k_3 u_3 - f_{3x}^{(4)}$$

In matrix form

$$\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_2 & -k_2 & 0 \\ -k_1 & -k_2 & k_1+k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} = 2 \text{ kN} \\ F_{4x} \end{Bmatrix}$$

For rest of solution, see solutions of Problem 2.15.

**2.23**

$$I = a_1 + a_2 x$$

$$I(0) = a_1 = I_1$$

$$I(L) = a_1 + a_2 L = I_2$$

$$a_2 = \frac{I_2 - I_1}{L}$$

$$\therefore I = I_1 + \frac{I_2 - I_1}{L} x$$

Now  $V = IR$

$$V = -V_1 = R(I_2 - I_1)$$

$$V = V_2 = R(I_2 - I_1)$$

$$\begin{Bmatrix} V_1 \\ V_2 \end{Bmatrix} = R \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix}$$

