Chapter 2

Problems

1. (a) $S = \{(r, r), (r, g), (r, b), (g, r), (g, g), (g, b), (b, r), b, g), (b, b)\}$

   (b) $S = \{(r, g), (r, b), (g, r), (g, b), (b, r), (b, g)\}$

2. $S = \{(n, x_1, \ldots, x_{n-1}), n \geq 1, x_i \neq 6, i = 1, \ldots, n-1\}$, with the interpretation that the outcome is $(n, x_1, \ldots, x_{n-1})$ if the first 6 appears on roll n, and x_i appears on roll, $i, i = 1, \ldots, n-1$. The event $(\cup_{n=1}^{\infty} E_n)^c$ is the event that 6 never appears.

3. $EF = \{(1, 2), (1, 4), (1, 6), (2, 1), (4, 1), (6, 1)\}$.

   $E \cup F$ occurs if the sum is odd or if at least one of the dice lands on 1. $FG = \{(1, 4), (4, 1)\}$. EF^c is the event that neither of the dice lands on 1 and the sum is odd. $EFG = FG$.

4. $A = \{1, 0001, 0000001, \ldots\}$ $B = \{01, 00001, 00000001, \ldots\}$

   $(A \cup B)^c = \{00000 \ldots, 001, 000001, \ldots\}$

5. (a) $2^5 = 32$

   (b) $W = \{(1, 1, 1, 1, 1), (1, 1, 1, 1, 0), (1, 1, 1, 0, 1), (1, 1, 0, 1, 1), (1, 1, 1, 0, 0), (1, 1, 0, 1, 0)$
   $(1, 1, 0, 0, 1), (1, 1, 0, 0, 0), (1, 0, 1, 1, 1), (0, 1, 1, 1, 1), (1, 0, 1, 1, 0), (0, 1, 1, 1, 0)$
   $(0, 0, 1, 1, 1) (0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

   (c) 8

   (d) $AW = \{(1, 1, 1, 0, 0), (1, 1, 0, 0, 0)\}$

6. (a) $S = \{(1, g), (0, g), (1, f), (0, f), (1, s), (0, s)\}$

   (b) $A = \{(1, s), (0, s)\}$

   (c) $B = \{(0, g), (0, f), (0, s)\}$

   (d) $\{(1, s), (0, s), (1, g), (1, f)\}$

7. (a) 6^{15}

   (b) $6^{15} - 3^{15}$

   (c) 4^{15}

8. (a) .8

   (b) .3

   (c) 0

9.   Choose a customer at random. Let A denote the event that this customer carries an American Express card and V the event that he or she carries a VISA card.

$$P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74.$$

Therefore, 74 percent of the establishment's customers carry at least one of the two types of credit cards that it accepts.

10.   Let R and N denote the events, respectively, that the student wears a ring and wears a necklace.

(a) $P(R \cup N) = 1 - .6 = .4$

(b) $.4 = P(R \cup N) = P(R) + P(N) - P(RN) = .2 + .3 - P(RN)$

Thus, $P(RN) = .1$

11.   Let A be the event that a randomly chosen person is a cigarette smoker and let B be the event that she or he is a cigar smoker.
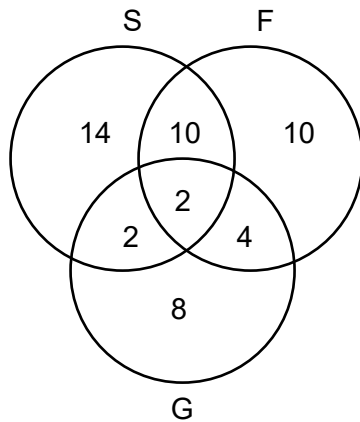
(a) $1 - P(A \cup B) = 1 - (.07 + .28 - .05) = .7$. Hence, 70 percent smoke neither.

(b) $P(A^c B) = P(B) - P(AB) = .07 - .05 = .02$. Hence, 2 percent smoke cigars but not cigarettes.

12.   (a) $P(S \cup F \cup G) = (28 + 26 + 16 - 12 - 4 - 6 + 2)/100 = 1/2$
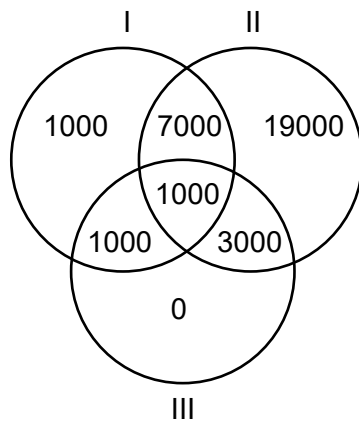
The desired probability is $1 - 1/2 = 1/2$.

(b) Use the Venn diagram below to obtain the answer 32/100.

(c) Since 50 students are not taking any of the courses, the probability that neither one is taking a course is $\binom{50}{2}/\binom{100}{2} = 49/198$ and so the probability that at least one is taking a course is 149/198.

13.

(a)   20,000
(b)   12,000
(c)   11,000
(d)   68,000
(e)   10,000

14. $P(M) + P(W) + P(G) - P(MW) - P(MG) - P(WG) + P(MWG) = .312 + .470 + .525 - .086 - .042 - .147 + .025 = 1.057$

15.   (a) $4\dbinom{13}{5} \Big/ \dbinom{52}{5}$

(b) $13\dbinom{4}{2}\dbinom{12}{3}\dbinom{4}{1}\dbinom{4}{1}\dbinom{4}{1} \Big/ \dbinom{52}{5}$

(c) $\dbinom{13}{2}\dbinom{4}{2}\dbinom{4}{2}\dbinom{44}{1} \Big/ \dbinom{52}{5}$

(d) $13\dbinom{4}{3}\dbinom{12}{2}\dbinom{4}{1}\dbinom{4}{1} \Big/ \dbinom{52}{5}$

(e) $13\dbinom{4}{4}\dbinom{48}{1} \Big/ \dbinom{52}{5}$

16.   (a) $\dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5}$

(b) $\dfrac{6\dbinom{5}{2}5 \cdot 4 \cdot 3}{6^5}$

(c) $\dfrac{\dbinom{6}{2}4\dbinom{5}{2}\dbinom{3}{2}}{6^5}$

(d) $\dfrac{6 \cdot 5 \cdot 4\dbinom{5}{3}}{21}$

(e) $\dfrac{6 \cdot 5\dbinom{5}{3}}{6^5}$

(f) $\dfrac{6 \cdot 5\dbinom{5}{4}}{6^5}$

(g) $\dfrac{6}{6^5}$

17. $\dfrac{\dbinom{15}{8}\dbinom{10}{8}\dbinom{7}{1}}{\dbinom{25}{16}\dbinom{9}{1}} = .1102$

18. $\dfrac{2 \cdot 4 \cdot 16}{52 \cdot 51}$

19. $4/36 + 4/36 + 1/36 + 1/36 = 5/18$

20.   Let A be the event that you are dealt blackjack and let B be the event that the dealer is dealt blackjack. Then,

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

$$= \frac{4 \cdot 4 \cdot 16}{52 \cdot 51} + \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49}$$

$$= .0983$$

where the preceding used that $P(A) = P(B) = 2 \times \dfrac{4 \cdot 16}{52 \cdot 51}$. Hence, the probability that neither is dealt blackjack is .9017.

21.   (a) $p_1 = 4/20$, $p_2 = 8/20$, $p_3 = 5/20$, $p_4 = 2/20$, $p_5 = 1/20$

(b) There are a total of $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 2 \cdot 4 + 1 \cdot 5 = 48$ children. Hence,

$$q_1 = 4/48, \ q_2 = 16/48, \ q_3 = 15/48, \ q_4 = 8/48, \ q_5 = 5/48$$

22.   The ordering will be unchanged if for some k, $0 \le k \le n$, the first k coin tosses land heads and the last $n - k$ land tails. Hence, the desired probability is $(n + 1/2^n$

23.   The answer is 5/12, which can be seen as follows:

$$1 = P\{\text{first higher}\} + P\{\text{second higher}\} + p\{\text{same}\}$$
$$= 2P\{\text{second higher}\} + p\{\text{same}\}$$
$$= 2P\{\text{second higher}\} + 1/6$$

Another way of solving is to list all the outcomes for which the second is higher. There is 1 outcome when the second die lands on two, 2 when it lands on three, 3 when it lands on four, 4 when it lands on five, and 5 when it lands on six. Hence, the probability is $(1 + 2 + 3 + 4 + 5)/36 = 5/12$.

25. $P(E_n) = \left(\dfrac{26}{36}\right)^{n-1} \dfrac{6}{36}, \quad \displaystyle\sum_{n=1}^{\infty} P(E_n) = \dfrac{2}{5}$

27.   Imagine that all 10 balls are withdrawn

$$P(A) = \frac{3 \cdot 9! + 7 \cdot 6 \cdot 3 \cdot 7! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 5! + 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 3!}{10!}$$

28. $$P\{\text{same}\} = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}}$$

$$P\{\text{different}\} = \binom{5}{1}\binom{6}{1}\binom{8}{1} \Big/ \binom{19}{3}$$

If sampling is with replacement

$$P\{\text{same}\} = \frac{5^3 + 6^3 + 8^3}{(19)^3}$$

$$P\{\text{different}\} = P(RBG) + P\{BRG\} + P(RGB) + \ldots + P(GBR)$$

$$= \frac{6 \cdot 5 \cdot 6 \cdot 8}{(19)^3}$$

29.   (a) $$\frac{n(n-1) + m(m-1)}{(n+m)(n+m-1)}$$

(b) Putting all terms over the common denominator $(n+m)^2(n+m-1)$ shows that we must prove that

$$n^2(n+m-1) + m^2(n+m-1) \geq n(n-1)(n+m) + m(m-1)(n+m)$$

which is immediate upon multiplying through and simplifying.

30.   (a) $$\frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} = 1/18$$

(b) $$\frac{\binom{7}{3}\binom{8}{3}3!}{\binom{8}{4}\binom{9}{4}4!} - 1/18 = 1/6$$

(c) $$\frac{\binom{7}{3}\binom{8}{4} + \binom{7}{4}\binom{8}{3}}{\binom{8}{4}\binom{9}{4}} = 1/2$$

31. $P(\{\text{complete}\} =$

        $P\{\text{same}\} =$

32. $\dfrac{g(b+g-1)!}{(b+g)!} = \dfrac{g}{b+g}$

33. $\dfrac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = \dfrac{70}{323}$

34. $\binom{32}{13} \Big/ \binom{52}{13}$

35.   (a) $\dfrac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}}$

        (b) $1 - \dfrac{\binom{34}{7}}{\binom{46}{7}} - \dfrac{\binom{12}{1}\binom{34}{6}}{\binom{46}{7}}$

        (c) $\dfrac{\binom{12}{7}+\binom{16}{7}+\binom{18}{7}}{\binom{46}{7}}$

        (d) $P(R_3 \cup B_3) = P(R_3) + P(B_3) - P(R_3 B_3) = \dfrac{\binom{12}{3}\binom{34}{4}}{\binom{46}{7}} + \dfrac{\binom{16}{3}\binom{30}{4}}{\binom{46}{7}} - \dfrac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}$

36.   (a) $\binom{4}{2} \Big/ \binom{52}{2} \approx .0045,$

        (b) $13\binom{4}{2} \Big/ \binom{52}{2} = 1/17 \approx .0588$

37.   (a) $\dbinom{7}{5} \Big/ \dbinom{10}{5} = 1/12 \approx .0833$

        (b) $\dbinom{7}{4}\dbinom{3}{1} \Big/ \dbinom{10}{5} + 1/12 = 1/2$

38. $1/2 = \dbinom{3}{2} \Big/ \dbinom{n}{2}$ or $n(n-1) = 12$ or $n = 4$.

39. $\dfrac{5 \cdot 4 \cdot 3}{5 \cdot 5 \cdot 5} = \dfrac{12}{25}$

40.   .8134; .1148

41. $1 - \dfrac{5^4}{6^4}$

42. $1 - \left(\dfrac{35}{36}\right)^n$

43. $\dfrac{2(n-1)(n-2)}{n!} = \dfrac{2}{n}$ in a line

        $\dfrac{2n(n-2)!}{n!} = \dfrac{2}{n-1}$ if in a circle, $n \geq 2$

44.   (a) If A is first, then A can be in any one of 3 places and B's place is determined, and the others can be arranged in any of 3! ways. As a similar result is true, when B is first, we see that the probability in this case is $2 \cdot 3 \cdot 3!/5! = 3/10$

        (b) $2 \cdot 2 \cdot 3!/5! = 1/5$

        (c) $2 \cdot 3!/5! = 1/10$

45. $1/n$ if discard, $\dfrac{(n-1)^{k-1}}{n^k}$ if do not discard

46.   If n in the room,

        $$P\{\text{all different}\} = \dfrac{12 \cdot 11 \cdot \quad \cdot (13 - n)}{12 \cdot 12 \cdot \quad \cdot 12}$$

        When $n = 5$ this falls below 1/2. (Its value when $n = 5$ is .3819)

47. $\dfrac{\dbinom{8}{2}\dbinom{5}{2}}{\dbinom{14}{5}} = .1399$

48. $\dbinom{12}{4}\dbinom{8}{4}\dfrac{(20)!}{(3!)^4(2!)^4}\Big/(12)^{20}$

49. $\dbinom{6}{3}\dbinom{6}{3}\Big/\dbinom{12}{6}$

50. $\dbinom{13}{5}\dbinom{39}{8}\dbinom{8}{8}\dbinom{31}{5}\Big/\dbinom{52}{13}\dbinom{39}{13}$

51. $\dbinom{n}{m}(n-1)^{n-m}/N^n$

52. (a) $\dfrac{20\cdot18\cdot16\cdot14\cdot12\cdot10\cdot8\cdot6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

   (b) $\dfrac{\dbinom{10}{1}\dbinom{9}{6}\dfrac{8!}{2!}2^6}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13}$

53. Let A_i be the event that couple i sit next to each other. Then

$$P(\cup_{i=1}^4 A_i) = 4\frac{2\cdot7!}{8!} - 6\frac{2^2\cdot6!}{8!} + 4\frac{2^3\cdot5!}{8!} - \frac{2^4\cdot4!}{8!}$$

and the desired probability is 1 minus the preceding.

54. $P(S\cup H\cup D\cup C) = P(S) + P(H) + P(D) + P(C) - P(SH) - \ldots - P(SHDC)$

$$= \frac{4\dbinom{39}{13}}{\dbinom{52}{13}} - \frac{6\dbinom{26}{13}}{\dbinom{52}{13}} + \frac{4\dbinom{13}{13}}{\dbinom{52}{13}}$$

$$= \frac{4\dbinom{39}{13} - 6\dbinom{26}{13} + 4}{\dbinom{52}{13}}$$

55.   (a) $P(S \cup H \cup D \cup C) = P(S) + \ldots - P(SHDC)$

$$= \frac{4\binom{2}{2}}{\binom{52}{13}} - \frac{6\binom{2}{2}\binom{2}{2}\binom{48}{9}}{\binom{52}{13}} + \frac{4\binom{2}{2}^3\binom{46}{7}}{\binom{52}{13}} - \frac{\binom{2}{2}^4\binom{44}{5}}{\binom{52}{13}}$$

$$= \frac{4\binom{50}{11} - 6\binom{48}{9} + 4\binom{46}{7} - \binom{44}{5}}{\binom{52}{13}}$$

(b) $P(1 \cup 2 \cup \ldots \cup 13) = \dfrac{13\binom{48}{9}}{\binom{52}{13}} - \dfrac{\binom{13}{2}\binom{44}{5}}{\binom{52}{13}} + \dfrac{\binom{13}{3}\binom{40}{1}}{\binom{52}{13}}$

56.   Player B. If Player A chooses spinner (a) then B can choose spinner (c). If A chooses (b) then B chooses (a). If A chooses (c) then B chooses (b). In each case B wins probability 5/9.

Theoretical Exercises

5. $F_i = E_i \overset{i=1}{\underset{j=1}{\cap}} E_j^c$

6.   (a) EF^cG^c

(b) EF^cG

(c) $E \cup F \cup G$

(d) $EF \cup EG \cup FG$

(e) EFG

(f) $E^cF^cG^c$

(g) $E^cF^cG^c \cup EF^cG^c \cup E^cFG^c \cup E^cF^cG$

(h) $(EFG)^c$

(i) $EFG^c \cup EF^cG \cup E^cFG$

(j) S

8.   The number of partitions that has $n + 1$ and a fixed set of i of the elements $1, 2, \ldots, n$ as a subset is T_{n-i}. Hence, (where $T_0 = 1$). Hence, as there are $\binom{n}{i}$ such subsets.

$$T_{n+1} = \sum_{i=0}^{n}\binom{n}{i}T_{n-i} = 1 + \sum_{i=0}^{n-1}\binom{n}{i}T_{n-i} = 1 + \sum_{k=1}^{n}\binom{n}{k}T_k \, .$$

11. $1 \geq P(E \cup F) = P(E) + P(F) - P(EF)$

12. $P(EF^c \cup E^cF) = P(EF^c) + P(E^cF)$
$$= P(E) - P(EF) + P(F) - P(EF)$$

13. $E = EF \cup EF^c$

15. $\dfrac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}}$

16. $P(E_1 \ldots E_n) \geq P(E_1 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferonni's Ineq.

$$\geq \sum_{1}^{n-1} P(E_i) - (n-2) + P(E_n) - 1 \text{ by induction hypothesis}$$

19. $\dfrac{\dbinom{n}{r-1}\dbinom{m}{k-r}(n-r+1)}{\dbinom{n+m}{k-1}(n+m-k+1)}$

21.   Let $y_1, y_2, \ldots, y_k$ denote the successive runs of losses and $x_1, \ldots, x_k$ the successive runs of wins. There will be $2k$ runs if the outcome is either of the form $y_1, x_1, \ldots, y_k x_k$ or $x_1 y_1, \ldots x_k, y_k$ where all x_i, y_i are positive, with $x_1 + \ldots + x_k = n$, $y_1 + \ldots + y_k = m$. By Proposition 6.1 there are

$2\dbinom{n-1}{k-1}\dbinom{m-1}{k-1}$ number of outcomes and so

$$P\{2k \text{ runs}\} = 2\dbinom{n-1}{k-1}\dbinom{m-1}{k-1}\Big/\dbinom{m+n}{n}.$$

There will be $2k+1$ runs if the outcome is either of the form $x_1, y_1, \ldots, x_k, y_k, x_{k+1}$ or $y_1, x_1, \ldots, y_k, x_k y_{k+1}$ where all are positive and $\sum x_i = n$, $\sum y_i = m$. By Proposition 6.1 there are $\dbinom{n-1}{k}\dbinom{m-1}{k-1}$ outcomes of the first type and $\dbinom{n-1}{k-1}\dbinom{m-1}{k}$ of the second.