Instructor's Manual with Solutions

A First Course in Mathematical Modeling

FIFTH EDITION

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PART I

ORGANIZING A MODELING COURSE

ISSUES

Organizing a modeling course is a significant, educational challenge. A number of resource materials must be gathered: an appropriate text, supplemental references both for students as well as the instructor, sources and scenarios for student projects, and, possibly, computer software. Furthermore, there are crucial pedagogical issues that must be resolved in designing the course, including the following:

1. Course objectives
2. Course prerequisites
3. Course content
4. Number and type of student projects
5. Individual versus group projects
6. The role of computation
7. Grading considerations
8. Opportunities for follow-on modeling courses.

We believe that a textbook can only serve as a base for a modeling course, which must then be tailored to meet the specific needs of students, as well as overall objectives in the curriculum. Moreover, a modeling course needs to be flexible and dynamic to allow for each individual instructor to take advantage of his or her particular mathematical expertise, experiences, and modeling preferences.

OBJECTIVES

The overall goal of our course is to provide a thorough introduction to the entire modeling process while affording students the opportunity to practice:

1. **Creative and Empirical Model Construction:** Given a real-world scenario, the student must identify a problem, make assumptions and collect data, propose a model, test the assumptions, refine the model as necessary, fit the model to data if appropriate, and analyze the underlying mathematical structure of the model in order to appraise the sensitivity of the conclusions in relation to the assumptions. Furthermore, the student should be able to generalize the construction to related scenarios.

2. **Model Analysis:** Given a model, the student must work backward to uncover the implicit underlying assumptions, assess critically how well the assumptions reflect the scenario at hand, and estimate the sensitivity of the conclusions when the assumptions are not precisely met.

3. **Model Research:** The student investigates an area of interest to gain knowledge, understanding, and an ability to use what already has been created or discovered. Model research provides for determining the 'state of the art' in a subject area.

To accomplish our goals, we provide students with a diversity of scenarios for practicing all three of these facets of modeling. In addition, normal and routine exercises are assigned to test the student's understanding of the instructional material we present. Thus, our textbook provides expository material and a framework around which a diversity of additional materials can be organized in support of a *modeling course.*

COURSE PREREQUISITES

There are strong arguments to require such courses as advanced calculus, linear algebra, differential equations, probability, numerical analysis, and optimization as prerequisites to an introductory modeling course. Certainly the level and sophistication of the mathematics that students are capable of using increases significantly as more advanced courses are added to their programs. However, our desire is to gain the modeling experience as early as possible in the student's career. Although some unfamiliar mathematical ideas are taught as part of the modeling process, the emphasis is on using mathematics already known to the student after completing high school. As outlined in the preface to the text, a course can be constructed requiring only high school mathematics as a prerequisite. Some sections do require an introductory calculus course as a prerequisite or corequisite, as detailed in the preface to the text. In our modeling courses, we emphasize teaching students how to use mathematics they already know in a context of significant applications with which they can readily identify. This approach stimulates student interest in mathematics and motivates them to study more advanced topics such as those mentioned above. Moreover, our students are eager to see meaningful applications of the mathematics they have learned.

COURSE CONTENT

Many modeling courses select from an inventory of specific model types which can be adapted to a variety of situations. Certainly model selection is a valid step in the problem-solving process and it is important that students learn to use what already has been created. However, our experience is that undergraduate students seldom comprehend the assumptions inherent in type models. We want our students to realize the necessity of making assumptions, the need to determine the appropriateness of the assumptions, and the importance of investigating how sensitive the conclusions are to the assumptions. Consequently, while we do discuss how to fit type models in the text, we have chosen to emphasize model construction, leaving the study of type models for more advanced courses.

We feel that model construction promotes student creativity, demonstrates the artistic nature of model building, and develops an appreciation for how mathematics can be used effectively in various settings. Since the student needs practice in the first several steps of the problem-solving process–identifying the problem, making assumptions, determining interrelationships between the variables and submodels–it is tempting to compose an entire course of creative model construction. However, there are serious difficulties with such a course. Typically, students are very anxious, at least initially, because they don't know how to begin the modeling process. After all, they have probably never before attacked an open-ended problem. When they are successful in constructing a model, they usually find the procedure enjoyable and exhilarating. Nevertheless, they tend to "burn out" if an entire quarter or semester is dedicated to creative model construction. Moreover, a course consisting entirely of creative model construction cannot address other important aspects of modeling, like experimentation and simulation. Furthermore, there are difficulties with such a course for the instructor. Preparation of the course requires enormous effort in researching and generating scenarios to be modeled. Grading is difficult and tends to be subjective because each student approaches each project in a different way, yet students need constant feedback on their work. These difficulties then contribute to the anxieties of the student who is overly concerned about being graded in class under a time constraint in an area perceived to be relatively subjective. Under these conditions, success of the course becomes highly instructor dependent and circumstantial.

For the above reasons, we have chosen to design a course consisting of a mixture of both *creative* and *empirical* modeling projects, along with projects in model analysis and model research. We begin by interactively constructing graphical models in class to engage students immediately in model analysis, which is relatively familiar to them. The transition to creative model construction commences with the students learning to make assumptions about a real-world behavior and by providing them with data to check their assumptions (initially with simple proportionality arguments). We then apply the modeling process to construct interactively in class relatively simple submodels and models in a variety of settings covering many disciplines. Students begin to see that situations arise where it may be very difficult or impossible to construct an analytic model, yet predictive capabilities are highly desirable. This perception motivates the study of empirical modeling. Students find empirical model construction more procedural and reproducible than creative model construction, and we find they welcome the mixture. In the text, some scenarios are modeled several ways creatively, and several ways empirically, so students can experience the alternatives that may be available.

STUDENT PROJECTS

We have developed our modeling courses to promote progressive development in the student's modeling capabilities. To achieve our objectives, we require each student to complete at least 6 **significant** problem assignments or projects to be handed-in for a grade. Each student is assigned a mixture of problems/projects in creative and empirical model construction, model analysis, and model research. We purposely select problems/projects which address scenarios for which there are no unique solutions. Several of the projects include **real** data that the student is either given or can **readily** collect.

If the course is taught early in the student's program we recommend a combination of individual and group projects. Individual work is essential if the student is to develop adequate modeling skills. By way of contrast, group projects are exhilarating and allow for the experience of the synergistic effect that takes place in a 'brainstorming' session. For example, during the instruction of Chapter 2 on proportionality, we give a choice of 5 or 6 different scenarios for which students individually create a model. During the instruction of the next chapter on model fitting, teams are organized to test, refine, and fit these models to data. The resulting group model is typically substantially superior to any of the individual models. A similar procedure is followed during the instruction of the simulation chapter where students are required to develop models individually followed by a team effort to refine the models and implement them on the computer.

Finally, we choose projects from a wide variety of disciplines including biology, economics and the physical sciences. We make certain that these projects require little overhead to be paid by the student in order to understand, and successfully attack, the problem. We also allow students to choose a project that can be turned in at the end of the course. Students are free to choose from a diversity of possibilities such as completing UMAP modules, researching a model studied earlier, developing a model in a scenario of interest to them, or analyzing a model presented in another subject they are studying. This chosen project gives the student an opportunity to pursue a subject in some detail. Typically, our students choose a project of interest from one of the Project sections in the text. We require that students complete one of the following for their selected projects:

1. **Model Construction:** Develop a scenario of interest to the student and develop a model to address the situation posed. For example, the student may pose a relevant economic question and construct a graphical model to explain the behavior being studied.

2. **Model Analysis:** Analyze a model of interest by identifying the underlying assumptions and discussing their applicability. Determine the mathematical conclusions of the model and the sensitivity of the conclusions to the assumptions made. Finally, interpret the mathematical conclusions for real-world scenarios.

3. **Model Research:** Research a model studied in class to determine the 'state of the art' in that area. For example, many UMAP modules are available that treat scenarios discussed in class in greater detail. The student may also study a section of the text not covered in class and complete the problems from the corresponding problem set.

THE ROLE OF COMPUTATION

We emphasize that computing and programming capabilities are not required for our modeling courses. However, beginning with Chapter One, Modeling Change, computation does play a role of increasing importance. The use of computers can significantly enhance a modeling course: in a demonstrational mode to facilitate student understanding of a concept (such as a graphical solution of a linear program) or in a computational mode to reduce the tediousness in carrying out certain numerical procedures. We have found the integration of a computer in a supportive role adds significantly to student interest and to the realism of our modeling course.

We provide our students with packaged software for making scatterplots, fitting models according to various criteria, solving linear programs by the Simplex method, constructing divided difference tables and cubic spline models, solving initial value problems by various methods, and graphing functions.

A TYPICAL COURSE

In this edition, we have provided several options for organizing a course. We have detailed the prerequisites in Figure 2 of the Preface. We incorporate lecture/discussion lessons, hour exams, and computer workshop sessions as part of our modeling courses. Additionally, time is provided for students to complete modeling projects. Detailed discussions of the various lessons and suggestions on how many lessons to devote to each section appear in the Teaching Suggestions portion of this Manual, arranged according to the chapters of our text.

GRADING CONSIDERATIONS

The difficulty in grading a modeling course has dissuaded many instructors from teaching modeling, and discouraged many students from taking the course as well. Given the objectives of our course, both creative and technical skills are to be acquired by the student. We have found it best to have the creative requirements accomplished outside the classroom, without the added pressure of a fixed, short time constraint. We use the classroom exams strictly for testing techniques we expect the student to master. These classroom exams test the understanding of basic concepts or straightforward applications of particular techniques, such as fitting a cubic spline (see the section on Sample Tests in this Manual). The more interpretive applications of the modeling techniques are treated in problems and projects assigned for work outside the classroom.

The classroom exams are similar to those in other mathematics courses and present no added difficulty in grading. For most of the student projects we are able to establish a grading scale that assigns weights to various parts of each problem. Homework is collected often, principally for purposes of feedback but also for a grade. (Some homework is collected but not graded, to assist students in allowing their minds to roam without fear of being 'wrong' while at the same time providing some guidance and feedback.)

THE USE OF UMAP MODULES

We have found material provided by The Consortium For Mathematics and Its Applications (COMAP) to be outstanding and particularly well suited to the course we propose. COMAP started under a grant from the National Science Foundation and has as its goal the production of instructional materials to introduce applications of mathematics into the undergraduate curriculum.

The individual modules may be used in a variety of ways. First, they may be used as instructional material to support several lessons. (We have incorporated several modules in the text in precisely this manner.) In this mode a student completes the self-study module by working through its exercises (the detailed solutions provided with the module can be removed conveniently before it is issued). Another option is to put together a block of instruction covering, for example, linear programming or difference equations, using as instructional material one or more UMAP modules suggested in the projects sections of the text. The modules also provide excellent sources for 'model research' since they cover a wide variety of applications of mathematics in many fields. In this mode, a student is given an appropriate module to be researched and is asked to complete and report on the module. Finally, the modules are excellent resources for scenarios for which students can practice model construction. In this mode the teacher writes a scenario for a student project based on an application addressed in a particular module and uses the module as background material, perhaps having the student complete the module at a later date. Information may be obtained by contacting COMAP Inc., Suite 3B, 175 Middlesex Turnpike, Bedford, MA 01730, 800-722-6627.

FOLLOW-ON MODELING COURSES

Many of our students express an interest in taking an additional modeling course. While we have tailored many courses to the individual needs of the students' programs, a very successful and popular course is one which combines studies from various UMAP modules selected to fit a student's particular interests, coupled with significant advanced student projects. The proportion of time devoted to the modules versus the student projects varies, depending on the nature of the projects that are available.

We are fortunate to have real-world projects readily available at our schools and appropriate to assign for advanced student work. There are, however, several other excellent sources which can be used as background material for student projects The comprehensive four-volume series *Modules in Applied Mathematics*, edited by William F. Lucas and published by Springer-Verlag, provides important and realistic applications of mathematics appropriate for undergraduates. The volumes treat differential equations models, political and related models, discrete and system models, and life science models. Another book, *Case Studies in Mathematical Modelling,* edited by D. J. James and J. J. McDonald and published by Halsted Press, provides case studies explicitly designed to facilitate the development of mathematical models. Finally, scenarios with an industrial flavor are contained in the excellent modular series edited by J. L. Agnew and M. S. Keener of Oklahoma State University.

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PART II

TEACHING SUGGESTIONS

CHAPTER ONE

Modeling Change

SUGGESTED SYLLABUS

Class Hours	Topic	Text
1	Modeling Change with Difference Equations	Sect 1.1
1	Approximating Change with Difference Equations	Sect 1.2
1	Solutions to Dynamical Systems	Sect 1.3
1	Systems of Difference Equations	Sect 1.4

OBJECTIVES

The major objectives of this chapter are:

1. To build and solve models involving change that takes place in discrete intervals.
2. To prepare the students for modeling change taking place continuously in Chapters 11 and 12.
3. To introduce numerical solutions by iterating difference equations.
4. To extend the modeling process by modeling interactive systems early.

DISCUSSION

We have found that freshman students model dynamical systems quite naturally. Since they can iterate the systems they build given an initial value, they gain intuition by graphing their results and analyzing the long-term behavior. Further, by experimenting with different initial values, they begin to appreciate the sensitivity of the model's conclusions to the initial conditions. Finally, the experience of this chapter prepares them for modeling with differential equations and systems of differential equations, which students generally find more difficult.

We strongly suggest beginning with behavior that can be modeled exactly, such as the accumulation of money in a savings account. Moving to annuities or mortgages retains student interest while introducing more sophisticated models. Since the students can readily enumerate these sequences even before building the model, they gain confidence in their work. Once the students are confident with exactly modeling behavior, we move to approximating change with discrete systems. Only an elementary notion of proportionality is needed for the scenarios in this chapter. (The concepts of proportionality and geometric similarity are studied in more detail in Chapter Two). A powerful advantage of studying discrete systems early is that the resulting models can be iterated if starting values are known. We have found that students have no difficulty iterating the systems. Finally, introducing systems of difference equations permits the students to model interesting systems with rather elementary mathematics. Many of the scenarios introduced here are revisited in Chapters 11 and 12.

A spreadsheet is of great use to iterate and graph the models the students build. We have found that even if the students have not used a spreadsheet before, they learn the few commands they need for this chapter quickly. They do not have difficulties extending their knowledge of spreadsheets to handle systems of difference equations in Section 1.4. We added a new modeling example and problems on *SIR* models.

We add a few additional examples for nonlinear DDS here for the instructors use. We build nonlinear discrete dynamical systems to describe the change in behavior of the quantities we study. We also will study systems of DDS to describe the changes in various systems that act together in some way or ways. We define a nonlinear DDS– If the function of a_n involves powers of a_n (like a_n^2), or a functional relationship (like $\frac{a_n}{a_{n-1}}$), we will say that the discrete dynamical system is nonlinear. A sequence is a function whose domain is the set of non-negative integers ($n = 0, 1, 2, \ldots$). We will restrict our model solution to the numerical and graphical solutions. Analytical solutions may be studied in more advanced mathematics courses.

Example 1. Growth of a Yeast Culture

We often model population growth by assuming that the change in population is directly proportional to the current size of the given population. This produces a simple, first order DDS similar to those seen earlier. It might appear reasonable at first examination, but the long-term behavior of growth without bound is disturbing. Why would growth without bound of a yeast culture in a jar (or controlled space) be alarming?

There are certain factors that affect population growth. Things include resources (food, oxygen, space, etc.) These resources can support some maximum population. As this number is approached, the change (or growth rate) should decrease and the population should never exceed its resource supported amount.

Problem Identification: Predict the growth of yeast in a controlled environment as a function of the resources available and the current population.

Assumptions and Variables:

We assume that the population size is best described by the weight of the biomass of the culture. We define y_n as the population size of the yeast culture after period n. There exists a maximum carrying capacity, M, that is sustainable by the resources available. The yeast culture is growing under the conditions established.

Model:

$$y_{n+1} = y_n + k \cdot y_n \cdot (M - y_n)$$

where

y_n is the population size after period n

n is the time period measured in hours

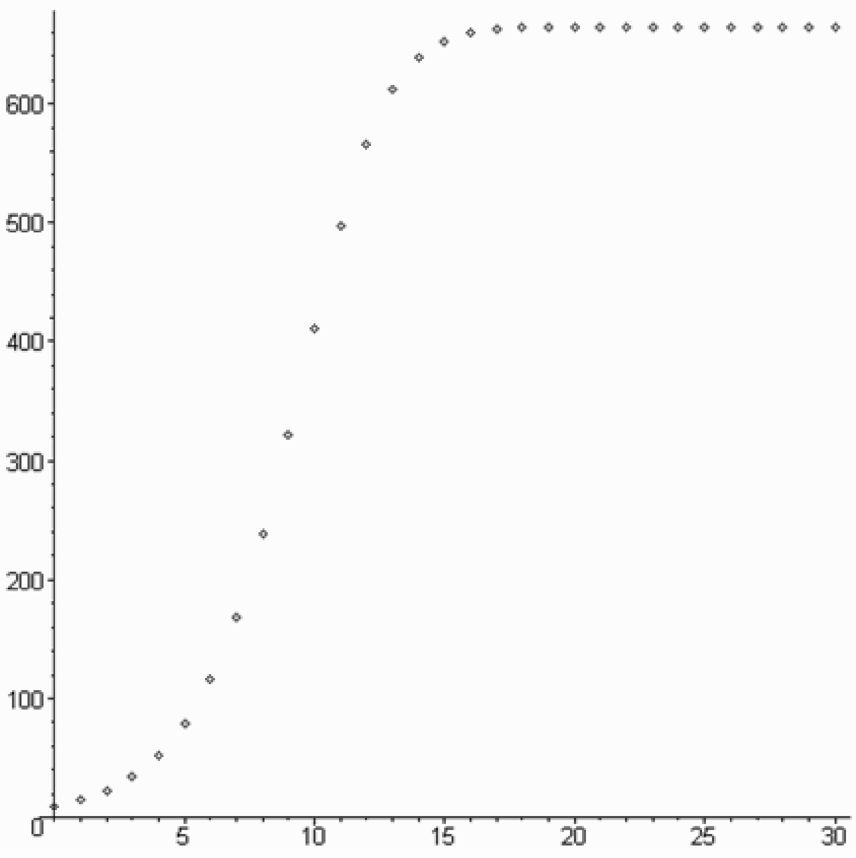
k is the growth (or decay) constant of proportionality

M is the carrying capacity of our system

In our experiment, we first plot y_n versus n and find a stable equilibrium value of approximately 665. Next, we plot $y_{n+1} - y_n$ versus $y_n(665 - y_n)$to find the slope, k; it is approximately 0.00082 , With $k = 0.00082$ and the carrying capacity in biomass is 665. This analysis is currently in the textbook. The final model is

$$y_{n+1} = y_n + 0.00082 \cdot y_n \cdot (665 - y_n)$$

Again, this is nonlinear because of the y_n^2 term obtained in the expansion. The solution is iterated via technology (there is no closed form analytical solution for this equation) from an initial condition, biomass, of 9.6:



Plot of DDS from growth of a yeast culture.

The model shows stability in that the population (biomass) of the yeast culture approaches 665 as n gets large. Thus, the population is eventually stable at approximately 665 units.

Example 2. Spread of a Contagious Disease

There are one thousand students in a college dormitory, and some students have been diagnosed with meningitis, a highly contagious disease. The health center wants to build a model to determine how fast the disease will spread.

Problem Identification: Predict the number of students affected with meningitis as a function of time.

Assumptions and Variables: Let $m(n)$ be the number of students affected with meningitis after n days. We assume all students are susceptible to the disease. The possible interactions of infected and susceptible students are proportional to their product (as an interaction term).

The model is,

$$m_{n+1} = m_n + k \cdot m_n(1000 - m_n)$$

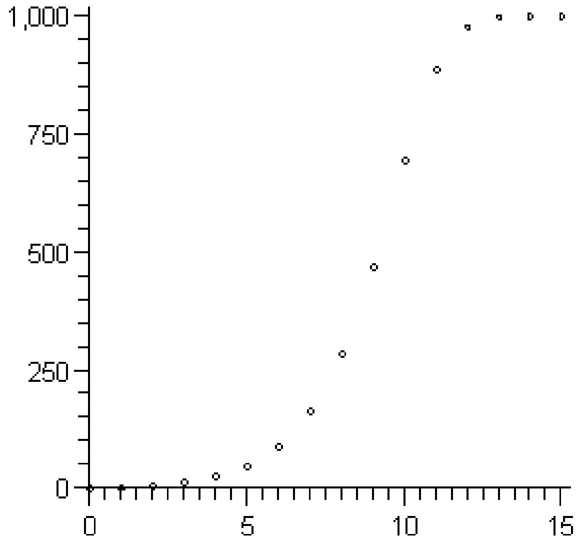
Two students returned from spring break with meningitis. The rate of spreading per day is characterized by $k = 0.0090.$ It is assumed that a vaccine can be in place and students vaccinated within 1-2 weeks.

$m_{n+1} = m_n + 0.00090 \cdot m_n(1000 - m_n)$

We iterated with technology:

2, 3.7964, 7.200188612, 13.63369992, 25.73673985,
48.30366391, 89.67704188, 163.1486049, 286.0266287,
469.8204854, 694.0007626, 885.1280963, 976.6368108,
997.1724263, 999.7100470, 999.9709290, 999.9970921,
999.9997092, 999.9999709, 999.9999971, 999.9999997,
1000.000000, 1000.000000, 1000.000000, 1000.000000,
1000.000000, 1000.000000, 1000.000000, 1000.000000,
1000.000000, 1000.000000

We see that without treatment everyone will eventually get the disease.



Plot of DDS for the spread of a disease.

Interpretation: The results show that most students will be affected within 2 weeks. Since only about 10% will be affected within one week, every effort must be made to get the vaccination at the school and get the students vaccinated within one week.

We also add an additional example for systems here for the instructor's use as needed. This is a military model concerning insurgencies.

Example 3. Modeling Military Interactions and Insurgencies

Insurgent forces have a strong foothold in the city of Urbania, a major metropolis in the center of the country of Ibestan. Intelligence estimates they currently have a force of about 1000 fighters. The local police force has approximately 1300 officers, many of which have had no formal training in law enforcement methods or modern tactics for addressing insurgent activity. Based on data collected over the past year, approximately 8% of insurgents switch sides and join the police each week, whereas about 11% of police switch sides and join the insurgents. Intelligence also estimates that around 120 new insurgents arrive from the neighboring country of Moronka each week. Recruiting efforts in Ibestan yield about 85 new police recruits each week as well. In armed conflict with insurgent forces, the local police are able to capture or kill approximately 10% of the insurgent force each week on average while losing about 3% of their force.

Problem Statement: Build a mathematical model of this insurgency. Determine the equilibrium state (if it exists) for this DDS.

We define the variables

P_n= the number of police in the system after time period n.

I_n= the number of insurgents in the system after time period n.

$n = 0, 1, 2, 3, \ldots$ weeks

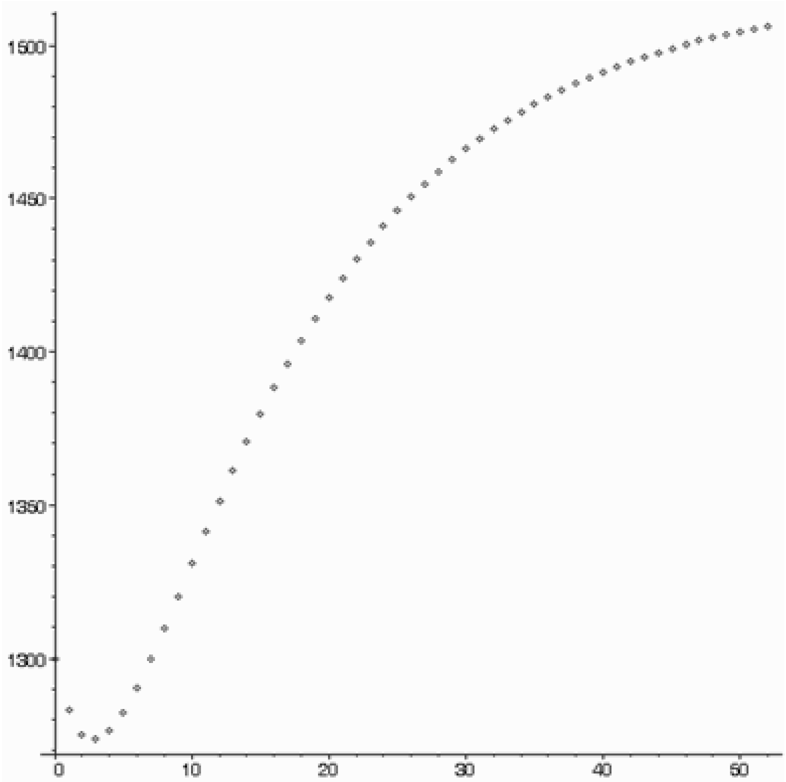
Model:

$P_{n+1} = P_n - 0.03P_n - 0.11P_n + 0.08I_n + 85, P_0 = 1300$

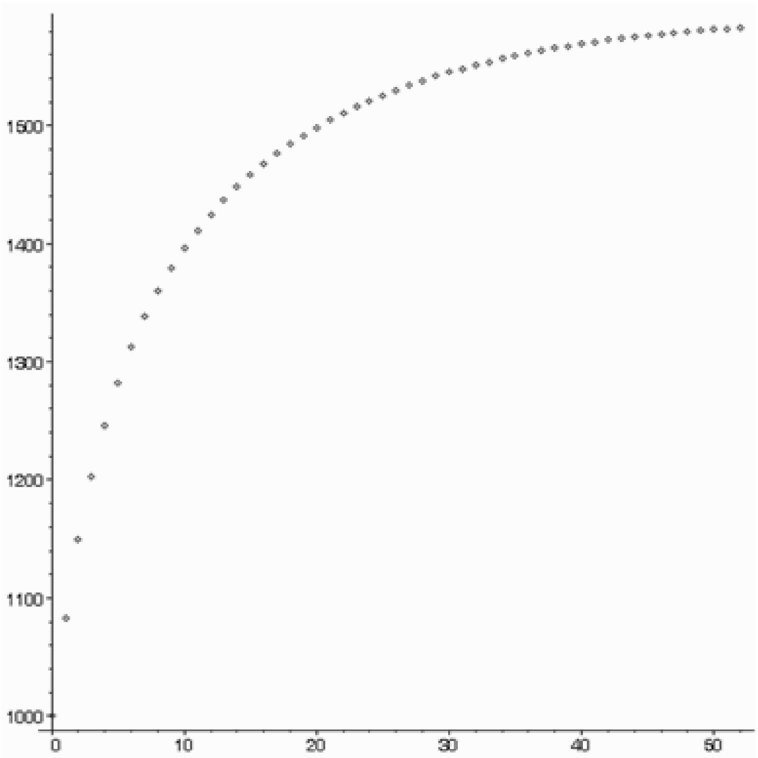
$I_{n+1} = I_n + 0.11P_n - 0.08I_n - 0.01I_n + 120, I_0 = 1000$

P_n	I_n
1300	1000
1283	1083
1275.02	1149.19
1273.452	1202.588
1276.376	1246.202
1282.38	1282.287
1290.429	1312.537
1299.772	1338.228
1309.862	1360.322
1320.307	1379.549
1330.828	1396.464
1341.229	1411.491
1351.377	1424.958
1361.18	1437.117
1370.585	1448.166
1379.556	1458.26
1388.079	1467.525
1396.15	1476.059
1403.774	1483.945
1410.961	1491.25
1417.726	1498.031

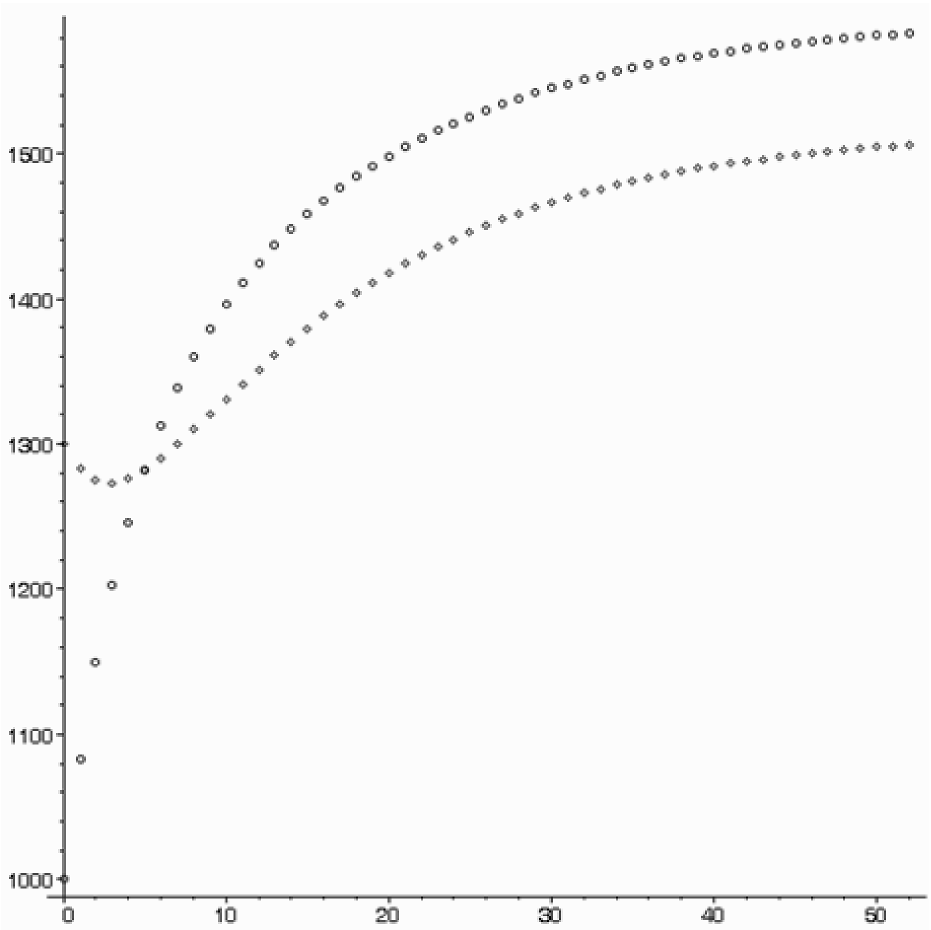
1424.087	1504.335
1430.062	1510.204
1435.669	1515.674
1440.93	1520.777
1445.862	1525.539



Plot of Insurgents I_n over time.



Plot of government, P_n over time



Overlay of the two plots.

Interpretation: In this insurgency model, we see that under the current conditions, that the insurgency overtakes the government within 5 to 10 weeks. If this is unacceptable, then we must modify conditions that affect the parameters in such a way to obtain a governmental victory. You might ask the students to experiment with these parameters provided.
We highly recommend assigning projects for this chapter. Students can be given a wide variety of scenarios to model and iterate. The problems and projects for this chapter can be augmented by those in Chapters 11 and 12. Having the students discuss their projects in class will demonstrate the power of dynamical systems to approximate a wide range of behaviors.

SUGGESTED PROBLEMS/PROJECTS

A spreadsheet is a useful companion for the following exercises:
Section 1.1: 1 – 6 and assign different students 7-10.
Section 1.2: 1 and 2. Assign different students a selection of the remaining problems.
Problems 3 through 10 will be revisited in Chapters 11 and 12.
Section 1.3: 1 – 4. Assign different students a selection of the remaining problems.
Section 1.4: Assign a mixture of all 7 problems to various students.

CHAPTER TWO
The Modeling Process, Proportionality, and Geometric Similarity

SUGGESTED SYLLABUS

Class hours	Topic	Text
1	Mathematical Models	Sect 2.1
1	Proportionality	Sect 2.2
1	Geometric Similarity	Sect 2.3
-	Automobile Mileage	Sect 2.4
-	Body Weight	Sect 2.5

OBJECTIVES

The major objectives of this chapter are:

The Modeling Process

1. To communicate to students that modeling is essentially a thinking **process** for tackling difficult open-ended problems.
2. To provide students with a specific procedure to follow when attacking new problems. Most students will personalize this procedure as they become increasingly familiar with the process.
3. To give each student an opportunity to practice the first several steps of the modeling process, both individually and in a group "brainstorming" session .
4. To introduce several of the scenarios for which models will be developed and tested further on in the course.

Proportionality and Geometric Similarity

1. To introduce the concepts of proportionality, geometric similarity, characteristic dimension, and shape factor(s).
2. To demonstrate the use of proportionality and geometric similarity in constructing explicative mathematical models.
3. To use proportionality in graphical model fitting to assess the reasonableness of a model and to ascertain how well it seems to fit a given set of data points.
4. To illustrate the importance of developing and testing submodels independently.

DISCUSSION: The Modeling Process

In this chapter we develop an overall picture of the modeling process and a structure for organizing the various components of mathematical modeling. The text materials describe a procedure that students may follow to formulate, validate, and use a mathematical model . We have found it useful for students to attack a few seemingly "non mathematical" problems to emphasize the thought process in mathematical modeling and to see how that process generalizes to problems they encounter in everyday life situations. The problems in Section 2.1 are especially suited for student group "brainstorming" sessions. It is important that each student realize that while certain steps of the modeling process are indeed artistic in nature, scientific techniques are needed to appraise the adequacy of that artistry. The computer flow chart in section 2.1 (Figure 2.7) of the text emphasizes the iterative nature of the modeling proc ess and shows the trade-offs between model simplification and refinement that blend of the artistic and scientific aspects of modeling. The remainder of the book presents foundational materials upon which follow-on quantitative courses can build (such as differential equations, linear programming, probability and statistics, numerical analysis, and so forth).

Although the problems may appear easy at first glance, students often have a difficult time getting started (which is reasonable since most of them have never attempted an open-ended problem). Students find it helpful if we develop several scenarios in detail using the methodology presented in the text. Consequently, we spend relatively little classroom time discussing Section 2.1 (we assign the section as reading) and concentrate on *interactively* developing in class one or more of the scenarios that appear in Section 2.1. Then having the students work together in teams is especially beneficial for the 2.1 Problems. We have also found it useful to develop a scenario that students have not seen before. (The most successful scenarios seem to be those that relate to their everyday lives, such as "How should final exams be scheduled?", or scenarios for which historical solutions can be offered like "How did you or the consultants approach this problem?").

In choosing problems to address, we suggest a mix that will illustrate the various aspects of the modeling process: some problems where many candidates are available for the problem identification, some where the initial problem is so difficult it later becomes necessary to restrict the problem that has been identified, and so forth. We have followed this strategy in selecting the illustrative problems in the text.

During the classroom discussions developing the scenarios, students initially tend to be overwhelmed by the number of variables associated with the problem. Typically they will try for far too much detail. We often suggest they start with major submodels and then refine as necessary. Subsequently, each submodel can be developed in telescopic fashion until the desired level of detail is obtained. A good discussion of the scenario, problem identification, classification of variables, and a discussion of major submodels is ample for the average student on this first go around. Normally we choose scenarios for which the student may later develop a proportionality model and, ultimately, test and fit it to collected data.

SUGGESTED PROBLEMS/PROJECTS

The problems and projects in this chapter are especially suited for group work. We typically assign one or two problems to each group and ask the group to turn in a single written report summarizing their conclusions.

It may also be very beneficial at this point to assign a project to individual students (since each student needs practice). There are a number of challenging projects given in section 2.1. The following are suggestions for additional scenarios:

1. How should final examinations be scheduled? What is the objective of the scheduling procedure?
2. Design an auditorium for a physics lecture. Discuss the solution presented in the paper "The Frank C. Wiley Lecture Halls: A new concept in the design of lecture auditoria," by A. A. Bartlett, *American Journal of Physics*, 41, 1973, 1233-1240.
3. How should the flow of water through a series of dams be controlled? How should irrigation benefits versus flood risks be modeled? (As water is lowered in the reservoir the capacity to store water increases, but the water supply for irrigation decreases.)

4. What are the priorities and objectives for course registration and class changes in a college or university? How can the scheduling operation be made more efficient?
5. How should insurance companies design screening tables for overweight people?

DISCUSSION: Modeling Using Proportionality

The concept of proportionality is used beginning with graphical model fitting and empirical one-term model fitting to test the reasonableness and validity of a model. More immediately, however, students now get an opportunity to actually build and test mathematical models using proportionality and geometric similarity. We have found it helpful to provide students with data to test their proportionality arguments in these early modeling attempts. Later on we motivate **model fitting** by asking the students to assess how well the constructed model fits the data. This assessment suggests that analytic techniques are needed to fit the model and that different criteria might be used to measure the adequacy of the fit. The concepts of geometric similarity, characteristic dimension, and shape factor(s) are needed in the study of dimensional analysis and similitude in Chapter 9. If you have covered Chapter One, you may want to move through the ideas of proportionality more quickly.

Proportionality: We have found it helpful to introduce the ideas of proportionality first. Later, we discuss geometric similarity and characteristic dimension coupled with a geometric interpretation of each concept. We then provide simple examples before proceeding to more sophisticated modeling exercises. You may have difficulty convincing students that all straight line graphs are not proportionality relationships. A good example for getting the point across is that of loading students onto a canoe versus an aircraft carrier; in neither case is the total volume of water displaced proportional to the number of students. However, proportionality is a much better assumption in the case of the canoe.

Geometric Similarity: We develop the volume and surface area formulas for a few regularly shaped objects before generalizing to irregularly shaped objects. We work with several characteristic dimensions as well. A simple example is in the case of the area of a circle: the area is proportional to the square of the radius, the diameter, the circumference, the length of arc between two corresponding points on the circle, or any other convenient linear measurement associated with the circle. Thus **any** of these measures can be chosen as a characteristic dimension. Some students have trouble realizing that any surface area (such as total surface area, cross-sectional area, or the surface area of a particular face of some surface) is proportional to the square of a characteristic dimension for objects that are geometrically similar, whatever characteristic dimension may have been chosen. All of these concepts are fundamental and are used throughout the remainder of the course. Thus a thorough understanding by students is essential.

ILLUSTRATIVE EXAMPLES

Vehicular Stopping Distance:

There are many assumptions in this model and it is probably worth developing in class. We especially emphasize the idea of developing and testing **submodels** independently.

A Bass Fishing Derby:

The purpose of this model is to provide students with some practice using the concept of geometric similarity in an example that is not too difficult. While students do seem to grasp the concept of geometric similarity for regularly shaped objects, they often have great difficulty generalizing the idea to irregularly shaped objects. Before introducing irregularly shaped objects, we have found it worthwhile to draw some circles on the blackboard and ask for the area in terms of the radius, diameter, and circumference. Students then realize that the area is proportional to the square of each of those measures so any of them may be chosen as a characteristic dimension. For emphasis we ask our students to consider the plot of radius versus diameter and radius versus circumference.

From the example of the circle, students can infer in the bass problem that the models $W \propto \ell^3$ and $W \propto g^3$ coincide if the bass are in fact geometrically similar. We ask our students to plot g versus ℓ to check this assumption. We also ask them which of the two dimensions is increasing more rapidly. Next we consider the $W \propto g^2\ell$ model which assumes that the transverse cross-sectional areas are geometrically similar. From the Mean Value Theorem for integral calculus, it can be assumed that there is an average cross section which when multiplied by the length of the bass gives its volume. Choosing the girth as the characteristic dimension and assuming that the average cross section is proportional to the square of the girth then gives the model. Finally, the model $W \propto g^2\ell$ can be developed in a similar fashion by assuming that the longitudinal cross sectional areas are geometrically similar.

USING A COMPUTER

Realism can be incorporated into the course by using mainframe computers, micro computers, or hand-held calculators to ease the analysis of larger sets of data. However, we emphasize that computers are **not required** for this course. The first several problems assigned can be with small data sets to ensure that students know what to plot to test the model $y \propto x^2$, for instance. They can then progress to larger data sets by using a computer (if one is available) for scatterplots and transformed scatterplots. We have used various computational devices and software in our courses and have even found it worthwhile to devote a class period during this chapter to familiarize students with any software available for their use. See the CD and or the website for technology information.

SUGGESTED PROBLEMS/PROJECTS

The problems and projects in this chapter provide the first opportunity for students to engage in creative model construction. Our students typically have questions concerning variables they may have overlooked, or they may be uncertain of adequate precision in their problem identification. One technique we have used to ease their anxieties is to require them first to submit a single problem or project addressing only the first several steps of the modeling process. This procedure helps the student onto a productive track in his or her first attempt at model construction. (We do not give a grade on this initial submission, only constructive comments.) Once on the right track, the student can begin the task of relating the variables for the various submodels, finally assembling them into a model.

Section 2.1: Assign a mix of problems to various students.
Section 2.2: Assign a mix of problems
Section 2.3: Assign a mix of problems to various students.
Projects: Ideas for projects appear in Sections 2.3 and the applications sections.

CHAPTER THREE
Model Fitting

SUGGESTED SYLLABUS

Class hours	Topic	Text
1	Fitting Models to Data Graphically	Sect 3.1
1	Analytic Methods of Model Fitting	Sect 3.2
1	Applying the Least-Squares Criterion	Sect 3.3
1	Choosing a Best Model	Sect 3.4

OBJECTIVES

The major objectives of this chapter are:

1. To introduce the idea of fitting a selected model type (e.g., a quadratic curve) to a collection of data.
2. To compare and contrast various criteria of "best-fit" including the least-squares and Chebyshev criteria.
3. To develop and apply the least-squares criterion to specific model types including the straight line, power curve, and quadratic function.
4. To distinguish between the least-squares fit for transformed data versus the original data.

A secondary objective is to motivate the eventual study of optimization (especially linear programming), statistics, and numerical analysis.

DISCUSSION

Model fitting is an important subject that is often overlooked in modeling text books. If the idea is addressed, it is only through the least-squares criterion as the sole method for fitting models. Motivated by our desire to appeal as much as possible to the student's geometric intuition, we begin with fitting models visually, via a graph. Two criteria for fitting models that appeal graphically are:

1. Minimizing the largest absolute deviation, and
2. Minimizing the sum of the absolute deviations.

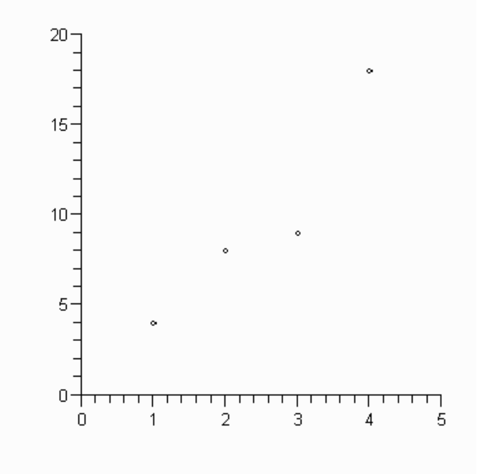
Because these two criteria cannot be applied using only the methods of calculus, we leave the presentation of their solution techniques to discuss later in the course. We also present the idea of transformed least-squares and demonstrate how a transformation may distort the distance concept. Depending on the background of your students and objectives of your course, you can downplay the first two criteria (having students only formulate a problem or two) and concentrate on applying the least-squares criterion. Or, if you have more advanced students, you may want them actually to solve problems using the first two criteria as well, including solving a linear program geometrically resulting from a Chebyshev fit (see Chapter 7 Discussion). Statistical indicators of goodness of fit, such as the coefficient of determination, are purposely avoided because it is our experience that students tend to apply such measures blindly. Instead, we have the student consider the various curve-fitting criteria, plot the residuals, and then consider each model on its own merit.

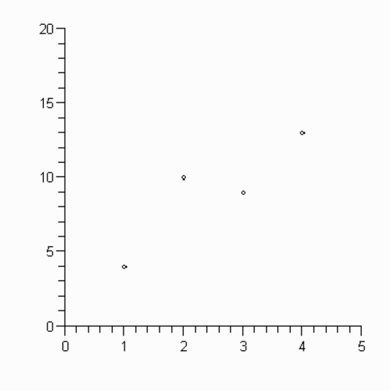
You will find the presentation in this chapter to be more like that of a standard mathematics text. Typically we assign more problems to our students than we do in the previous chapters. We begin with problems with small data sets and progress to larger ones where students can take real advantage of using a calculator or computer. (At this point it can be very beneficial to incorporate into the course a workshop on using various software that might be available for curve-fitting.) We do not have our students solve the linear programs resulting from applying the Chebyshev criterion in Section 3.2 so we move very quickly through that material and simply formulate the model.

SECTION BY SECTION ANALYSIS

Fitting Models to Data Graphically:

The purpose of this section is to have the students gain an intuitive feel for the process of fitting models to data before proceeding with analytic techniques. We begin by plotting some scatterplots on the board that represent approximately proportionality relationships, like these graphs:





We then ask the students to fit straight lines to these graphs and ask them how they did it. Most of them say they attempted to minimize the sum of deviations or the largest deviation (or words to that effect). Since they cannot visualize the sum of the squared deviations geometrically, no student ordinarily offers this criterion though some may be aware of it. Our goal is to arrive at the least-squares criterion in a logical way as an alternative to the criteria the students have verbalized, because applying the least-squares criterion requires only calculus.

The most difficult concept for students in this section is that of the transformation. They may have difficulties algebraically making the appropriate transformation. More importantly, the concept of distorting distance via no-orthogonal transformations is difficult for many students. In class you may want to emphasize the presentation in the text to ensure that students understand that the parameters determined in the transformed space are only approximations (perhaps bad ones) to their counterparts in the original space. In subsequent sections, we develop some measures of goodness of fit. It is important for students to realize that if measures are made to compare various models on the same data set, the measures must be made in the same space.