Instructor's Resource Manual

Differential Equations with Boundary Value Problems

EIGHTH EDITION

and

A First Course in Differential Equations

TENTH EDITION

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9. Writing the boundary-value problem in the form $x(dy/dx)+y^2=1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2-1)(dx/dy)+x=0$, we see that it is linear in x.

10. Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in u.

11. From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

12. From $y=\frac{6}{5}-\frac{6}{5}e^{-20t}$ we obtain $dy/dt=24e^{-20t}$, so that

$$\frac{dy}{dt}+20y=24e^{-20t}+20\left(\frac{6}{5}-\frac{6}{5}e^{-20t}\right)=24.$$

13. From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

14. From $y=-\cos x\ln(\sec x+\tan x)$ we obtain $y'=-1+\sin x\ln(\sec x+\tan x)$ and

$y''=\tan x+\cos x\ln(\sec x+\tan x)$. Then $y''+y=\tan x$.

15. The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1 + 2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}] \\ &= y - x + 2(y-x)(x+2)^{-1/2} \\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2} \\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

16. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

17. The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ or } x \neq 2\}$. From $y' = 2x/(4-x^2)^2$ we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

18. The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x)$ we have

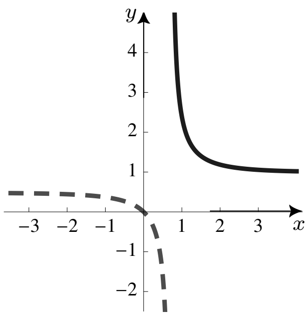
$$2y' = (1 - \sin x)^{-3/2}\cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another interval is $(5\pi/2, 9\pi/2)$ and so on.

19. Writing $\ln(2X-1) - \ln(X-1) = t$ and differentiating implicitly we obtain

$$\begin{aligned}\frac{2}{2X-1}\frac{dX}{dt} - \frac{1}{X-1}\frac{dX}{dt} &= 1 \\ \left(\frac{2}{2X-1} - \frac{1}{X-1}\right)\frac{dX}{dt} &= 1 \\ \frac{2X - 2 - 2X + 1}{(2X-1)(X-1)}\frac{dX}{dt} &= 1 \\ \frac{dX}{dt} &= -(2X-1)(X-1) = (X-1)(1-2X).\end{aligned}$$

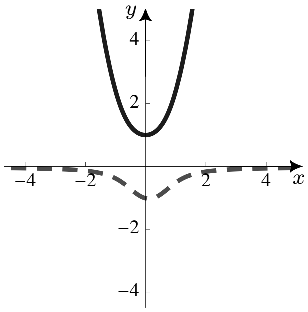
Exponentiating both sides of the implicit solution we obtain



$$\begin{aligned}\frac{2X-1}{X-1}&=e^t\\ 2X-1&=Xe^t-e^t\\ e^t-1&=(e^t-2)X\\ X&=\frac{e^t-1}{e^t-2}\,.\end{aligned}$$

Solving $e^t-2=0$ we get $t=\ln 2$. Thus, the solution is defined on $(-\infty,\ln 2)$ or on $(\ln 2,\infty)$. The graph of the solution defined on $(-\infty,\ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2,\infty)$ is solid.

20. Implicitly differentiating the solution, we obtain



$$\begin{aligned}-2x^2\,\frac{dy}{dx}-4xy+2y\,\frac{dy}{dx}&=0\\ -x^2\,dy-2xy\,dx+y\,dy&=0\\ 2xy\,dx+(x^2-y)dy&=0.\end{aligned}$$

Using the quadratic formula to solve $y^2-2x^2y-1=0$ for y, we get $y=\left(2x^2\pm\sqrt{4x^4+4}\right)/2=x^2\pm\sqrt{x^4+1}$. Thus, two explicit solutions are $y_1=x^2+\sqrt{x^4+1}$ and $y_2=x^2-\sqrt{x^4+1}$. Both solutions are defined on $(-\infty,\infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

21. Differentiating $P=c_1e^t/\left(1+c_1e^t\right)$ we obtain

$$\begin{aligned}\frac{dP}{dt}&=\frac{\left(1+c_1e^t\right)c_1e^t-c_1e^t\cdot c_1e^t}{\left(1+c_1e^t\right)^2}=\frac{c_1e^t}{1+c_1e^t}\,\frac{\left[\left(1+c_1e^t\right)-c_1e^t\right]}{1+c_1e^t}\\ &=\frac{c_1e^t}{1+c_1e^t}\left[1-\frac{c_1e^t}{1+c_1e^t}\right]=P(1-P).\end{aligned}$$

22. Differentiating $y=e^{-x^2}\displaystyle\int_0^x e^{t^2}dt+c_1e^{-x^2}$ we obtain

$$y'=e^{-x^2}e^{x^2}-2xe^{-x^2}\int_0^x e^{t^2}dt-2c_1xe^{-x^2}=1-2xe^{-x^2}\int_0^x e^{t^2}dt-2c_1xe^{-x^2}.$$

Substituting into the differential equation, we have

$$y'+2xy=1-2xe^{-x^2}\int_0^x e^{t^2}dt-2c_1xe^{-x^2}+2xe^{-x^2}\int_0^x e^{t^2}dt+2c_1xe^{-x^2}=1.$$

23. From $y=c_1e^{2x}+c_2xe^{2x}$ we obtain $\dfrac{dy}{dx}=(2c_1+c_2)e^{2x}+2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2}=(4c_1+4c_2)e^{2x}+4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=(4c_1+4c_2-8c_1-4c_2+4c_1)e^{2x}+(4c_2-8c_2+4c_2)xe^{2x}=0.$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$
$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\qquad + (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2 \\ &= 12x^2. \end{aligned}$$

25. From $y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ we obtain $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$ so that $xy' - 2y = 0$.

26. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} y(x) = 5$ and $\lim_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

27. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $y' + 2y = 0$ implies

$$me^{mx} + 2e^{mx} = (m + 2)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = -2$. Thus $y = e^{-2x}$ is a solution.

28. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $5y' = 2y$ implies

$$5me^{mx} = 2e^{mx} \quad \text{or} \quad m = \frac{2}{5}.$$

Thus $y = e^{2x/5} > 0$ is a solution.

29. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $y'' - 5y' + 6y = 0$ implies

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = (m - 2)(m - 3)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = 2$ and $m = 3$. Thus $y = e^{2x}$ and $y = e^{3x}$ are solutions.

30. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $2y'' + 7y' - 4y = 0$ implies

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = (2m - 1)(m + 4)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = \frac{1}{2}$ and $m = -4$. Thus $y = e^{x/2}$ and $y = e^{-4x}$ are solutions.

31. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $xy'' + 2y' = 0$ implies

$$\begin{aligned} xm(m-1)x^{m-2} + 2mx^{m-1} &= [m(m-1)+2m]x^{m-1} = (m^2+m)x^{m-1} \\ &= m(m+1)x^{m-1} = 0. \end{aligned}$$

Since $x^{m-1} > 0$ for $x > 0$, $m = 0$ and $m = -1$. Thus $y = 1$ and $y = x^{-1}$ are solutions.

32. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $x^2y'' - 7xy' + 15y = 0$ implies

$$\begin{aligned} x^2m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m &= [m(m-1) - 7m + 15]x^m \\ &= (m^2 - 8m + 15)x^m = (m-3)(m-5)x^m = 0. \end{aligned}$$

Since $x^m > 0$ for $x > 0$, $m = 3$ and $m = 5$. Thus $y = x^3$ and $y = x^5$ are solutions.

In Problems 33–36 we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$.

33. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

34. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

35. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

36. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

37. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}\,.$$

38. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}\,.$$

Discussion Problems

39. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all functions $y = \phi(x)$.

40. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since, if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

41. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is $f'(x) = ke^{kx}$. The differential equations are $y' = y$ and $y' = ky$, respectively.

42. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c\sin x$ or $y = c\cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

43. We first note that $\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x\big|_{x=\pi} = \cos\pi = -1,$$

but

$$\left.\sqrt{1-y^2}\,\right|_{x=\pi} = \sqrt{1-\sin^2\pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1-y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

44. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t. \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

45. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

46. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

47. Differentiating $(x^3 + y^3)/xy = 3c$ we obtain

$$\frac{xy(3x^2 + 3y^2y') - (x^3 + y^3)(xy' + y)}{x^2y^2} = 0$$
$$3x^3y + 3xy^3y' - x^4y' - x^3y - xy^3y' - y^4 = 0$$
$$(3xy^3 - x^4 - xy^3)y' = -3x^3y + x^3y + y^4$$
$$y' = \frac{y^4 - 2x^3y}{2xy^3 - x^4} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

48. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3 - x^3) = 0$. This gives $x = 0$ and $2y^3 = x^3$. Substituting $y^3 = x^3/2$ into $x^3 + y^3 = 3xy$ we get

$$x^3 + \frac{1}{2}x^3 = 3x\left(\frac{1}{2^{1/3}}\,x\right)$$
$$\frac{3}{2}x^3 = \frac{3}{2^{1/3}}x^2$$
$$x^3 = 2^{2/3}x^2$$
$$x^2(x - 2^{2/3}) = 0.$$

Thus, there are vertical tangent lines at $x = 0$ and $x = 2^{2/3}$, or at $(0,\,0)$ and $(2^{2/3},\,2^{1/3})$. Since $2^{2/3} \approx 1.59$, the estimates of the domains in Problem 46 were close.

49. The derivatives of the functions are $\phi_1'(x) = -x/\sqrt{25 - x^2}$ and $\phi_2'(x) = x/\sqrt{25 - x^2}$, neither of which is defined at $x = \pm 5$.

50. To determine if a solution curve passes through $(0, 3)$ we let $t = 0$ and $P = 3$ in the equation $P = c_1e^t/(1 + c_1e^t)$. This gives $3 = c_1/(1 + c_1)$ or $c_1 = -\frac{3}{2}$. Thus, the solution curve

$$P = \frac{(-3/2)e^t}{1 - (3/2)e^t} = \frac{-3e^t}{2 - 3e^t}$$

passes through the point $(0, 3)$. Similarly, letting $t = 0$ and $P = 1$ in the equation for the one-parameter family of solutions gives $1 = c_1/(1 + c_1)$ or $c_1 = 1 + c_1$. Since this equation has no solution, no solution curve passes through $(0, 1)$.

51. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y = \int(\int f(x)dx)dx + c_1x + c_2$.

52. Solving for y' using the quadratic formula we obtain the two differential equations

$$y' = \frac{1}{x}\left(2 + 2\sqrt{1 + 3x^6}\right) \quad \text{and} \quad y' = \frac{1}{x}\left(2 - 2\sqrt{1 + 3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx = f(x, y)$.

53. The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

54. Differentiating $y = c_1x + c_2x^2$ we get $y' = c_1 + 2c_2x$ and $y'' = 2c_2$. Then $c_2 = \frac{1}{2}y''$ and $c_1 = y' - xy''$, so

$$y = c_1x + c_2x^2 = (y' - xy'')x + \frac{1}{2}y''x^2 = xy' - \frac{1}{2}x^2y''.$$

The differential equation is $\frac{1}{2}x^2y'' - xy' + y = 0$ or $x^2y'' - 2xy' + 2y = 0$.

55. **(a)** Since e^{-x^2} is positive for all values of x, $dy/dx > 0$ for all x, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

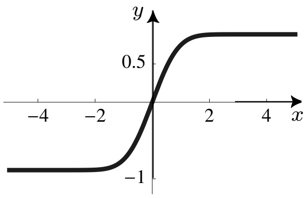
(b) $\lim_{x\to-\infty} \frac{dy}{dx} = \lim_{x\to-\infty} e^{-x^2} = 0$ and $\lim_{x\to\infty} \frac{dy}{dx} = \lim_{x\to\infty} e^{-x^2} = 0$. Since $\frac{dy}{dx}$ approaches 0 as x approaches $-\infty$ and ∞, the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. x

(d)



56. **(a)** The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

57. **(a)** The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

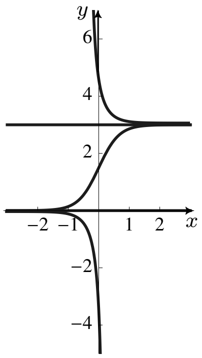
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Solving $d^2y/dx^2 = 0$ we obtain $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

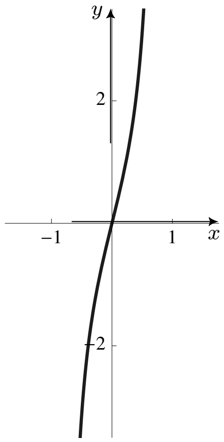


58. **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c.

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2+4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



Computer Lab Assignments

59. In *Mathematica* use

```
Clear[y]
y[x_]:= x Exp[5x] Cos[2x]
y[x]
y''''[x] - 20 y'''[x] + 158 y''[x] - 580 y'[x] + 841 y[x] // Simplify
```

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

60. In *Mathematica* use

```
Clear[y]
y[x_]:= 20 Cos[5 Log[x]]/x - 3 Sin[5 Log[x]]/x
y[x]
x^3 y'''[x] + 2x^2 y''[x] + 20x y'[x] - 78y[x] // Simplify
```

The output will show $y(x) = \dfrac{20\cos(5\ln x)}{x} - \dfrac{3\sin(5\ln x)}{x}$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

1. Solving $-1/3 = 1/(1+c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1-4e^{-x})$.

2. Solving $2 = 1/(1+c_1 e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2-e^{-(x+1)})$.

3. Letting $x = 2$ and solving $1/3 = 1/(4+c)$ we get $c = -1$. The solution is $y = 1/(x^2-1)$. This solution is defined on the interval $(1, \infty)$.

4. Letting $x = -2$ and solving $1/2 = 1/(4+c)$ we get $c = -2$. The solution is $y = 1/(x^2-2)$. This solution is defined on the interval $(-\infty, -\sqrt{2})$.

5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2+1)$. This solution is defined on the interval $(-\infty, \infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4+c)$ we get $c = -1/2$. The solution is $y = 1/(x^2-1/2) = 2/(2x^2-1)$. This solution is defined on the interval $(-1/\sqrt{2}, 1/\sqrt{2})$.

In Problems 7–10 we use $x = c_1\cos t + c_2\sin t$ and $x' = -c_1\sin t + c_2\cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1$$
$$c_2 = 8.$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0$$
$$-c_1 = 1.$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}\,c_1 + \frac{1}{2}\,c_2 = \frac{1}{2}$$
$$-\frac{1}{2}\,c_1 + \frac{\sqrt{3}}{2}\,c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

10. From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = \sqrt{2}$$
$$-\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = 2\sqrt{2}\,.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is

$$x = -\cos t + 3\sin t.$$

In Problems 11–14 we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is

$$y = \frac{3}{2}\,e^x - \frac{1}{2}\,e^{-x}.$$

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}\,e^x - \frac{1}{2}\,e^2e^{-x} = \frac{1}{2}\,e^x - \frac{1}{2}\,e^{2-x}.$$

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$
$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is

$$y = 5e^{-1}e^{-x} = 5e^{-1-x}.$$

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. A lso, any constant multiple of x^2 is a solution.

17. For $f(x, y) = y^{2/3}$ we have Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x, y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ or where $x < 0$.

20. For $f(x, y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x, y) = \dfrac{x^2}{1 + y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28 we identify $f(x,y)=\sqrt{y^2-9}$ *and* $\partial f/\partial y = y/\sqrt{y^2-9}$. *We see that* f *and* $\partial f/\partial y$ *are both continuous in the regions of the plane determined by* $y<-3$ *and* $y>3$ *with no restrictions on* x.

25. Since $4>3$, $(1,4)$ is in the region defined by $y>3$ and the differential equation has a unique solution through $(1,4)$.

26. Since $(5,3)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(5,3)$.

27. Since $(2,-3)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(2,-3)$.

28. Since $(-1,1)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(-1,1)$.

29. (a) A one-parameter family of solutions is $y=cx$. Since $y'=c$, $xy'=xc=y$ and $y(0)=c\cdot 0=0$.

(b) Writing the equation in the form $y'=y/x$, we see that R cannot contain any point on the y-axis. Thus, any rectangular region disjoint from the y-axis and containing (x_0,y_0) will determine an interval around x_0 and a unique solution through (x_0,y_0). Since $x_0=0$ in part (a), we are not guaranteed a unique solution through $(0,0)$.

(c) The piecewise-defined function which satisfies $y(0)=0$ is not a solution since it is not differentiable at $x=0$.

30. (a) Since $\dfrac{d}{dx}\tan(x+c)=\sec^2(x+c)=1+\tan^2(x+c)$, we see that $y=\tan(x+c)$ satisfies the differential equation.

(b) Solving $y(0)=\tan c=0$ we obtain $c=0$ and $y=\tan x$. Since $\tan x$ is discontinuous at $x=\pm\pi/2$, the solution is not defined on $(-2,2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2,\pi/2)$.

31. (a) Since $\dfrac{d}{dx}\left(-\dfrac{1}{x+c}\right)=\dfrac{1}{(x+c)^2}=y^2$, we see that $y=-\dfrac{1}{x+c}$ is a solution of the differential equation.

(b) Solving $y(0)=-1/c=1$ we obtain $c=-1$ and $y=1/(1-x)$. Solving $y(0)=-1/c=-1$ we obtain $c=1$ and $y=-1/(1+x)$. Being sure to include $x=0$, we see that the interval of existence of $y=1/(1-x)$ is $(-\infty,1)$, while the interval of existence of $y=-1/(1+x)$ is $(-1,\infty)$.

(c) By inspection we see that $y=0$ is a solution on $(-\infty,\infty)$.

32. **(a)** Applying $y(1) = 1$ to $y = -1/(x+c)$ gives

$$1 = -\frac{1}{1+c} \quad \text{or} \quad 1 + c = -1.$$

Thus $c = -2$ and

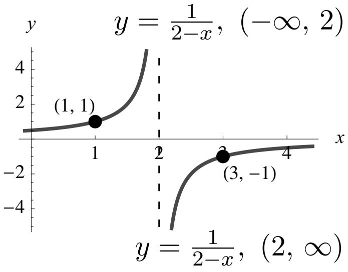
$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

(b) Applying $y(3) = -1$ to $y = -1/(x+c)$ gives

$$-1 = -\frac{1}{3+c} \quad \text{or} \quad 3 + c = 1.$$

Thus $c = -2$ and

$$y = -\frac{1}{x-2} = \frac{1}{2-x}.$$

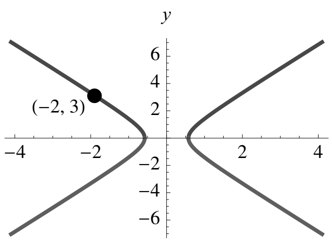


(c) No, they are not the same solution. The interval I of definition for the solution in part (a) is $(-\infty, 2)$; whereas the interval I of definition for the solution in part (b) is $(2, \infty)$. See the figure.

33. **(a)** Differentiating $3x^2 - y^2 = c$ we get $6x - 2yy' = 0$ or $yy' = 3x$.

(b) Solving $3x^2 - y^2 = 3$ for y we get

$$\begin{aligned} y &= \phi_1(x) = \sqrt{3(x^2-1)}, & 1 < x < \infty, \\ y &= \phi_2(x) = -\sqrt{3(x^2-1)}, & 1 < x < \infty, \\ y &= \phi_3(x) = \sqrt{3(x^2-1)}, & -\infty < x < -1, \\ y &= \phi_4(x) = -\sqrt{3(x^2-1)}, & -\infty < x < -1. \end{aligned}$$

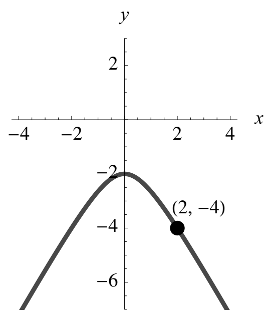


(c) Only $y = \phi_3(x)$ satisfies $y(-2) = 3$.

34. **(a)** Setting $x = 2$ and $y = -4$ in $3x^2 - y^2 = c$ we get $12 - 16 = -4 = c$, so the explicit solution is

$$y = -\sqrt{3x^2 + 4}, \quad -\infty < x < \infty.$$

(b) Setting $c = 0$ we have $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, both defined on $(-\infty, \infty)$ and both passing through the origin.



In Problems 35–38 *we consider the points on the graphs with* x*-coordinates* $x_0 = -1$, $x_0 = 0$, *and* $x_0 = 1$. *The slopes of the tangent lines at these points are compared with the slopes given by* $y'(x_0)$ *in* (a) *through* (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

In Problems 39-44 $y = c_1 \cos 2x + c_2 \sin 2x$ *is a two parameter family of solutions of the second-order differential equation* $y'' + 4y = 0$. *In some of the problems we will use the fact that* $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$.

39. From the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{4}\right) = 3$ we obtain

$$y(0) = c_1 = 0$$

$$y\left(\frac{\pi}{4}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = c_2 = 3.$$

Thus, $c_1 = 0$, $c_2 = 3$, and the solution of the boundary-value problem is $y = 3\sin 2x$.

40. From the boundary conditions $y(0) = 0$ and $y(\pi) = 0$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 0.$$

Thus, $c_1 = 0$, c_2 is unrestricted, and the solution of the boundary-value problem is $y = c_2 \sin 2x$, where c_2 is any real number.

41. From the boundary conditions $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 0$ we obtain

$$y'(0) = 2c_2 = 0$$

$$y'\left(\frac{\pi}{6}\right) = -2c_1 \sin\left(\frac{\pi}{3}\right) = -\sqrt{3}\, c_1 = 0.$$

Thus, $c_2 = 0$, $c_1 = 0$, and the solution of the boundary-value problem is $y = 0$.

42. From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \frac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \frac{5}{2}\sin 2x$.

43. From the boundary conditions $y(0) = 0$ and $y(\pi) = 2$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 2.$$

Since $0 \neq 2$, this is not possible and there is no solution.

44. From the boundary conditions $y' = \left(\frac{\pi}{2}\right) = 1$ and $y'(\pi) = 0$ we obtain

$$y'\left(\frac{\pi}{2}\right) = -2c_2 = 1$$

$$y'(\pi) = 2c_2 = 0.$$

Since $0 \neq -1$, this is not possible and there is no solution.

Discussion Problems

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

46. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1)dx = 2x^3 - x^2 + c_1 x + c_2.$$

At $x = 1$ the y-coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \quad \text{or} \quad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \quad \text{or} \quad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

47. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the red curve.

48. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

49. The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

50. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \geq 0 \end{cases} = x \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \geq 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

51. At $t = 0$, $dP/dt = 0.15P(0) + 20 = 0.15(100) + 20 = 35$. Thus, the population is increasing at a rate of 3,500 individuals per year. If the population is 500 at time $t = T$ then

$$\left.\frac{dP}{dt}\right|_{t=T} = 0.15P(T) + 20 = 0.15(500) + 20 = 95.$$

Thus, at this time, the population is increasing at a rate of 9,500 individuals per year.

1.3 Differential Equations as Mathematical Models

Population Dynamics

1. $\dfrac{dP}{dt} = kP + r; \qquad \dfrac{dP}{dt} = kP - r$

2. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P$.

3. Let b be the rate of births and d the rate of deaths. Then $b = k_1P$ and $d = k_2P^2$. Since $dP/dt = b - d$, the differential equation is $dP/dt = k_1P - k_2P^2$.

4. $\dfrac{dP}{dt} = k_1P - k_2P^2 - h, \; h > 0$

Newton's Law of cooling/Warming

5. From the graph in the text we estimate $T_0 = 180°$ and $T_m = 75°$. We observe that when $T = 85$, $dT/dt \approx -1$. From the differential equation we then have

$$k = \frac{dT/dt}{T - T_m} = \frac{-1}{85 - 75} = -0.1.$$

6. By inspecting the graph in the text we take T_m to be $T_m(t) = 80 - 30\cos \pi t/12$. Then the temperature of the body at time t is determined by the differential equation

$$\frac{dT}{dt} = k\left[T - \left(80 - 30\cos\frac{\pi}{12}t\right)\right], \quad t > 0.$$

Spread of a Disease/Technology

7. The number of students with the flu is x and the number not infected is $1000 - x$, so $dx/dt = kx(1000 - x)$.

8. By analogy, with the differential equation modeling the spread of a disease, we assume that the rate at which the technological innovation is adopted is proportional to the number of people who have adopted the innovation and also to the number of people, $y(t)$, who have not yet adopted it. Then $x + y = n$, and assuming that initially one person has adopted the innovation, we have

$$\frac{dx}{dt} = kx(n - x), \quad x(0) = 1.$$

Mixtures

9. The rate at which salt is leaving the tank is

$$R_{out}\ (3 \text{ gal/min})\left(\frac{A}{300} \text{ lb/gal}\right) = \frac{A}{100} \text{ lb/min}.$$

Thus $dA/dt = -A/100$ (where the minus sign is used since the amount of salt is decreasing). The initial amount is $A(0) = 50$.

10. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min}) \cdot (2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the solution is pumped out at a slower rate, it is accumulating at the rate of $(3-2)$gal/min $= 1$ gal/min. After t minutes there are $300+t$ gallons of brine in the tank. The rate at which salt is leaving is

$$R_{out} = (2 \text{ gal/min}) \cdot \left(\frac{A}{300+t} \text{ lb/gal}\right) = \frac{2A}{300+t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{2A}{300+t}.$$

11. The rate at which salt is entering the tank is

$$R_{in} = (3 \text{ gal/min})(2 \text{ lb/gal}) = 6 \text{ lb/min}.$$

Since the tank loses liquid at the net rate of

$$3 \text{ gal/min} - 3.5 \text{ gal/min} = -0.5 \text{ gal/min},$$

after t minutes the number of gallons of brine in the tank is $300 - \frac{1}{2}t$ gallons. Thus the rate at which salt is leaving is

$$R_{out} = \left(\frac{A}{300 - t/2} \text{ lb/gal}\right)(3.5 \text{ gal/min}) = \frac{3.5A}{300 - t/2} \text{ lb/min} = \frac{7A}{600-t} \text{ lb/min}.$$

The differential equation is

$$\frac{dA}{dt} = 6 - \frac{7A}{600-t} \quad \text{or} \quad \frac{dA}{dt} + \frac{7}{600-t}A = 6.$$

12. The rate at which salt is entering the tank is

$$R_{in} = (c_{in} \text{ lb/gal})(r_{in} \text{ gal/min}) = c_{in}r_{in} \text{ lb/min}.$$

Now let $A(t)$ denote the number of pounds of salt and $N(t)$ the number of gallons of brine in the tank at time t. The concentration of salt in the tank as well as in the outflow is $c(t) = x(t)/N(t)$. But the number of gallons of brine in the tank remains steady, is increased, or is decreased depending on whether $r_{in} = r_{out}$, $r_{in} > r_{out}$, or $r_{in} < r_{out}$. In any case, the number of gallons of brine in the tank at time t is $N(t) = N_0 + (r_{in} - r_{out})t$. The output rate of salt is then

$$R_{out} = \left(\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/gal}\right)(r_{out} \text{ gal/min}) = r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \text{ lb/min}.$$

The differential equation for the amount of salt, $dA/dt = R_{in} - R_{out}$, is

$$\frac{dA}{dt} = c_{in}r_{in} - r_{out}\frac{A}{N_0 + (r_{in} - r_{out})t} \quad \text{or} \quad \frac{dA}{dt} + \frac{r_{out}}{N_0 + (r_{in} - r_{out})t}A = c_{in}r_{in}.$$

Draining a Tank

13. The volume of water in the tank at time t is $V = A_w h$. The differential equation is then

$$\frac{dh}{dt} = \frac{1}{A_w}\frac{dV}{dt} = \frac{1}{A_w}\left(-cA_h\sqrt{2gh}\right) = -\frac{cA_h}{A_w}\sqrt{2gh}\,.$$

Using $A_h = \pi\left(\dfrac{2}{12}\right)^2 = \dfrac{\pi}{36}$, $A_w = 10^2 = 100$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/36}{100}\sqrt{64h} = -\frac{c\pi}{450}\sqrt{h}\,.$$

14. The volume of water in the tank at time t is $V = \frac{1}{3}\pi r^2 h$ where r is the radius of the tank at height h. From the figure in the text we see that $r/h = 8/20$ so that $r = \frac{2}{5}h$ and $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{75}\pi h^3$. Differentiating with respect to t we have $dV/dt = \frac{4}{25}\pi h^2\, dh/dt$ or

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\frac{dV}{dt}\,.$$

From Problem 13 we have $dV/dt = -cA_h\sqrt{2gh}$ where $c = 0.6$, $A_h = \pi\left(\frac{2}{12}\right)^2$, and $g = 32$. Thus $dV/dt = -2\pi\sqrt{h}/15$ and

$$\frac{dh}{dt} = \frac{25}{4\pi h^2}\left(-\frac{2\pi\sqrt{h}}{15}\right) = -\frac{5}{6h^{3/2}}\,.$$

Series Circuits

15. Since $i = dq/dt$ and $L\,d^2q/dt^2 + R\,dq/dt = E(t)$, we obtain $L\,di/dt + Ri = E(t)$.

16. By Kirchhoff's second law we obtain $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$.

Falling Bodies and Air Resistance

17. From Newton's second law we obtain $m\dfrac{dv}{dt} = -kv^2 + mg$.

Newton's Second Law and Archimedes' Principle

18. Since the barrel in Figure 1.3.17(b) in the text is submerged an additional y feet below its equilibrium position the number of cubic feet in the additional submerged portion is the volume of the circular cylinder: $\pi\times(\text{radius})^2\times\text{height}$ or $\pi(s/2)^2 y$. Then we have from Archimedes' principle

$$\begin{aligned}\text{upward force of water on barrel} &= \text{weight of water displaced}\\ &= (62.4)\times(\text{volume of water displaced})\\ &= (62.4)\pi(s/2)^2 y = 15.6\pi s^2 y.\end{aligned}$$

It then follows from Newton's second law that

$$\frac{w}{g}\frac{d^2y}{dt^2} = -15.6\pi s^2 y \qquad \text{or} \qquad \frac{d^2y}{dt^2} + \frac{15.6\pi s^2 g}{w}\,y = 0,$$

where $g = 32$ and w is the weight of the barrel in pounds.

Newton's Second Law and Hooke's Law

19. The net force acting on the mass is

$$F = ma = m\,\frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + mg - ks.$$

Since the condition of equilibrium is $mg = ks$, the differential equation is

$$m\,\frac{d^2x}{dt^2} = -kx.$$

20. From Problem 19, without a damping force, the differential equation is $m\,d^2x/dt^2 = -kx$. With a damping force proportional to velocity, the differential equation becomes

$$m\,\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0.$$

Newton's Second Law and Rocket Motion

21. Since the positive direction is taken to be upward, and the acceleration due to gravity g is positive, (14) in Section 1.3 becomes

$$m\,\frac{dv}{dt} = -mg - kv + R.$$

This equation, however, only applies if m is constant. Since in this case m includes the variable amount of fuel we must use (17) in Exercises 1.3:

$$F = \frac{d}{dt}(mv) = m\,\frac{dv}{dt} + v\,\frac{dm}{dt}\,.$$

Thus, replacing $m\,dv/dt$ with $m\,dv/dt + v\,dm/dt$, we have

$$m\,\frac{dv}{dt} + v\,\frac{dm}{dt} = -mg - kv + R \qquad \text{or} \qquad m\,\frac{dv}{dt} + v\,\frac{dm}{dt} + kv = -mg + R.$$

22. Here we are given that the variable mass of the rocket is $m(t) = m_p + m_\nu + m_f(t)$, where m_p and m_ν are the constant masses of the payload and vehicle, respectively, and $m_f(t)$ is the variable mass of the fuel.

(a) Since

$$\frac{d}{dt}\,m(t) = \frac{d}{dt}\left(m_p + m_\nu + m_f(t)\right) = \frac{d}{dt}\,m_f(t),$$

the rates at which the mass of the rocket and the mass of the fuel change are the same.

(b) If the rocket loses fuel at a constant rate λ then we take $dm/dt = -\lambda$. We use $-\lambda$ instead of λ because the fuel is decreasing over time. We next divide the resulting differential equation in Problem 21 by m, obtaining

$$\frac{dv}{dt} + \frac{v}{m}(-\lambda) + \frac{kv}{m} = -g + \frac{R}{m} \qquad \text{or} \qquad \frac{dv}{dt} + \frac{k-\lambda}{m}v = -g + \frac{R}{m}.$$

Integrating $dm/dt = -\lambda$ with respect to t we have $m(t) = -\lambda + C$. Since $m(0) = m_0$, $C = m_0$ and $m(t) = -\lambda t + m_0$. The differential equation then may be written as

$$\frac{dv}{dt} + \frac{k-\lambda}{m_0 - \lambda t}v = -g + \frac{R}{m_0 - \lambda t}.$$

(c) We integrate $dm_f/dt = -\lambda$ to obtain $m_f(t) = -\lambda t + C$. Since $m_f(0) = C$ we have $m_f(t - \lambda t + m_f(0)$. At burnout $m_f(t_b) = -\lambda t_b + m_f(0) = 0$, so $t_b = m_f(0)/\lambda$.

Newton's Second Law and the Law of Universal Gravitation

23. From $g = k/R^2$ we find $k = gR^2$. Using $a = d^2r/dt^2$ and the fact that the positive direction is upward we get

$$\frac{d^2r}{dt^2} = -a = -\frac{k}{r^2} = -\frac{gR^2}{r^2} \qquad \text{or} \qquad \frac{d^2r}{dt^2} + \frac{gR^2}{r^2} = 0.$$

24. The gravitational force on m is $F = -kM_r m/r^2$. Since $M_r = 4\pi\delta r^3/3$ and $M = 4\pi\delta R^3/3$ we have $M_r = r^3M/R^3$ and

$$F = -k\frac{M_r m}{r^2} = -k\frac{r^3Mm/R^3}{r^2} = -k\frac{mM}{R^3}r.$$

Now from $F = ma = d^2r/dt^2$ we have

$$m\frac{d^2r}{dt^2} = -k\frac{mM}{R^3}r \quad \text{or} \quad \frac{d^2r}{dt^2} = -\frac{kM}{R^3}r.$$

Additional Mathematical Models

25. The differential equation is $\dfrac{dA}{dt} = k(M - A)$ where $k > 0$.

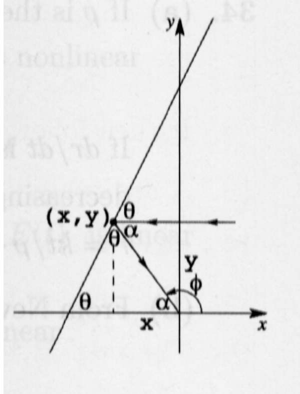
26. The differential equation is $\dfrac{dA}{dt} = k_1(M - A) - k_2A$.

27. The differential equation is $x'(t) = r - kx(t)$ where $k > 0$.

28. By the Pythagorean Theorem the slope of the tangent line is $y' = \dfrac{-y}{\sqrt{s^2 - y^2}}$.

29. We see from the figure that $2\theta + \alpha = \pi$. Thus

$$\frac{y}{-x} = \tan\alpha = \tan(\pi - 2\theta) = -\tan 2\theta = -\frac{2\tan\theta}{1 - \tan^2\theta}.$$



Since the slope of the tangent line is $y' = \tan\theta$ we have $y/x = 2y'/[1 - (y')^2]$ or $y - y(y')^2 = 2xy'$, which is the quadratic equation $y(y')^2 + 2xy' - y = 0$ in y'. Using the quadratic formula, we get

$$y' = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}.$$

Since $dy/dx > 0$, the differential equation is

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y} \qquad \text{or} \qquad y\frac{dy}{dx} - \sqrt{x^2 + y^2} + x = 0.$$

Discussion Problems

30. The differential equation is $dP/dt = kP$, so from Problem 41 in Exercises 1.1, $P = e^{kt}$, and a one-parameter family of solutions is $P = ce^{kt}$.

31. The differential equation in (3) is $dT/dt = k(T - T_m)$. When the body is cooling, $T > T_m$, so $T - T_m > 0$. Since T is decreasing, $dT/dt < 0$ and $k < 0$. When the body is warming, $T < T_m$, so $T - T_m < 0$. Since T is increasing, $dT/dt > 0$ and $k < 0$.

32. The differential equation in (8) is $dA/dt = 6 - A/100$. If $A(t)$ attains a maximum, then $dA/dt = 0$ at this time and $A = 600$. If $A(t)$ continues to increase without reaching a maximum, then $A'(t) > 0$ for $t > 0$ and A cannot exceed 600. In this case, if $A'(t)$ approaches 0 as t increases to infinity, we see that $A(t)$ approaches 600 as t increases to infinity.

33. This differential equation could describe a population that undergoes periodic fluctuations.

34. **(a)** As shown in Figure 1.3.24(b) in the text, the resultant of the reaction force of magnitude F and the weight of magnitude mg of the particle is the centripetal force of magnitude $m\omega^2 x$. The centripetal force points to the center of the circle of radius x on which the particle rotates about the y-axis. Comparing parts of similar triangles gives

$$F\cos\theta = mg \quad \text{and} \quad F\sin\theta = m\omega^2 x.$$

(b) Using the equations in part (a) we find

$$\tan\theta = \frac{F\sin\theta}{F\cos\theta} = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{\omega^2 x}{g}.$$

35. From Problem 23, $d^2r/dt^2 = -gR^2/r^2$. Since R is a constant, if $r = R + s$, then $d^2r/dt^2 = d^2s/dt^2$ and, using a Taylor series, we get

$$\frac{d^2s}{dt^2} = -g\frac{R^2}{(R+s)^2} = -gR^2(R+s)^{-2} \approx -gR^2\left[R^{-2} - 2sR^{-3} + \cdots\right] = -g + \frac{2gs}{R^3} + \cdots .$$

Thus, for R much larger than s, the differential equation is approximated by $d^2s/dt^2 = -g$.

36. (a) If ρ is the mass density of the raindrop, then $m = \rho V$ and

$$\frac{dm}{dt} = \rho\,\frac{dV}{dt} = \rho\,\frac{d}{dt}\left[\frac{4}{3}\pi r^3\right] = \rho\left(4\pi r^2\frac{dr}{dt}\right) = \rho S\,\frac{dr}{dt}\,.$$

If dr/dt is a constant, then $dm/dt = kS$ where $\rho\,dr/dt = k$ or $dr/dt = k/\rho$. Since the radius is decreasing, $k < 0$. Solving $dr/dt = k/\rho$ we get $r = (k/\rho)t + c_0$. Since $r(0) = r_0$, $c_0 = r_0$ and $r = kt/\rho + r_0$.

(b) From Newton's second law, $\dfrac{d}{dt}[mv] = mg$, where v is the velocity of the raindrop. Then

$$m\,\frac{dv}{dt} + v\,\frac{dm}{dt} = mg \qquad \text{or} \qquad \rho\left(\frac{4}{3}\pi r^3\right)\frac{dv}{dt} + v(k4\pi r^2) = \rho\left(\frac{4}{3}\pi r^3\right)g.$$

Dividing by $4\rho\pi r^3/3$ we get

$$\frac{dv}{dt} + \frac{3k}{\rho r}v = g \qquad \text{or} \qquad \frac{dv}{dt} + \frac{3k/\rho}{kt/\rho + r_0}v = g, \quad k < 0.$$

37. We assume that the plow clears snow at a constant rate of k cubic miles per hour. Let t be the time in hours after noon, $x(t)$ the depth in miles of the snow at time t, and $y(t)$ the distance the plow has moved in t hours. Then dy/dt is the velocity of the plow and the assumption gives

$$wx\frac{dy}{dt} = k,$$

where w is the width of the plow. Each side of this equation simply represents the volume of snow plowed in one hour. Now let t_0 be the number of hours before noon when it started snowing and let s be the constant rate in miles per hour at which x increases. Then for $t > -t_0$, $x = s(t + t_0)$. The differential equation then becomes

$$\frac{dy}{dt} = \frac{k}{ws}\,\frac{1}{t + t_0}.$$

Integrating, we obtain

$$y = \frac{k}{ws}\,[\ln(t + t_0) + c],$$

where c is a constant. Now when $t = 0$, $y = 0$ so $c = -\ln t_0$ and

$$y = \frac{k}{ws}\ln\left(1 + \frac{t}{t_0}\right).$$

Finally, from the fact that when $t = 1$, $y = 2$ and when $t = 2$, $y = 3$, we obtain

$$\left(1 + \frac{2}{t_0}\right)^2 = \left(1 + \frac{1}{t_0}\right)^3.$$

Expanding and simplifying gives $t_0^2 + t_0 - 1 = 0$. Since $t_0 > 0$, we find $t_0 \approx 0.618$ hours ≈ 37 minutes. Thus it started snowing at about 11:23 in the morning.

38. (1): $\dfrac{dP}{dt} = kP$ is linear (2): $\dfrac{dA}{dt} = kA$ is linear

(3): $\dfrac{dT}{dt} = k(T - T_m)$ is linear (5): $\dfrac{dx}{dt} = kx(n + 1 - x)$ is nonlinear

(6): $\dfrac{dX}{dt} = k(\alpha - X)(\beta - X)$ is nonlinear (8): $\dfrac{dA}{dt} = 6 - \dfrac{A}{100}$ is linear

(10): $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$ is nonlinear (11): $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$ is linear

(12): $\dfrac{d^2s}{dt^2} = -g$ is linear (14): $m\dfrac{dv}{dt} = mg - kv$ is linear

(15): $m\dfrac{d^2s}{dt^2} + k\dfrac{ds}{dt} = mg$ is linear

(16): $\dfrac{dy}{dx} = \dfrac{W}{T_1}$ linearity or nonlinearity is determined by the manner in which W and T_1 involve x.

1.R Chapter 1 in Review

1. $\dfrac{d}{dx}c_1e^{10x} = 10c_1e^{10x}$; $\dfrac{dy}{dx} = 10y$

2. $\dfrac{d}{dx}(5 + c_1e^{-2x}) = -2c_1e^{-2x} = -2(5 + c_1e^{-2x} - 5)$; $\dfrac{dy}{dx} = -2(y - 5)$ or $\dfrac{dy}{dx} = -2y + 10$

3. $\dfrac{d}{dx}(c_1 \cos kx + c_2 \sin kx) = -kc_1 \sin kx + kc_2 \cos kx$;

$\dfrac{d^2}{dx^2}(c_1 \cos kx + c_2 \sin kx) = -k^2c_1 \cos kx - k^2c_2 \sin kx = -k^2(c_1 \cos kx + c_2 \sin kx)$;

$\dfrac{d^2y}{dx^2} = -k^2y$ or $\dfrac{d^2y}{dx^2} + k^2y = 0$

4. $\dfrac{d}{dx}(c_1 \cosh kx + c_2 \sinh kx) = kc_1 \sinh kx + kc_2 \cosh kx$;

$\dfrac{d^2}{dx^2}(c_1 \cosh kx + c_2 \sinh kx) = k^2c_1 \cosh kx + k^2c_2 \sinh kx = k^2(c_1 \cosh kx + c_2 \sinh kx)$;

$\dfrac{d^2y}{dx^2} = k^2y$ or $\dfrac{d^2y}{dx^2} - k^2y = 0$

5. $y = c_1e^x + c_2xe^x; \quad y' = c_1e^x + c_2xe^x + c_2e^x; \quad y'' = c_1e^x + c_2xe^x + 2c_2e^x;$

$y'' + y = 2(c_1e^x + c_2xe^x) + 2c_2e^x = 2(c_1e^x + c_2xe^x + c_2e^x) = 2y'; \quad y'' - 2y' + y = 0$

6. $y' = -c_1e^x \sin x + c_1e^x \cos x + c_2e^x \cos x + c_2e^x \sin x;$

$y'' = -c_1e^x \cos x - c_1e^x \sin x - c_1e^x \sin x + c_1e^x \cos x - c_2e^x \sin x + c_2e^x \cos x + c_2e^x \cos x + c_2e^x \sin x$

$= -2c_1e^x \sin x + 2c_2e^x \cos x;$

$y'' - 2y' = -2c_1e^x \cos x - 2c_2e^x \sin x = -2y; \quad y'' - 2y' + 2y = 0$

7. a, d **(8.)** c **(9.)** b **(10.)** a, c **(11.)** b **(12.)** a, b, d

13. A few solutions are $y = 0$, $y = c$, and $y = e^x$. In general, $y = c_1 + c_2e^x$ is a solution for any constants c_1 and c_2.

14. When y is a constant, then $y' = 0$. Thus, easy solutions to see are $y = 0$ and $y = 3$.

15. The slope of the tangent line at (x, y) is y', so the differential equation is $y' = x^2 + y^2$.

16. The rate at which the slope changes is $dy'/dx = y''$, so the differential equation is $y'' = -y'$ or $y'' + y' = 0$.

17. (a) The domain is all real numbers.

(b) Since $y' = 2/3x^{1/3}$, the solution $y = x^{2/3}$ is undefined at $x = 0$. This function is a solution of the differential equation on $(-\infty, 0)$ and also on $(0, \infty)$.

18. (a) Differentiating $y^2 - 2y = x^2 - x + c$ we obtain $2yy' - 2y' = 2x - 1$ or $(2y - 2)y' = 2x - 1$.

(b) Setting $x = 0$ and $y = 1$ in the solution we have $1 - 2 = 0 - 0 + c$ or $c = -1$. Thus, a solution of the initial-value problem is $y^2 - 2y = x^2 - x - 1$.

(c) Solving $y^2 - 2y - (x^2 - x - 1) = 0$ by the quadratic formula we get

$$y = \frac{2 \pm \sqrt{4 + 4(x^2 - x - 1)}}{2} = 1 \pm \sqrt{x^2 - x} = 1 \pm \sqrt{x(x-1)}\,.$$

Since $x(x - 1) \geq 0$ for $x \leq 0$ or $x \geq 1$, we see that neither $y = 1 + \sqrt{x(x-1)}$ nor $y = 1 - \sqrt{x(x-1)}$ is differentiable at $x = 0$. Thus, both functions are solutions of the differential equation, but neither is a solution of the initial-value problem.

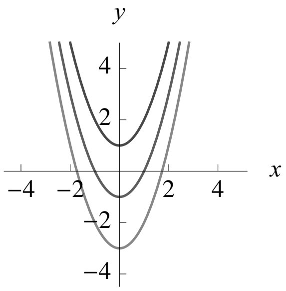
19. Setting $x = x_0$ and $y = 1$ in $y = -2/x + x$, we get

$$1 = -\frac{2}{x_0} + x_0 \qquad \text{or} \qquad x_0^2 - x_0 - 2 = (x_0 - 2)(x_0 + 1) = 0.$$

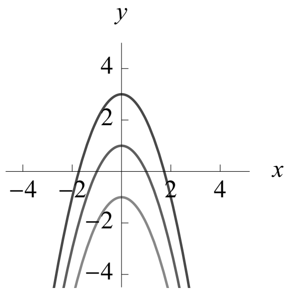
Thus, $x_0 = 2$ or $x_0 = -1$. Since $x \neq 0$ in $y = -2/x + x$, we see that $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(-1) = 1$ on the interval $(-\infty, 0)$ $(-1 < 0)$, and $y = -2/x + x$ is a solution of the initial-value problem $xy' + y = 2x$, $y(2) = 1$, on the interval $(0, \infty)$ $(2 > 0)$.

20. From the differential equation, $y'(1) = 1^2 + [y(1)]^2 = 1 + (-1)^2 = 2 > 0$, so $y(x)$ is increasing in some neighborhood of $x = 1$. From $y'' = 2x + 2yy'$ we have $y''(1) = 2(1) + 2(-1)(2) = -2 < 0$, so $y(x)$ is concave down in some neighborhood of $x = 1$.

21. **(a)**



$y = x^2 + c_1$



$y = -x^2 + c_2$

(b) When $y = x^2 + c_1$, $y' = 2x$ and $(y')^2 = 4x^2$. When $y = -x^2 + c_2$, $y' = -2x$ and $(y')^2 = 4x^2$.

(c) Pasting together x^2, $x \geq 0$, and $-x^2$, $x \leq 0$, we get

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x > 0. \end{cases}$$

22. The slope of the tangent line is $y' \Big|_{(-1,4)} = 6\sqrt{4} + 5(-1)^3 = 7$.

23. Differentiating $y = x \sin x + x \cos x$ we get

$$y' = x \cos x + \sin x - x \sin x + \cos x$$

and

$$\begin{aligned} y'' &= -x \sin x + \cos x + \cos x - x \cos x - \sin x - \sin x \\ &= -x \sin x - x \cos x + 2 \cos x - 2 \sin x. \end{aligned}$$

Thus

$$y'' + y = -x \sin x - x \cos x + 2 \cos x - 2 \sin x + x \sin x + x \cos x = 2 \cos x - 2 \sin x.$$

An interval of definition for the solution is $(-\infty, \infty)$.

24. Differentiating $y = x \sin x + (\cos x) \ln(\cos x)$ we get

$$\begin{aligned} y' &= x \cos x + \sin x + \cos x \left(\frac{-\sin x}{\cos x} \right) - (\sin x) \ln(\cos x) \\ &= x \cos x + \sin x - \sin x - (\sin x) \ln(\cos x) \\ &= x \cos x - (\sin x) \ln(\cos x) \end{aligned}$$

and,
$$\begin{aligned}
y'' &= -x\sin x + \cos x - \sin x\left(\frac{-\sin x}{\cos x}\right) - (\cos x)\ln(\cos x)\\
&= -x\sin x + \cos x + \frac{\sin^2 x}{\cos x} - (\cos x)\ln(\cos x)\\
&= -x\sin x + \cos x + \frac{1-\cos^2 x}{\cos x} - (\cos x)\ln(\cos x)\\
&= -x\sin x + \cos x + \sec x - \cos x - (\cos x)\ln(\cos x)\\
&= -x\sin x + \sec x - (\cos x)\ln(\cos x).
\end{aligned}$$

Thus

$$y'' + y = -x\sin x + \sec x - (\cos x)\ln(\cos x) + x\sin x + (\cos x)\ln(\cos x) = \sec x.$$

To obtain an interval of definition we note that the domain of $\ln x$ is $(0,\infty)$, so we must have $\cos x > 0$. Thus, an interval of definition is $(-\pi/2, \pi/2)$.

25. Differentiating $y = \sin(\ln x)$ we obtain $y' = \cos(\ln x)/x$ and $y'' = -[\sin(\ln x) + \cos(\ln x)]/x^2$. Then

$$x^2y'' + xy' + y = x^2\left(-\frac{\sin(\ln x)+\cos(\ln x)}{x^2}\right) + x\frac{\cos(\ln x)}{x} + \sin(\ln x) = 0.$$

An interval of definition for the solution is $(0,\infty)$.

26. Differentiating $y = \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)$ we obtain

$$\begin{aligned}
y' &= \cos(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right) + \ln(\cos(\ln x))\left(-\frac{\sin(\ln x)}{x}\right) + \ln x\frac{\cos(\ln x)}{x} + \frac{\sin(\ln x)}{x}\\
&= -\frac{\ln(\cos(\ln x))\sin(\ln x)}{x} + \frac{(\ln x)\cos(\ln x)}{x}
\end{aligned}$$

and

$$\begin{aligned}
y'' &= -x\left[\ln(\cos(\ln x))\frac{\cos(\ln x)}{x} + \sin(\ln x)\frac{1}{\cos(\ln x)}\left(-\frac{\sin(\ln x)}{x}\right)\right]\frac{1}{x^2}\\
&\quad + \ln(\cos(\ln x))\sin(\ln x)\frac{1}{x^2} + x\left[(\ln x)\left(-\frac{\sin(\ln x)}{x}\right) + \frac{\cos(\ln x)}{x}\right]\frac{1}{x^2} - (\ln x)\cos(\ln x)\frac{1}{x^2}\\
&= \frac{1}{x^2}\Big[-\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)\\
&\qquad - (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x)\Big].
\end{aligned}$$

Then

$$\begin{aligned}
x^2y'' + xy' + y &= -\ln(\cos(\ln x))\cos(\ln x) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \ln(\cos(\ln x))\sin(\ln x)\\
&\quad - (\ln x)\sin(\ln x) + \cos(\ln x) - (\ln x)\cos(\ln x) - \ln(\cos(\ln x))\sin(\ln x)\\
&\quad + (\ln x)\cos(\ln x) + \cos(\ln x)\ln(\cos(\ln x)) + (\ln x)\sin(\ln x)\\
&= \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) = \frac{\sin^2(\ln x)+\cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x).
\end{aligned}$$

To obtain an interval of definition, we note that the domain of $\ln x$ is $(0, \infty)$, so we must have $\cos(\ln x) > 0$. Since $\cos x > 0$ when $-\pi/2 < x < \pi/2$, we require $-\pi/2 < \ln x < \pi/2$. Since e^x is an increasing function, this is equivalent to $e^{-\pi/2} < x < e^{\pi/2}$. Thus, an interval of definition is $(e^{-\pi/2}, e^{\pi/2})$. Much of this problem is more easily done using a computer algebra system such as *Mathematica* or *Maple*.

27. Using implicit differentiation on $x^3y^3 = x^3 + 1$ we have

$$\begin{aligned} 3x^3y^2y' + 3x^2y^3 &= 3x^2 \\ xy^2y' + y^3 &= 1 \\ xy' + y &= \frac{1}{y^2}. \end{aligned}$$

28. Using implicit differentiation on $(x-5)^2 + y^2 = 1$ we have

$$\begin{aligned} 2(x-5) + 2yy' &= 0 \\ x - 5 + yy' &= 0 \\ y' &= -\frac{x-5}{y} \\ (y')^2 = \frac{(x-5)^2}{y^2} &= \frac{1-y^2}{y^2} = \frac{1}{y^2} - 1 \\ (y')^2 + 1 &= \frac{1}{y^2}. \end{aligned}$$

29. Using implicit differentiation on $y^3 + 3y = 1 - 3x$ we have

$$\begin{aligned} 3y^2y' + 3y' &= -3 \\ y^2y' + y' &= -1 \\ y' &= -\frac{1}{y^2+1}. \end{aligned}$$

Again, using implicit differentiation, we have

$$y'' = -\frac{-2yy'}{(y^2+1)^2} = 2yy'\left(\frac{1}{y^2+1}\right)^2 = 2yy'\left(-\frac{1}{y^2+1}\right)^2 = 2yy'\big(-y'\big)^2 = 2y(y')^3.$$

30. Using implicit differentiation on $y = e^{xy}$ we have

$$\begin{aligned} y' &= e^{xy}\big(xy' + y\big) \\ \big(1 - xe^xy\big)y' &= ye^{xy}. \end{aligned}$$

Since $y = e^{xy}$ we have

$$(1 - xy)y' = y \cdot y \qquad \text{or} \qquad (1 - xy)y' = y^2.$$

In Problems 31–34 we have $y' = 3c_1e^{3x} - c_2e^x - 2$.

31. The initial conditions imply

$$c_1 + c_2 = 0$$
$$3c_1 - c_2 - 2 = 0,$$

so $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. Thus $y = \frac{1}{2}e^{3x} - \frac{1}{2}e^{-x} - 2x$.

32. The initial conditions imply

$$c_1 + c_2 = 1$$
$$3c_1 - c_2 - 2 = -3,$$

so $c_1 = 0$ and $c_2 = 1$. Thus $y = e^{-x} - 2x$.

33. The initial conditions imply

$$c_1e^3 + c_2e^{-1} - 2 = 4$$
$$3c_1e^3 - c_2e^{-1} - 2 = -2,$$

so $c_1 = \frac{3}{2}e^{-3}$ and $c_2 = \frac{9}{2}e$. Thus $y = \frac{3}{2}e^{3x-3} + \frac{9}{2}e^{-x+1} - 2x$.

34. The initial conditions imply

$$c_1e^{-3} + c_2e + 2 = 0$$
$$3c_1e^{-3} - c_2e - 2 = 1,$$

so $c_1 = \frac{1}{4}e^3$ and $c_2 = -\frac{9}{4}e^{-1}$. Thus $y = \frac{1}{4}e^{3x+3} - \frac{9}{4}e^{-x-1} - 2x$.

35. From the graph we see that estimates for y_0 and y_1 are $y_0 = -3$ and $y_1 = 0$.

36. Figure 1.3.3 in the text can be used for reference in this problem. The differential equation is

$$\frac{dh}{dt} = -\frac{cA_0}{A_w}\sqrt{2gh}\,.$$

Using $A_0 = \pi(1/24)^2 = \pi/576$, $A_w = \pi(2)^2 = 4\pi$, and $g = 32$, this becomes

$$\frac{dh}{dt} = -\frac{c\pi/576}{4\pi}\sqrt{64h} = \frac{c}{288}\sqrt{h}\,.$$

37. Let $P(t)$ be the number of owls present at time t. Then $dP/dt = k(P - 200 + 10t)$.

38. Setting $A'(t) = -0.002$ and solving $A'(t) = -0.0004332A(t)$ for $A(t)$, we obtain

$$A(t) = \frac{A'(t)}{-0.0004332} = \frac{-0.002}{-0.0004332} \approx 4.6 \text{ grams.}$$

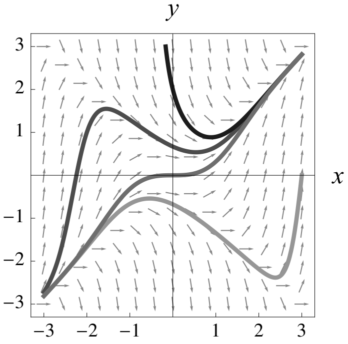
2 FIRST-ORDER DIFFERENTIAL EQUATIONS

2.1 Solution Curves Without a Solution

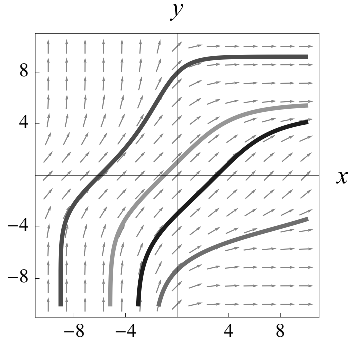
2.1.1 DIRECTION FIELDS

In Problems 1–4 the graph corresponding to the initial condition in Part (a) *is red,* Part (b) *is green,* Part (c) *is blue, and* Part (d) *is brown. The pictures are obtain using Mathematica with* **VectorPlot[{1, f[x, y]}, {x, lhs, rhs}, {y, down, up}, . . .]**.

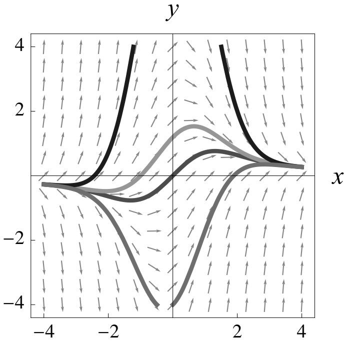
1.



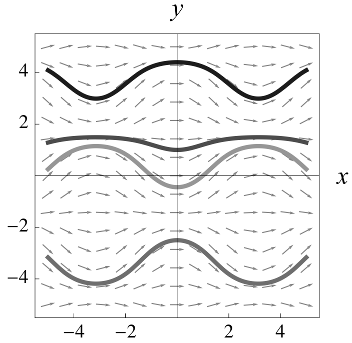
2.



3.

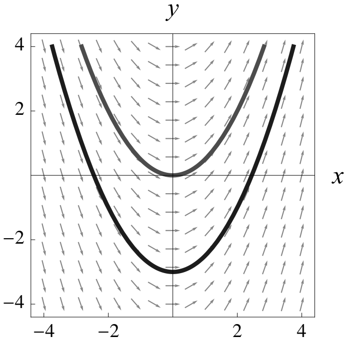


4.

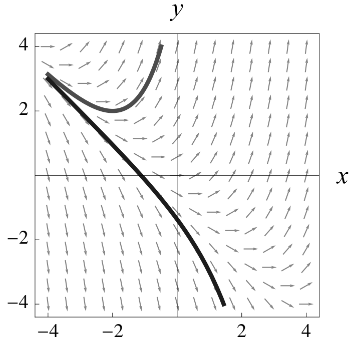


In Problems 5–12 the graph corresponding to the initial condition in Part (a) *is red, and* Part (b) *is blue. The pictures are obtain using Mathematica, as mentioned before* Problem 1.

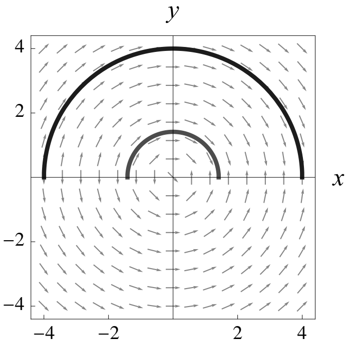
5.



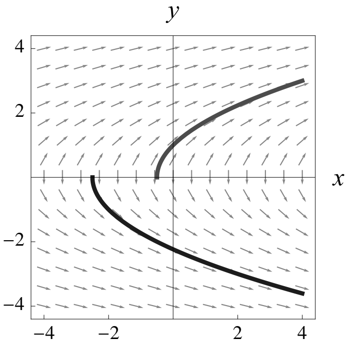
6.



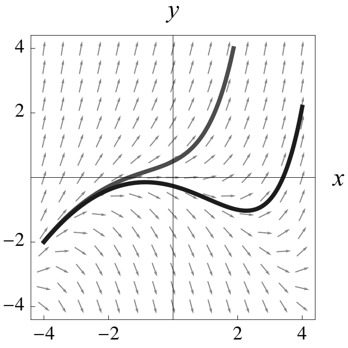
7.



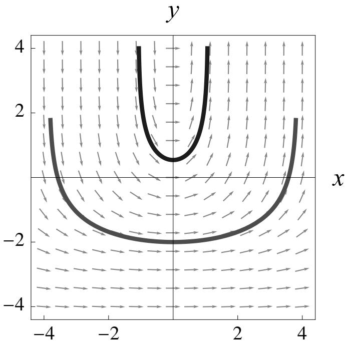
8.



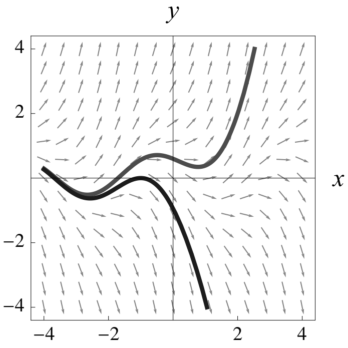
9.



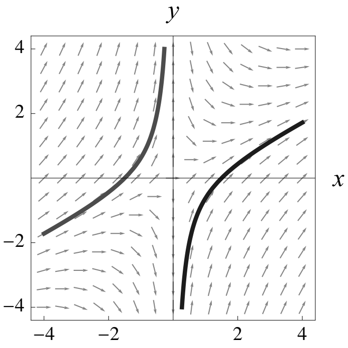
10.



11.

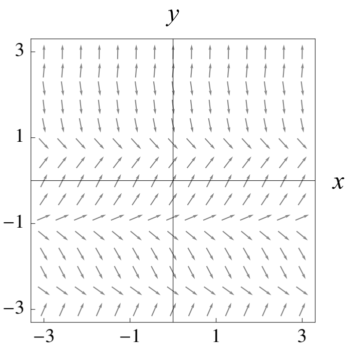


12.

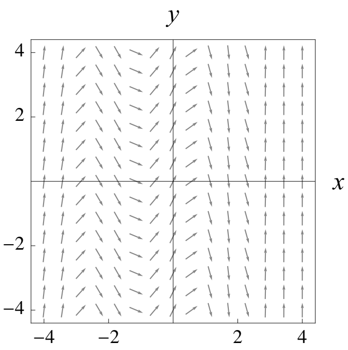


In Problems 13 *and* 14 *Mathematica was used, as mentioned before Problem* 1.

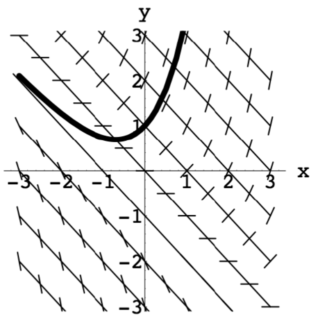
13.



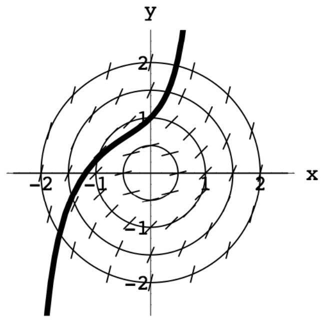
14.



15. (a) The isoclines have the form $y = -x + c$, which are straight lines with slope -1.



(b) The isoclines have the form $x^2 + y^2 = c$, which are circles centered at the origin.

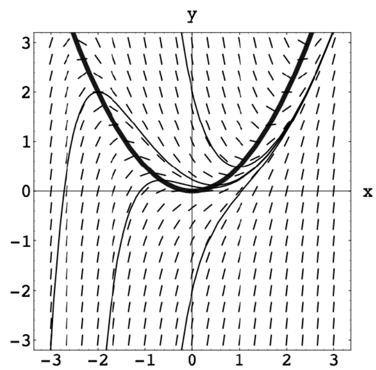


Discussion Problems

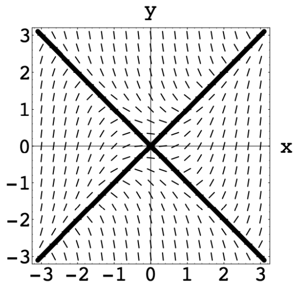
16. (a) When $x = 0$ or $y = 4$, $dy/dx = -2$ so the lineal elements have slope -2. When $y = 3$ or $y = 5$, $dy/dx = x - 2$, so the lineal elements at $(x, 3)$ and $(x, 5)$ have slopes $x - 2$.

(b) At $(0, y_0)$ the solution curve is headed down. If $y \to \infty$ as x increases, the graph must eventually turn around and head up, but while heading up it can never cross $y = 4$ where a tangent line to a solution curve must have slope -2. Thus, y cannot approach ∞ as x approaches ∞.

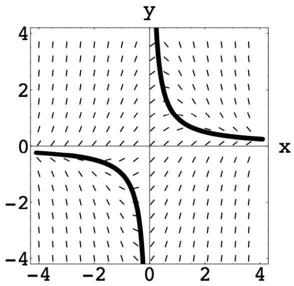
17. When $y < \frac{1}{2}x^2$, $y' = x^2 - 2y$ is positive and the portions of solution curves "outside" the nullcline parabola are increasing. When $y > \frac{1}{2}x^2$, $y' = x^2 - 2y$ is negative and the portions of the solution curves "inside" the nullcline parabola are decreasing.



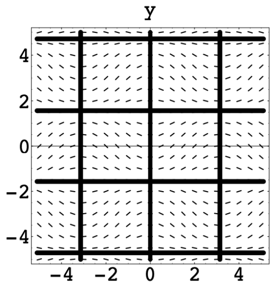
18. (a) Any horizontal lineal element should be at a point on a nullcline. In Problem 1 the nullclines are $x^2 - y^2 = 0$ or $y = \pm x$. In Problem 3 the nullclines are $1 - xy = 0$ or $y = 1/x$. In Problem 4 the nullclines are $(\sin x)\cos y = 0$ or $x = n\pi$ and $y = \pi/2 + n\pi$, where n is an integer. The graphs on the next page show the nullclines for the differential equations in Problems 1, 3, and 4 superimposed on the corresponding direction field.



Problem 1



Problem 3

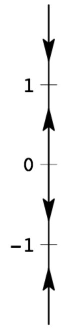


Problem 4

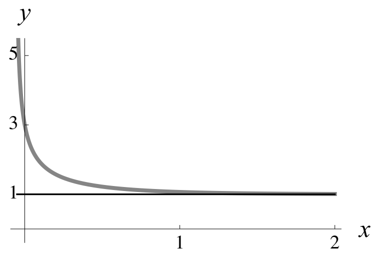
(b) An autonomous first-order differential equation has the form $y' = f(y)$. Nullclines have the form $y = c$ where $f(c) = 0$. These are the equilibrium solutions of the differential equation.

2.1.2 AUTONOMOUS FIRST-ORDER DEs

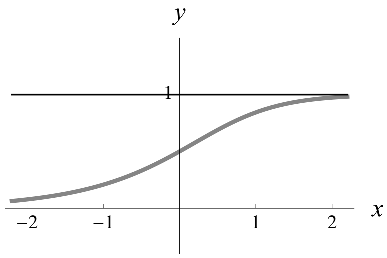
19. Writing the differential equation in the form $dy/dx = y(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right.



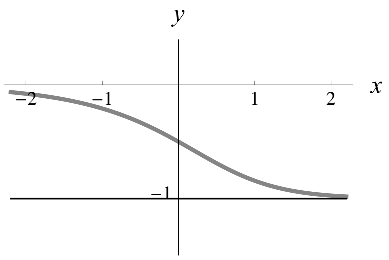
(a)



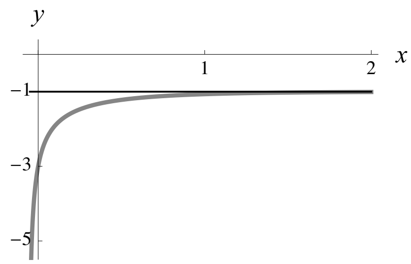
(b)



(c)

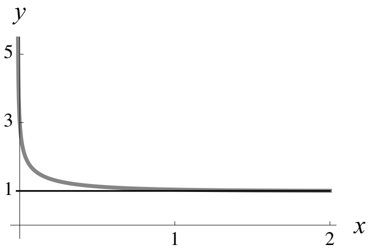


(d)

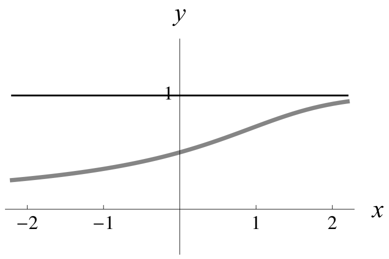


20. Writing the differential equation in the form $dy/dx = y^2(1-y)(1+y)$ we see that critical points are located at $y = -1$, $y = 0$, and $y = 1$. The phase portrait is shown at the right, and the graphs of the typical solutions are shown on the next page.

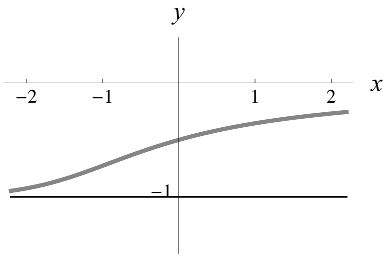
(a)



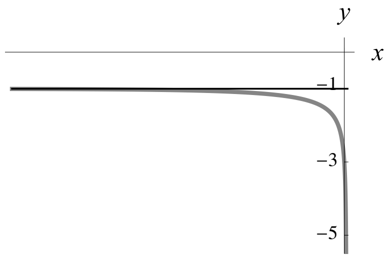
(b)



(c)

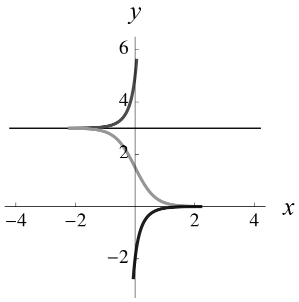


(d)

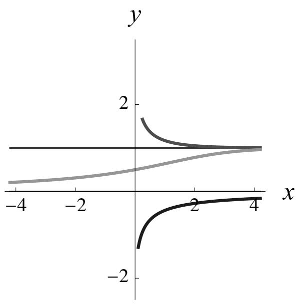


In Problems 21–28 graphs of typical solutions are shown. However, in some of the solutions, even though the upper and lower graphs either actually bend up or down, they display as straight line segments. This is a peculiarity of the Mathematica graphing routine and may be due to the fact that the **NDSolve** *function was used rather than* **DSolve**. **NDSolve** *uses a numerical routine (see Section* 2.6 *in the text), and involves sampling x-coordinates where the corresponding y-coordinates are approximated. It may be that the routine involved breaks down as the graph becomes nearly vertical, forcing the x-coordinates on the graph to becomes closer and closer together.*

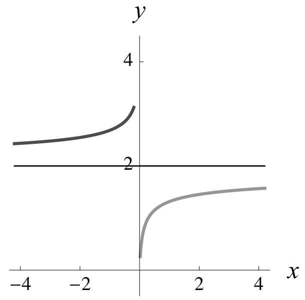
21. Solving $y^2 - 3y = y(y-3) = 0$ we obtain the critical points 0 and 3. From the phase portrait we see that 0 is asymptotically stable (attractor) and 3 is unstable (repeller).

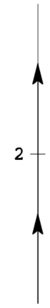


22. Solving $y^2 - y^3 = y^2(1-y) = 0$ we obtain the critical points 0 and 1. From the phase portrait we see that 1 is asymptotically stable (attractor) and 0 is semi-stable.

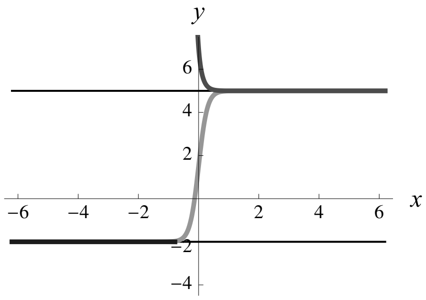


23. Solving $(y-2)^4 = 0$ we obtain the critical point 2. From the phase portrait we see that 2 is semi-stable.

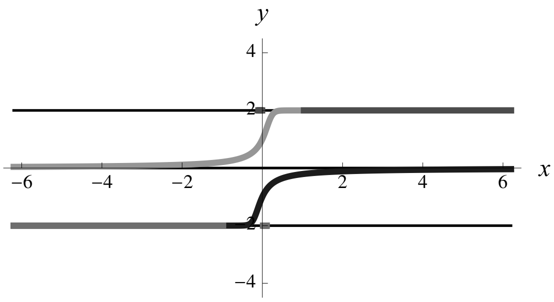


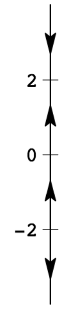


24. Solving $10 + 3y - y^2 = (5-y)(2+y) = 0$ we obtain the critical points -2 and 5. From the phase portrait we see that 5 is asymptotically stable (attractor) and -2 is unstable (repeller).

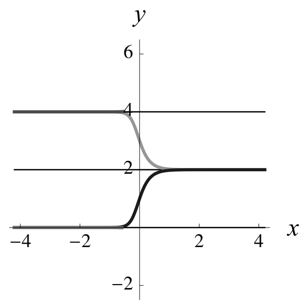


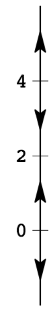
25. Solving $y^2(4-y^2) = y^2(2-y)(2+y) = 0$ we obtain the critical points -2, 0, and 2. From the phase portrait we see that 2 is asymptotically stable (attractor), 0 is semi-stable, and -2 is unstable (repeller).



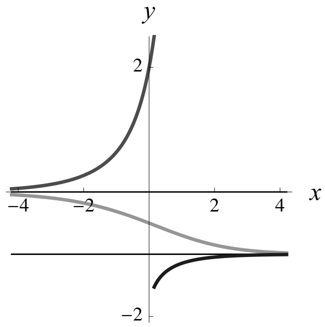


26. Solving $y(2-y)(4-y) = 0$ we obtain the critical points 0, 2, and 4. From the phase portrait we see that 2 is asymptotically stable (attractor) and 0 and 4 are unstable (repellers).

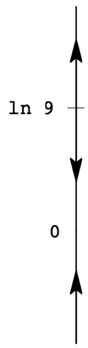


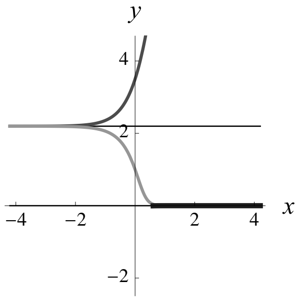


27. Solving $y\ln(y+2) = 0$ we obtain the critical points -1 and 0. From the phase portrait we see that -1 is asymptotically stable (attractor) and 0 is unstable (repeller).

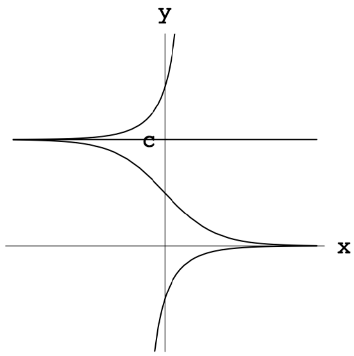


28. Solving $ye^y - 9y = y(e^y - 9) = 0$ (since e^y is always positive) we obtain the critical points 0 and $\ln 9$. From the phase portrait we see that 0 is asymptotically stable (attractor) and $\ln 9$ is unstable (repeller).

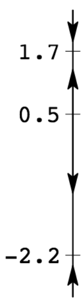


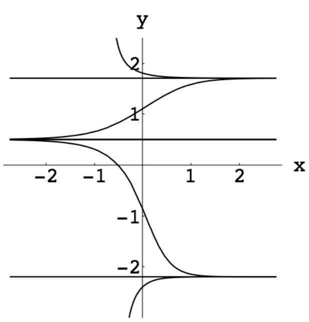


29. The critical points are 0 and c because the graph of $f(y)$ is 0 at these points. Since $f(y) > 0$ for $y < 0$ and $y > c$, the graph of the solution is increasing on $(-\infty, 0)$ and (c, ∞). Since $f(y) < 0$ for $0 < y < c$, the graph of the solution is decreasing on $(0, c)$.



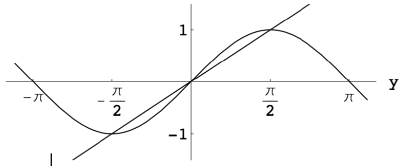
30. The critical points are approximately at -2, 2, 0.5, and 1.7. Since $f(y) > 0$ for $y < -2.2$ and $0.5 < y < 1.7$, the graph of the solution is increasing on $(-\infty, -2.2)$ and $(0.5, 1.7)$. Since $f(y) < 0$ for $-2.2 < y < 0.5$ and $y > 1.7$, the graph is decreasing on $(-2.2, 0.5)$ and $(1.7, \infty)$.





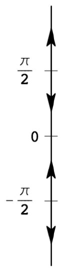
Discussion Problems

31. From the graphs of $z = (\pi/2)y$ and $z = \sin y$ we see that $(\pi/2)y - \sin y = 0$ has only three solutions. By inspection we see that the critical points are $-\pi/2$, 0, and $\pi/2$. From the graph at the right we see that



$$\frac{2}{\pi}y - \sin y \begin{cases} < 0 & \text{for} \quad y < -\pi/2 \\ > 0 & \text{for} \quad y > \pi/2 \end{cases}$$

$$\frac{2}{\pi}y - \sin y \begin{cases} > 0 & \text{for} \quad -\pi/2 < y < 0 \\ < 0 & \text{for} \quad 0 < y < \pi/2. \end{cases}$$



This enables us to construct the phase portrait shown at the right. From this portrait we see that $\pi/2$ and $-\pi/2$ are unstable (repellers), and 0 is asymptotically stable (attractor).

32. For $dy/dx = 0$ every real number is a critical point, and hence all critical points are nonisolated.

33. Recall that for $dy/dx = f(y)$ we are assuming that f and f' are continuous functions of y on some interval I. Now suppose that the graph of a nonconstant solution of the differential equation crosses the line $y = c$. If the point of intersection is taken as an initial condition we have two distinct solutions of the initial-value problem. This violates uniqueness, so the graph of any nonconstant solution must lie entirely on one side of any equilibrium solution. Since f is continuous it can only change signs at a point where it is 0. But this is a critical point. Thus, $f(y)$ is completely positive or completely negative in each region R_i. If $y(x)$ is oscillatory or has a relative extremum, then it must have a horizontal tangent line at some point (x_0, y_0). In this case y_0 would be a critical point of the differential equation, but we saw above that the graph of a nonconstant solution cannot intersect the graph of the equilibrium solution $y = y_0$.

34. By Problem 33, a solution $y(x)$ of $dy/dx = f(y)$ cannot have relative extrema and hence must be monotone. Since $y'(x) = f(y) > 0$, $y(x)$ is monotone increasing, and since $y(x)$ is bounded above by c_2, $\lim_{x\to\infty} y(x) = L$, where $L \le c_2$. We want to show that $L = c_2$. Since L is a horizontal asymptote of $y(x)$, $\lim_{x\to\infty} y'(x) = 0$. Using the fact that $f(y)$ is continuous we have

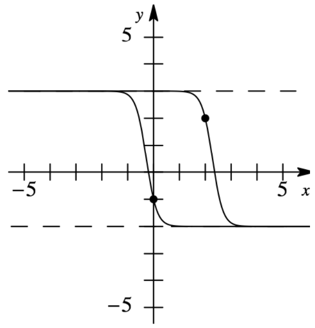
$$f(L) = f(\lim_{x\to\infty} y(x)) = \lim_{x\to\infty} f(y(x)) = \lim_{x\to\infty} y'(x) = 0.$$

But then L is a critical point of f. Since $c_1 < L \le c_2$, and f has no critical points between c_1 and c_2, $L = c_2$.

35. Assuming the existence of the second derivative, points of inflection of $y(x)$ occur where $y''(x) = 0$. From $dy/dx = f(y)$ we have $d^2y/dx^2 = f'(y)\,dy/dx$. Thus, the y-coordinate of a point of inflection can be located by solving $f'(y) = 0$. Points where $dy/dx = 0$ correspond to constant solutions of the differential equation.

36. Solving $y^2 - y - 6 = (y-3)(y+2) = 0$ we see that 3 and -2 are critical points. Now

$$d^2y/dx^2 = (2y-1)\,dy/dx = (2y-1)(y-3)(y+2),$$



so the only possible point of inflection is at $y = \frac{1}{2}$, although the concavity of solutions can be different on either side of $y = -2$ and $y = 3$. Since $y''(x) < 0$ for $y < -2$ and $\frac{1}{2} < y < 3$, and $y''(x) > 0$ for $-2 < y < \frac{1}{2}$ and $y > 3$, we see that solution curves are concave down for $y < -2$ and $\frac{1}{2} < y < 3$ and concave up for $-2 < y < \frac{1}{2}$ and $y > 3$. Points of inflection of solutions of autonomous differential equations will have the same y-coordinates because between critical points they are horizontal translates of each other.

37. If (1) in the text has no critical points it has no constant solutions. The solutions have neither an upper nor lower bound. Since solutions are monotonic, every solution assumes all real values.

Mathematical Models

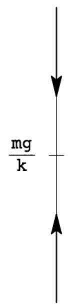
38. The critical points are 0 and b/a. From the phase portrait we see that 0 is an attractor and b/a is a repeller. Thus, if an initial population satisfies $P_0 > b/a$, the population becomes unbounded as t increases, most probably in finite time, i.e. $P(t) \to \infty$ as $t \to T$. If $0 < P_0 < b/a$, then the population eventually dies out, that is, $P(t) \to 0$ as $t \to \infty$. Since population $P > 0$ we do not consider the case $P_0 < 0$.

39. The only critical point of the autonomous differential equation is the positive number h/k. A phase portrait shows that this point is unstable, so h/k is a repeller. For any initial condition $P(0) = P_0 < h/k, dP/dt < 0$, which means $P(t)$ is monotonic decreasing and so the graph of $P(t)$ must cross the t-axis or the line $P = 0$ at some time $t_1 > 0$. But $P(t_1) = 0$ means the population is extinct at time t_1.

40. Writing the differential equation in the form

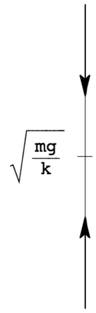
$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v\right)$$

we see that a critical point is mg/k. From the phase portrait we see that mg/k is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = mg/k$.



41. Writing the differential equation in the form

$$\frac{dv}{dt} = \frac{k}{m}\left(\frac{mg}{k} - v^2\right) = \frac{k}{m}\left(\sqrt{\frac{mg}{k}} - v\right)\left(\sqrt{\frac{mg}{k}} + v\right)$$

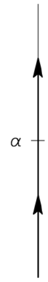


we see that the only physically meaningful critical point is $\sqrt{mg/k}$. From the phase portrait we see that $\sqrt{mg/k}$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} v = \sqrt{mg/k}$.

42. **(a)** From the phase portrait we see that critical points are α and β. Let $X(0) = X_0$.

- If $X_0 < \alpha$, we see that $X \to \alpha$ as $t \to \infty$.
- If $\alpha < X_0 < \beta$, we see that $X \to \alpha$ as $t \to \infty$.
- If $X_0 > \beta$, we see that $X(t)$ increases in an unbounded manner, but more specific behavior of $X(t)$ as $t \to \infty$ is not known.

(b) When $\alpha = \beta$ the phase portrait is as shown.



- If $X_0 < \alpha$, then $X(t) \to \alpha$ as $t \to \infty$.
- If $X_0 > \alpha$, then $X(t)$ increases in an unbounded manner. This could happen in a finite amount of time. That is, the phase portrait does not indicate that X becomes unbounded as $t \to \infty$.

(c) When $k = 1$ and $\alpha = \beta$ the differential equation is $dX/dt = (\alpha - X)^2$. For $X(t) = \alpha - 1/(t+c)$ we have $dX/dt = 1/(t+c)^2$ and

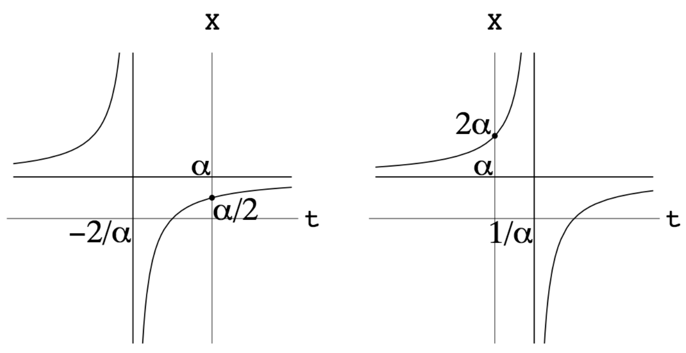
$$(\alpha - X)^2 = \left[\alpha - \left(\alpha - \frac{1}{t+c}\right)\right]^2 = \frac{1}{(t+c)^2} = \frac{dX}{dt}\,.$$

For $X(0) = \alpha/2$ we obtain

$$X(t) = \alpha - \frac{1}{t + 2/\alpha}\,.$$

For $X(0) = 2\alpha$ we obtain

$$X(t) = \alpha - \frac{1}{t - 1/\alpha}\,.$$



For $X_0 > \alpha$, $X(t)$ increases without bound up to $t = 1/\alpha$. For $t > 1/\alpha$, $X(t)$ increases but $X \to \alpha$ as $t \to \infty$

2.2 Separable Equations

In this section and ones following we will encounter an expression of the form $\ln|g(y)| = f(x) + c$. *To solve for* $g(y)$ *we exponentiate both sides of the equation. This yields* $|g(y)| = e^{f(x)+c} = e^c e^{f(x)}$ *which implies* $g(y) = \pm e^c e^{f(x)}$. *Letting* $c_1 = \pm e^c$ *we obtain* $g(y) = c_1 e^{f(x)}$.

1. From $dy = \sin 5x\, dx$ we obtain $y = -\frac{1}{5}\cos 5x + c$.

2. From $dy = (x+1)^2\, dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.

3. From $dy = -e^{-3x}\, dx$ we obtain $y = \frac{1}{3}e^{-3x} + c$.

4. From $\dfrac{1}{(y-1)^2}\, dy = dx$ we obtain $-\dfrac{1}{y-1} = x + c$ or $y = 1 - \dfrac{1}{x+c}$.

5. From $\dfrac{1}{y}\, dy = \dfrac{4}{x}\, dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1 x^4$.

6. From $\dfrac{1}{y^2}\, dy = -2x\, dx$ we obtain $-\dfrac{1}{y} = -x^2 + c$ or $y = \dfrac{1}{x^2 + c_1}$.

7. From $e^{-2y} dy = e^{3x} dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.

8. From $ye^y dy = \left(e^{-x} + e^{-3x}\right) dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.

9. From $\left(y + 2 + \dfrac{1}{y}\right) dy = x^2 \ln x\, dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.

10. From $\dfrac{1}{(2y+3)^2}\, dy = \dfrac{1}{(4x+5)^2}\, dx$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.

11. From $\dfrac{1}{\csc y}\, dy = -\dfrac{1}{\sec^2 x}\, dx$ or $\sin y\, dy = -\cos^2 x\, dx = -\frac{1}{2}(1 + \cos 2x)\, dx$ we obtain $-\cos y = -\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ or $4\cos y = 2x + \sin 2x + c_1$.

12. From $2y\, dy = -\dfrac{\sin 3x}{\cos^3 3x}\, dx$ or $2y\, dy = -\tan 3x \sec^2 3x\, dx$ we obtain $y^2 = -\frac{1}{6}\sec^2 3x + c$.

13. From $\dfrac{e^y}{(e^y+1)^2}\, dy = \dfrac{-e^x}{(e^x+1)^3}\, dx$ we obtain $-\left(e^y + 1\right)^{-1} = \frac{1}{2}\left(e^x + 1\right)^{-2} + c$.

14. From $\dfrac{y}{(1+y^2)^{1/2}}\, dy = \dfrac{x}{(1+x^2)^{1/2}}\, dx$ we obtain $\left(1 + y^2\right)^{1/2} = \left(1 + x^2\right)^{1/2} + c$.

15. From $\dfrac{1}{S}\, dS = k\, dr$ we obtain $S = ce^{kr}$.

16. From $\dfrac{1}{Q-70}\,dQ = k\,dt$ we obtain $\ln|Q-70| = kt + c$ or $Q - 70 = c_1e^{kt}$.

17. From $\dfrac{1}{P-P^2}dP = \left(\dfrac{1}{P} + \dfrac{1}{1-P}\right)dP = dt$ we obtain $\ln|P| - \ln|1-P| = t + c$ so that $\ln\left|\dfrac{P}{1-P}\right| = t + c$ or $\dfrac{P}{1-P} = c_1e^t$. Solving for P we have $P = \dfrac{c_1e^t}{1+c_1e^t}$.

18. From $\dfrac{1}{N}\,dN = \left(te^{t+2} - 1\right)dt$ we obtain $\ln|N| = te^{t+2} - e^{t+2} - t + c$ or $N = c_1e^{te^{t+2}-e^{t+2}-t}$.

19. From $\dfrac{y-2}{y+3}\,dy = \dfrac{x-1}{x+4}\,dx$ or $\left(1 - \dfrac{5}{y+3}\right)dy = \left(1 - \dfrac{5}{x+4}\right)dx$ we obtain $y - 5\ln|y+3| = x - 5\ln|x+4| + c$ or $\left(\dfrac{x+4}{y+3}\right)^5 = c_1e^{x-y}$.

20. From $\dfrac{y+1}{y-1}\,dy = \dfrac{x+2}{x-3}\,dx$ or $\left(1 + \dfrac{2}{y-1}\right)dy = \left(1 + \dfrac{5}{x-3}\right)dx$ we obtain $y + 2\ln|y-1| = x + 5\ln|x-3| + c$ or $\dfrac{(y-1)^2}{(x-3)^5} = c_1e^{x-y}$.

21. From $x\,dx = \dfrac{1}{\sqrt{1-y^2}}\,dy$ we obtain $\frac{1}{2}x^2 = \sin^{-1}y + c$ or $y = \sin\left(\dfrac{x^2}{2} + c_1\right)$.

22. From $\dfrac{1}{y^2}\,dy = \dfrac{1}{e^x + e^{-x}}\,dx = \dfrac{e^x}{(e^x)^2+1}\,dx$ we obtain $-\dfrac{1}{y} = \tan^{-1}e^x + c$ or $y = -\dfrac{1}{\tan^{-1}e^x + c}$.

23. From $\dfrac{1}{x^2+1}\,dx = 4\,dt$ we obtain $\tan^{-1}x = 4t + c$. Using $x(\pi/4) = 1$ we find $c = -3\pi/4$. The solution of the initial-value problem is $\tan^{-1}x = 4t - \dfrac{3\pi}{4}$ or $x = \tan\left(4t - \dfrac{3\pi}{4}\right)$.

24. From $\dfrac{1}{y^2-1}\,dy = \dfrac{1}{x^2-1}\,dx$ or $\dfrac{1}{2}\left(\dfrac{1}{y-1} - \dfrac{1}{y+1}\right)dy = \dfrac{1}{2}\left(\dfrac{1}{x-1} - \dfrac{1}{x+1}\right)dx$ we obtain $\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + \ln c$ or $\dfrac{y-1}{y+1} = \dfrac{c(x-1)}{x+1}$. Using $y(2) = 2$ we find $c = 1$. A solution of the initial-value problem is $\dfrac{y-1}{y+1} = \dfrac{x-1}{x+1}$ or $y = x$.

25. From $\dfrac{1}{y}\,dy = \dfrac{1-x}{x^2}\,dx = \left(\dfrac{1}{x^2} - \dfrac{1}{x}\right)dx$ we obtain $\ln|y| = -\dfrac{1}{x} - \ln|x| = c$ or $xy = c_1e^{-1/x}$. Using $y(-1) = -1$ we find $c_1 = e^{-1}$. The solution of the initial-value problem is $xy = e^{-1-1/x}$ or $y = e^{-(1+1/x)}/x$.

26. From $\dfrac{1}{1-2y}\,dy = dt$ we obtain $-\frac{1}{2}\ln|1-2y| = t + c$ or $1 - 2y = c_1e^{-2t}$. Using $y(0) = 5/2$ we find $c_1 = -4$. The solution of the initial-value problem is $1 - 2y = -4e^{-2t}$ or $y = 2e^{-2t} + \frac{1}{2}$.

27. Separating variables and integrating we obtain

$$\frac{dx}{\sqrt{1-x^2}} - \frac{dy}{\sqrt{1-y^2}} = 0 \quad \text{and} \quad \sin^{-1} x - \sin^{-1} y = c.$$

Setting $x = 0$ and $y = \sqrt{3}/2$ we obtain $c = -\pi/3$. Thus, an implicit solution of the initial-value problem is $\sin^{-1} x - \sin^{-1} y = -\pi/3$. Solving for y and using an addition formula from trigonometry, we get

$$y = \sin\left(\sin^{-1} x + \frac{\pi}{3}\right) = x\cos\frac{\pi}{3} + \sqrt{1-x^2}\sin\frac{\pi}{3} = \frac{x}{2} + \frac{\sqrt{3}\sqrt{1-x^2}}{2}.$$

28. From $\dfrac{1}{1+(2y)^2}\,dy = \dfrac{-x}{1+(x^2)^2}\,dx$ we obtain

$$\frac{1}{2}\tan^{-1} 2y = -\frac{1}{2}\tan^{-1} x^2 + c \quad \text{or} \quad \tan^{-1} 2y + \tan^{-1} x^2 = c_1.$$

Using $y(1) = 0$ we find $c_1 = \pi/4$. Thus, an implicit solution of the initial-value problem is $\tan^{-1} 2y + \tan^{-1} x^2 = \pi/4$. Solving for y and using a trigonometric identity we get

$$\begin{aligned}
2y &= \tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\
y &= \frac{1}{2}\tan\left(\frac{\pi}{4} - \tan^{-1} x^2\right) \\
&= \frac{1}{2}\left(\frac{\tan(\pi/4) - \tan(\tan^{-1} x^2)}{1 + \tan(\pi/4)\tan(\tan^{-1} x^2)}\right) \\
&= \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right).
\end{aligned}$$

29. Separating variables, integrating from 4 to x, and using t as a dummy variable of integration gives

$$\begin{aligned}
\int_4^x \frac{1}{y}\frac{dy}{dt}\,dt &= \int_4^x e^{-t^2}\,dt \\
\ln y(t)\Big|_4^x &= \int_4^x e^{-t^2}\,dt \\
\ln y(x) - \ln y(4) &= \int_4^x e^{-t^2}\,dt,
\end{aligned}$$

Using the initial condition we have

$$\ln y(x) = \ln y(4) + \int_4^x e^{-t^2}\,dt = \ln 1 + \int_4^x e^{-t^2}\,dt = \int_4^x e^{-t^2}\,dt.$$

Thus,

$$y(x) = e^{\int_4^x e^{-t^2}\,dt}.$$

30. Separating variables, integrating from -2 to x, and using t as a dummy variable of integration gives

$$\int_{-2}^{x} \frac{1}{y^2}\frac{dy}{dt}\,dt = \int_{-2}^{x} \sin t^2 dt$$

$$-y(t)^{-1}\Big|_{-2}^{x} = \int_{-2}^{x} \sin t^2 dt$$

$$-y(x)^{-1} + y(-2)^{-1} = \int_{-2}^{x} \sin t^2 dt$$

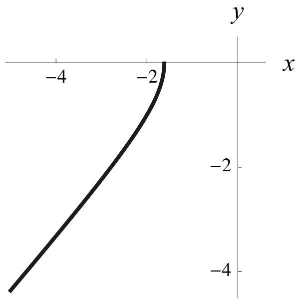
$$-y(x)^{-1} = -y(-2)^{-1} + \int_{-2}^{x} \sin t^2 dt$$

$$y(x)^{-1} = 3 - \int_{-2}^{x} \sin t^2 dt.$$

Thus

$$y(x) = \frac{1}{3 - \int_{-2}^{x} \sin t^2 dt}\,.$$

31. Separating variables we have $2y\,dy = (2x+1)dx$. Integrating gives $y^2 = x^2 + x + c$. When $y(-2) = -1$ we find $c = -1$, so $y^2 = x^2 + x - 1$ and $y = -\sqrt{x^2+x-1}$. The negative square root is chosen because of the initial condition.



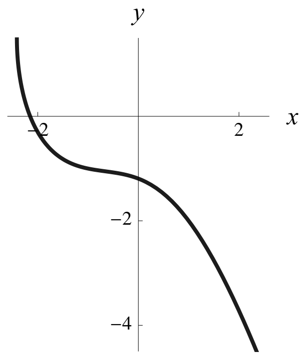
To obtain the exact interval of definition we want $x^2+x-1 > 0$. Since $y = x^2 + x - 1 = 0$ is a parabola opening up and $x^2+x-1 = 0$ when $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, we use $\left(-\infty, -\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)$ (because of the initial condition).

32. The problem should read

$$(2y-2)\frac{dy}{dx} = 3x^2 + 4x + 2, \quad y(-2) = 1.$$

Separating variables we have $(2y-2)dy = (3x^2+4x+2)dx$. Integrating gives $y^2 - 2y = x^3 + 2x^2 + 2x + c$. We complete the square by adding 1 to the left-hand side and absorbing the 1 into the constant on the right-hand side. This gives $(y-1)^2 = x^3 + 2x^2 + 2x + c_1$. From the initial condition we find that $c_1 = 4$, so the solution of the initial-value problem is

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}\,,$$



where the minus sign is determined by the initial condition.

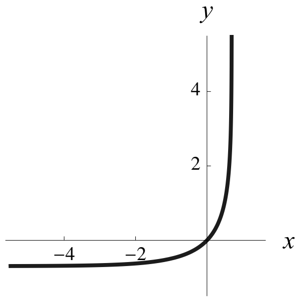
To obtain the exact interval of definition of the solution we want

$$x^3 + 2x^2 + 2x + 4 = (x^2 + 2)(x + 2) > 0 \qquad \text{or} \qquad x > -2.$$

Thus, the interval of definition of the solution is $(-2, \infty)$.

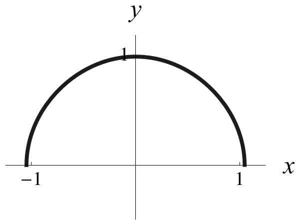
33. Writing the differential equation as $e^x dx = e^{-y} dy$ and integrating we have $e^x = -e^{-y} + c$. Using $y(0) = 0$ we find that $c = 2$ so that $y = -\ln(2 - e^x)$.

To find the interval of definition of this solution we note that $2 - e^x > 0$ so x must be in $(-\infty, \ln 2)$.



34. Integrating the differential equation we have $-\cos x + \frac{1}{2}y^2 = c$. Then $y(0) = 1$ implies that $c = -\frac{1}{2}$, and so $y = \sqrt{2\cos x - 1}$. We choose the positive square root because of the initial condition.

To find the interval of definition of the solution we note that



$$2\cos x - 1 > 0 \qquad \text{or} \qquad \cos x > \frac{1}{2}, \qquad \text{so} \qquad -\frac{\pi}{3} < x < \frac{\pi}{3},$$

and x must be in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

35. (a) The equilibrium solutions $y(x) = 2$ and $y(x) = -2$ satisfy the initial conditions $y(0) = 2$ and $y(0) = -2$, respectively. Setting $x = \frac{1}{4}$ and $y = 1$ in $y = 2(1 + ce^{4x})/(1 - ce^{4x})$ we obtain

$$1 = 2\frac{1 + ce}{1 - ce}, \quad 1 - ce = 2 + 2ce, \quad -1 = 3ce, \quad \text{and} \quad c = -\frac{1}{3e}.$$

The solution of the corresponding initial-value problem is

$$y = 2\left(\frac{1 - \frac{1}{3}e^{4x-1}}{1 + \frac{1}{3}e^{4x-1}}\right) = 2\left(\frac{3 - e^{4x-1}}{3 + e^{4x-1}}\right).$$

(b) Separating variables and integrating yields

$$\frac{1}{4}\ln|y - 2| - \frac{1}{4}\ln|y + 2| + \ln c_1 = x$$
$$\ln|y - 2| - \ln|y + 2| + \ln c = 4x$$
$$\ln\left|\frac{c(y - 2)}{y + 2}\right| = 4x$$
$$c\,\frac{y - 2}{y + 2} = e^{4x}.$$

Solving for y we get $y = 2(c + e^{4x})/(c - e^{4x})$. The initial condition $y(0) = -2$ implies $2(c + 1)/(c - 1) = -2$ which yields $c = 0$ and $y(x) = -2$. The initial condition $y(0) = 2$

does not correspond to a value of c, and it must simply be recognized that $y(x) = 2$ is a solution of the initial-value problem. Setting $x = \frac{1}{4}$ and $y = 1$ in $y = 2(c + e^{4x})/(c - e^{4x})$ leads to $c = -3e$. Thus, a solution of the initial-value problem is

$$y = 2\,\frac{-3e + e^{4x}}{-3e - e^{4x}} = 2\,\frac{3 - e^{4x-1}}{3 + e^{4x-1}}\,.$$

36. Separating variables, we have

$$\frac{dy}{y^2 - y} = \frac{dx}{x} \qquad \text{or} \qquad \int \frac{dy}{y(y-1)} = \ln|x| + c.$$

Using partial fractions, we obtain

$$\begin{aligned} \int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy &= \ln|x| + c \\ \ln|y-1| - \ln|y| &= \ln|x| + c \\ \ln\left|\frac{y-1}{xy}\right| &= c \\ \frac{y-1}{xy} &= e^c = c_1. \end{aligned}$$

Solving for y we get $y = 1/(1 - c_1 x)$. We note by inspection that $y = 0$ is a singular solution of the differential equation.

(a) Setting $x = 0$ and $y = 1$ we have $1 = 1/(1 - 0)$, which is true for all values of c_1. Thus, solutions passing through $(0, 1)$ are $y = 1/(1 - c_1 x)$.

(b) Setting $x = 0$ and $y = 0$ in $y = 1/(1 - c_1 x)$ we get $0 = 1$. Thus, the only solution passing through $(0, 0)$ is $y = 0$.

(c) Setting $x = \frac{1}{2}$ and $y = \frac{1}{2}$ we have $\frac{1}{2} = 1/(1 - \frac{1}{2}\,c_1)$, so $c_1 = -2$ and $y = 1/(1 + 2x)$.

(d) Setting $x = 2$ and $y = \frac{1}{4}$ we have $\frac{1}{4} = 1/(1 - 2c_1)$, so $c_1 = -\frac{3}{2}$ and $y = 1/(1 + \frac{3}{2}x) = 2/(2 + 3x)$.

37. Singular solutions of $dy/dx = x\sqrt{1 - y^2}$ are $y = -1$ and $y = 1$. A singular solution of $(e^x + e^{-x})dy/dx = y^2$ is $y = 0$.

38. Differentiating $\ln(x^2 + 10) + \csc y = c$ we get

$$\begin{aligned} \frac{2x}{x^2 + 10} - \csc y \, \cot y \, \frac{dy}{dx} &= 0, \\ \frac{2x}{x^2 + 10} - \frac{1}{\sin y} \cdot \frac{\cos y}{\sin y}\,\frac{dy}{dx} &= 0, \end{aligned}$$

or

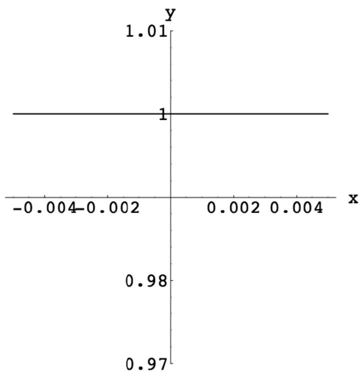
$$2x \sin^2 y \, dx - (x^2 + 10) \cos y \, dy = 0.$$

Writing the differential equation in the form

$$\frac{dy}{dx} = \frac{2x\sin^2 y}{(x^2+10)\cos y}$$

we see that singular solutions occur when $\sin^2 y = 0$, or $y = k\pi$, where k is an integer.

39. The singular solution $y = 1$ satisfies the initial-value problem.

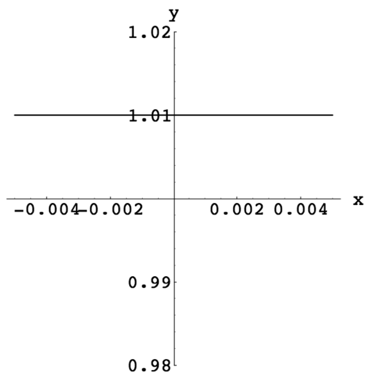


40. Separating variables we obtain $\dfrac{dy}{(y-1)^2} = dx$. Then

$$-\frac{1}{y-1} = x + c \quad \text{and} \quad y = \frac{x+c-1}{x+c}.$$

Setting $x = 0$ and $y = 1.01$ we obtain $c = -100$. The solution is

$$y = \frac{x-101}{x-100}.$$

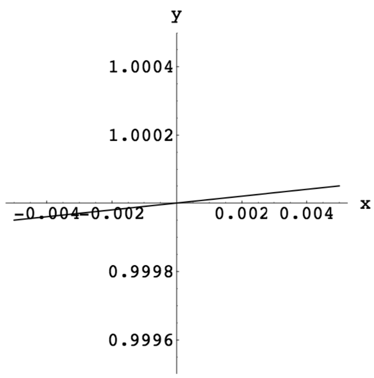


41. Separating variables we obtain $\dfrac{dy}{(y-1)^2+0.01} = dx$. Then

$$10\tan^{-1}10(y-1) = x+c \quad \text{and} \quad y = 1 + \frac{1}{10}\tan\frac{x+c}{10}.$$

Setting $x = 0$ and $y = 1$ we obtain $c = 0$. The solution is

$$y = 1 + \frac{1}{10}\tan\frac{x}{10}.$$

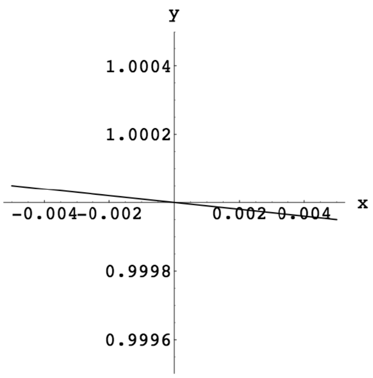


42. Separating variables we obtain $\dfrac{dy}{(y-1)^2+0.01} = dx$. Then

$10\tan^{-1} 10(y-1) = x+c$ and $y = 1+\dfrac{1}{10}\tan\dfrac{x+c}{10}$.

Setting $x = 0$ and $y = 1$ we obtain $c = 0$. The solution is

$$y = 1 + \frac{1}{10}\tan\frac{x}{10}.$$



Alternatively, we can use the fact that

$$\int \frac{dy}{(y-1)^2 - 0.01} = -\frac{1}{0.1}\tanh^{-1}\frac{y-1}{0.1} = -10\tanh^{-1} 10(y-1).$$

We use the inverse hyperbolic tangent because $|y-1| < 0.1$ or $0.9 < y < 1.1$. This follows from the initial condition $y(0) = 1$. Solving the above equation for y we get $y = 1 + 0.1\tanh(x/10)$.

43. Separating variables, we have

$$\frac{dy}{y-y^3} = \frac{dy}{y(1-y)(1+y)} = \left(\frac{1}{y} + \frac{1/2}{1-y} - \frac{1/2}{1+y}\right) dy = dx.$$

Integrating, we get

$$\ln|y| - \frac{1}{2}\ln|1-y| - \frac{1}{2}\ln|1+y| = x + c.$$

When $y > 1$, this becomes

$$\ln y - \frac{1}{2}\ln(y-1) - \frac{1}{2}\ln(y+1) = \ln\frac{y}{\sqrt{y^2-1}} = x + c.$$

Letting $x = 0$ and $y = 2$ we find $c = \ln(2/\sqrt{3})$. Solving for y we get $y_1(x) = 2e^x/\sqrt{4e^{2x}-3}$, where $x > \ln(\sqrt{3}/2)$.

When $0 < y < 1$ we have

$$\ln y - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{y}{\sqrt{1-y^2}} = x + c.$$

Letting $x = 0$ and $y = \frac{1}{2}$ we find $c = \ln(1/\sqrt{3})$. Solving for y we get $y_2(x) = e^x/\sqrt{e^{2x}+3}$, where $-\infty < x < \infty$.

When $-1 < y < 0$ we have

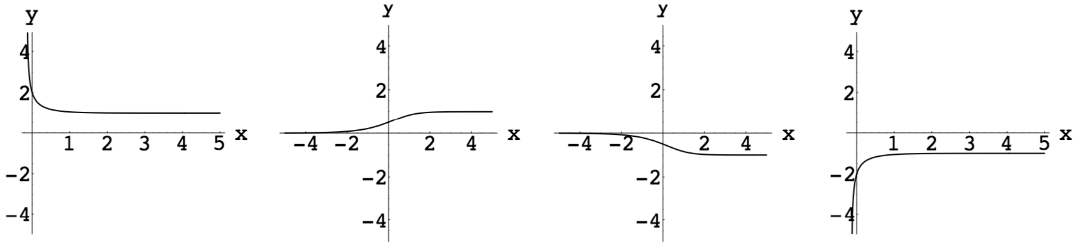
$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(1+y) = \ln\frac{-y}{\sqrt{1-y^2}} = x + c.$$

Letting $x = 0$ and $y = -\frac{1}{2}$ we find $c = \ln(1/\sqrt{3})$. Solving for y we get $y_3(x) = -e^x/\sqrt{e^{2x}+3}$, where $-\infty < x < \infty$.

When $y < -1$ we have

$$\ln(-y) - \frac{1}{2}\ln(1-y) - \frac{1}{2}\ln(-1-y) = \ln\frac{-y}{\sqrt{y^2-1}} = x + c.$$

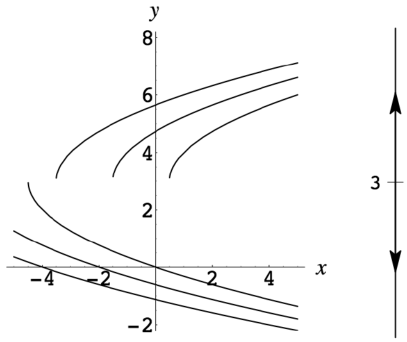
Letting $x = 0$ and $y = -2$ we find $c = \ln(2/\sqrt{3}\,)$. Solving for y we get $y_4(x) = -2e^x/\sqrt{4e^{2x}-3}\,$, where $x > \ln(\sqrt{3}/2)$.



44. **(a)** The second derivative of y is

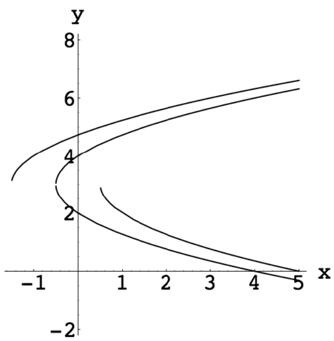
$$\frac{d^2y}{dx^2} = -\frac{dy/dx}{(y-3)^2} = -\frac{1/(y-3)}{(y-3)^2} = -\frac{1}{(y-3)^3}\,.$$

The solution curve is concave down when $d^2y/dx^2 < 0$ or $y > 3$, and concave up when $d^2y/dx^2 > 0$ or $y < 3$. From the phase portrait we see that the solution curve is decreasing when $y < 3$ and increasing when $y > 3$.



(b) Separating variables and integrating we obtain

$$\begin{aligned}(y-3)\,dy &= dx\\ \frac{1}{2}y^2 - 3y &= x + c\\ y^2 - 6y + 9 &= 2x + c_1\\ (y-3)^2 &= 2x + c_1\\ y &= 3 \pm \sqrt{2x + c_1}\,.\end{aligned}$$



The initial condition dictates whether to use the plus or minus sign.

When $y_1(0) = 4$ we have $c_1 = 1$ and $y_1(x) = 3 + \sqrt{2x+1}\,$.

When $y_2(0) = 2$ we have $c_1 = 1$ and $y_2(x) = 3 - \sqrt{2x+1}\,$.

When $y_3(1) = 2$ we have $c_1 = -1$ and $y_3(x) = 3 - \sqrt{2x-1}\,$.

When $y_4(-1) = 4$ we have $c_1 = 3$ and $y_4(x) = 3 + \sqrt{2x+3}\,$.

45. We separate variable and rationalize the denominator:

$$dy = \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx = \frac{1-\sin x}{1-\sin^2 x}\,dx = \frac{1-\sin x}{cos^2 x}\,dx$$

$$= (\sec^2 x - \tan x \sec x)\,dx.$$

Integrating, we have $y = \tan x - \sec x + C$.

46. Separating variables we have $\sqrt{y}\,dy = \sin\sqrt{x}\,dx$. Then

$$\int \sqrt{y}\,dy = \int \sin\sqrt{x}\,dx \qquad \text{and} \qquad \frac{2}{3}y^{3/2} = \int \sin\sqrt{x}\,dx.$$

To integrate $\sin\sqrt{x}$ we first make the substitution $u = \sqrt{x}$. then $du = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2u}\,du$ and

$$\int \sin\sqrt{x}\,dx = \int (\sin u)(2u)du = 2\int u\sin u\,du.$$

Using integration by parts we find

$$\int u\sin u\,du = -u\cos u + \sin u = -\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}.$$

Thus

$$\frac{2}{3}y = \int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

and

$$y = 3^{2/3}\left(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x} + C\right).$$

47. Separating variables we have $dy/(\sqrt{y} + y) = dx/(\sqrt{x} + x)$. To integrate $\int dx/(\sqrt{x} + x)$ we substitute $u^2 = x$ and get

$$\int \frac{2u}{u + u^2}\,du = \int \frac{2}{1+u}\,du = 2\ln|1+u| + c = 2\ln(1 + \sqrt{x}\,) + c.$$

Integrating the separated differential equation we have

$$2\ln(1 + \sqrt{y}\,) = 2\ln(1 + \sqrt{x}\,) + c \quad \text{or} \quad \ln(1 + \sqrt{y}\,) = \ln(1 + \sqrt{x}\,) + \ln c_1.$$

Solving for y we get $y = [c_1(1 + \sqrt{x}\,) - 1]^2$.

48. Separating variables and integrating we have

$$\int \frac{dy}{y^{2/3}(1 - y^{1/3})} = \int dx$$

$$\int \frac{y^{2/3}}{1 - y^{1/3}}\,dy = x + c_1$$

$$-3\ln\left|1 - y^{1/3}\right| = x + c_1$$

$$\ln\left|1 - y^{1/3}\right| = -\frac{x}{3} + c_2$$

$$\left|1 - y^{1/3}\right| = c_3e^{-x/3}$$

$$1 - y^{1/3} = c_4e^{-x/3}$$

$$y^{1/3} = 1 + c_5e^{-x/3}$$

$$y = \left(1 + c_5e^{-x/3}\right)^3.$$

49. Separating variables we have $y\,dy = e^{\sqrt{x}}\,dx$. If $u = \sqrt{x}$, then $u^2 = x$ and $2u\,du = dx$. Thus, $\int e^{\sqrt{x}}\,dx = \int 2ue^u\,du$ and, using integration by parts, we find

$$\int y\,dy = \int e^{\sqrt{x}}\,dx \qquad \text{so} \qquad \frac{1}{2}y^2 = \int 2ue^u\,du = -2e^u + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C,$$

and
$$y = 2\sqrt{\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + C}\,.$$

To find C we solve $y(1) = 4$.

$$y(1) = 2\sqrt{\sqrt{1}\,e^{\sqrt{1}} - e^{\sqrt{1}} + C} = 2\sqrt{C} = 4 \qquad \text{so} \qquad C = 4,$$

and the solution of the initial-value problem is $y = 2\sqrt{\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}} + 4}$.

50. Separating variables we have $y\,dy = x\tan^{-1}x\,dx$. Integrating both sides and using integration by parts with $u = \tan^{-1}x$ and $dv = x\,dx$ we have

$$\int y\,dy = x\tan^{-1}x\,dx$$
$$\frac{1}{2}y^2 = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$$
$$y^2 = x^2\tan^{-1}x - x + \tan^{-1}x + C_1$$
$$y = \sqrt{x^2\tan^{-1}x - x + \tan^{-1}x + C_1}$$

To find C_1 we solve $y(0) = 3$.

$$y(0) = \sqrt{0^2\tan^{-1}0 - 0 + \tan^{-1}0 + C_1} = \sqrt{C_1} = 3 \qquad \text{so} \qquad C_1 = 9,$$

and the solution of the initial-value problem is $y = \sqrt{x^2\tan^{-1}x - x + \tan^{-1}x + 9}$.

Discussion Problems

51. **(a)** While $y_2(x) = -\sqrt{25 - x^2}$ is defined at $x = -5$ and $x = 5$, $y_2'(x)$ is not defined at these values, and so the interval of definition is the open interval $(-5, 5)$.

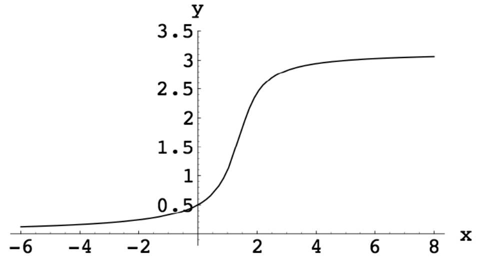
(b) At any point on the x-axis the derivative of $y(x)$ is undefined, so no solution curve can cross the x-axis. Since $-x/y$ is not defined when $y = 0$, the initial-value problem has no solution.

52. **(a)** Separating variables and integrating we obtain $x^2 - y^2 = c$. For $c \neq 0$ the graph is a hyperbola centered at the origin. All four initial conditions imply $c = 0$ and $y = \pm x$. Since the differential equation is not defined for $y = 0$, solutions are $y = \pm x$, $x < 0$ and $y = \pm x$, $x > 0$. The solution for $y(a) = a$ is $y = x$, $x > 0$; for $y(a) = -a$ is $y = -x$; for $y(-a) = a$ is $y = -x$, $x < 0$; and for $y(-a) = -a$ is $y = x$, $x < 0$.

(b) Since x/y is not defined when $y = 0$, the initial-value problem has no solution.

(c) Setting $x = 1$ and $y = 2$ in $x^2 - y^2 = c$ we get $c = -3$, so $y^2 = x^2 + 3$ and $y(x) = \sqrt{x^2+3}$, where the positive square root is chosen because of the initial condition. The domain is all real numbers since $x^2 + 3 > 0$ for all x.

53. Separating variables we have $dy/\left(\sqrt{1+y^2}\,\sin^2 y\right) = dx$ which is not readily integrated (even by a CAS). We note that $dy/dx \geq 0$ for all values of x and y and that $dy/dx = 0$ when $y = 0$ and $y = \pi$, which are equilibrium solutions.



54. **(a)** The solution of $y' = y$, $y(0) = 1$, is $y = e^x$. Using separation of variables we find that the solution of $y' = y\left[1 + 1/(x \ln x)\right]$, $y(e) = 1$, is $y = e^{x-e} \ln x$. Solving the two solutions simultaneously we obtain

$$e^x = e^{x-e} \ln x, \qquad \text{so} \qquad e^e = \ln x \qquad \text{and} \qquad x = e^{e^e}.$$

(b) Since $y = e^{\left(e^{e^e}\right)} \approx 2.33 \times 10^{1{,}656{,}520}$, the y-coordinate of the point of intersection of the two solution curves has over 1.65 million digits.

55. We are looking for a function $y(x)$ such that

$$y^2 + \left(\frac{dy}{dx}\right)^2 = 1.$$

Using the positive square root gives

$$\frac{dy}{dx} = \sqrt{1-y^2}$$

$$\frac{dy}{\sqrt{1-y^2}} = dx$$

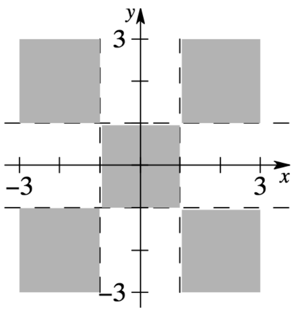
$$\sin^{-1} y = x + c.$$

Thus a solution is $y = \sin(x + c)$. If we use the negative square root we obtain

$$y = \sin(c - x) = -\sin(x - c) = -\sin(x + c_1).$$

Note that when $c = c_1 = 0$ and when $c = c_1 = \pi/2$ we obtain the well known particular solutions $y = \sin x$, $y = -\sin x$, $y = \cos x$, and $y = -\cos x$. Note also that $y = 1$ and $y = -1$ are singular solutions.

56. **(a)**



(b) For $|x| > 1$ and $|y| > 1$ the differential equation is $dy/dx = \sqrt{y^2-1}\,/\sqrt{x^2-1}$. Separating variables and integrating, we obtain

$$\frac{dy}{\sqrt{y^2-1}} = \frac{dx}{\sqrt{x^2-1}} \quad \text{and} \quad \cosh^{-1} y = \cosh^{-1} x + c.$$

Setting $x = 2$ and $y = 2$ we find $c = \cosh^{-1} 2 - \cosh^{-1} 2 = 0$ and $\cosh^{-1} y = \cosh^{-1} x$. An explicit solution is $y = x$.

Mathematical Model

57. Since the tension T_1 (or magnitude T_1) acts at the lowest point of the cable, we use symmetry to solve the problem on the interval $[0,\ L/2]$. The assumption that the roadbed is uniform (that is, weighs a constant ρ pounds per horizontal foot) implies $W = \rho x$, where x is measured in feet and $0 \le x \le L/2$. Therefore (10) in the text becomes $dy/dx = (\rho/T_1)x$. This last equation is a separable equation of the form given in (1) of Section 2.2 in the text. Integrating and using the initial condition $y(0) = a$ shows that the shape of the cable is a parabola: $y(x) = (\rho/2T_1)x^2 + a$. In terms of the sag h of the cable and the span L, we see from Figure 2.2.5 in the text that $y(L/2) = h + a$. By applying this last condition to $y(x) = (\rho/2T_1)x^2 + a$ enables us to express $\rho/2T_1$ in terms of h and L: $y(x) = (4h/L^2)x^2 + a$. Since $y(x)$ is an even function of x, the solution is valid on $-L/2 \le x \le L/2$.

Computer Lab Assignments

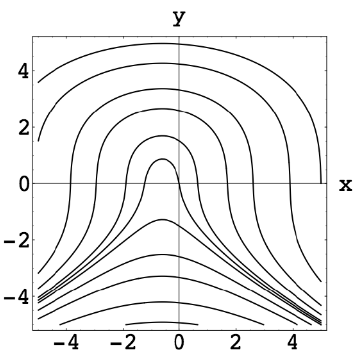
58. **(a)** Separating variables and integrating, we have

$$(3y^2 + 1)dy = -(8x + 5)dx$$

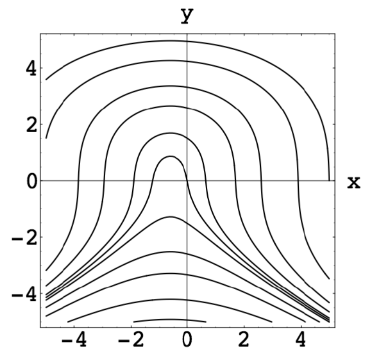
and

$$y^3 + y = -4x^2 - 5x + c.$$

Using a CAS we show various contours of $f(x, y) = y^3+y+4x^2+5x$. The plots, shown on $[-5,\ 5]\times[-5,\ 5]$, correspond to c-values of 0, ±5, ±20, ±40, ±80, and ±125.



(b) The value of c corresponding to $y(0) = -1$ is $f(0, -1) = -2$; to $y(0) = 2$ is $f(0, 2) = 10$; to $y(-1) = 4$ is $f(-1, 4) = 67$; and to $y(-1) = -3$ is -31.



59. **(a)** An implicit solution of the differential equation $(2y + 2)dy - (4x^3 + 6x)dx = 0$ is

$$y^2 + 2y - x^4 - 3x^2 + c = 0.$$

The condition $y(0) = -3$ implies that $c = -3$. Therefore $y^2 + 2y - x^4 - 3x^2 - 3 = 0$.

(b) Using the quadratic formula we can solve for y in terms of x:

$$y = \frac{-2 \pm \sqrt{4 + 4(x^4 + 3x^2 + 3)}}{2}.$$

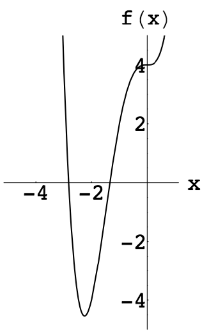
The explicit solution that satisfies the initial condition is then

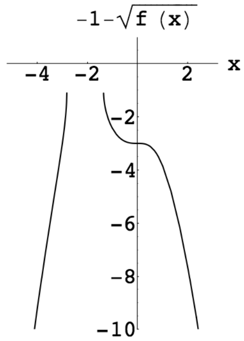
$$y = -1 - \sqrt{x^4 + 3x^3 + 4}.$$

(c) From the graph of $f(x) = x^4 + 3x^3 + 4$ below we see that $f(x) \le 0$ on the approximate interval $-2.8 \le x \le -1.3$. Thus the approximate domain of the function

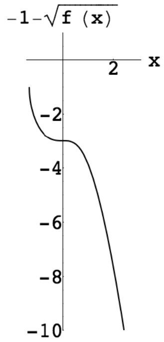
$$y = -1 - \sqrt{x^4 + 3x^3 + 4} = -1 - \sqrt{f(x)}$$

is $x \le -2.8$ or $x \ge -1.3$. The graph of this function is shown below.





(d) Using the root finding capabilities of a CAS, the zeros of f are found to be -2.82202 and -1.3409. The domain of definition of the solution $y(x)$ is then $x > -1.3409$. The equality has been removed since the derivative dy/dx does not exist at the points where $f(x) = 0$. The graph of the solution $y = \phi(x)$ is given on the right.



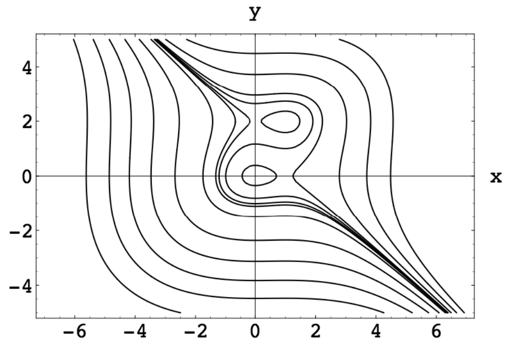
60. **(a)** Separating variables and integrating, we have

$$(-2y + y^2)dy = (x - x^2)dx$$

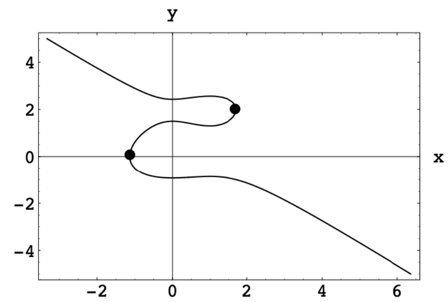
and

$$-y^2 + \frac{1}{3}y^3 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + c.$$

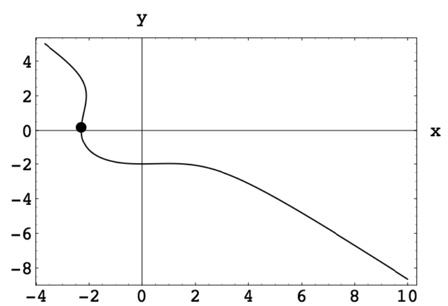
Using a CAS we show some contours of $f(x, y) = 2y^3 - 6y^2 + 2x^3 - 3x^2$. The plots shown on $[-7, 7] \times [-5, 5]$ correspond to c-values of -450, -300, -200, -120, -60, -20, -10, -8.1, -5, -0.8, 20, 60, and 120.



(b) The value of c corresponding to $y(0) = \frac{3}{2}$ is $f\left(0, \frac{3}{2}\right) = -\frac{27}{4}$. The portion of the graph between the dots corresponds to the solution curve satisfying the initial condition. To determine the interval of definition we find dy/dx for $2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}$. Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting $y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -\frac{27}{4}$ and using a CAS to solve for x we get $x = -1.13232$. Similarly, letting $y = 2$, we find $x = 1.71299$. The largest interval of definition is approximately $(-1.13232, 1.71299)$.



(c) The value of c corresponding to $y(0) = -2$ is $f(0, -2) = -40$. The portion of the graph to the right of the dot corresponds to the solution curve satisfying the initial condition. To determine the interval of definition we find dy/dx for $2y^3-6y^2+2x^3-3x^2 = -40$. Using implicit differentiation we get $y' = (x - x^2)/(y^2 - 2y)$, which is infinite when $y = 0$ and $y = 2$. Letting



$y = 0$ in $2y^3 - 6y^2 + 2x^3 - 3x^2 = -40$ and using a CAS to solve for x we get $x = -2.29551$. The largest interval of definition is approximately $(-2.29551, \infty)$.

2.3 Linear Equations

1. For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$. There is no transient term.

2. For $y' + 2y = 0$ an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\dfrac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$. The transient term is ce^{-2x}.

3. For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^{x}$ so that $\dfrac{d}{dx}\left[e^{x}y\right] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is ce^{-x}.

4. For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\,dx} = e^{4x}$ so that $\dfrac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is ce^{-4x}.

5. For $y' + 3x^2y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\dfrac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$. The transient term is ce^{-x^3}.

6. For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ so that $\dfrac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

7. For $y' + \dfrac{1}{x}y = \dfrac{1}{x^2}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\dfrac{d}{dx}\left[xy\right] = \dfrac{1}{x}$ and $y = \dfrac{1}{x}\ln x + \dfrac{c}{x}$ for $0 < x < \infty$. The entire solution is transient.

8. For $y' - 2y = x^2 + 5$ an integrating factor is $e^{-\int 2\,dx} = e^{-2x}$ so that $\dfrac{d}{dx}\left[e^{-2x}y\right] = x^2e^{-2x} + 5e^{-2x}$ and $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + ce^{2x}$ for $-\infty < x < \infty$. There is no transient term.

9. For $y' - \dfrac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \dfrac{1}{x}$ so that $\dfrac{d}{dx}\left[\dfrac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$. There is no transient term.

10. For $y' + \dfrac{2}{x}y = \dfrac{3}{x}$ an integrating factor is $e^{\int (2/x)dx} = x^2$ so that $\dfrac{d}{dx}\left[x^2y\right] = 3x$ and $y = \frac{3}{2} + cx^{-2}$ for $0 < x < \infty$. The transient term is cx^{-2}.

11. For $y' + \dfrac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int (4/x)dx} = x^4$ so that $\dfrac{d}{dx}\left[x^4y\right] = x^6 - x^4$ and $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$. The transient term is cx^{-4}.

12. For $y' - \dfrac{x}{(1+x)}\, y = x$ an integrating factor is $e^{-\int [x/(1+x)]dx} = (x+1)e^{-x}$ so that $\dfrac{d}{dx}\left[(x+1)e^{-x}y\right] = x(x+1)e^{-x}$ and $y = -x - \dfrac{2x+3}{x+1} + \dfrac{ce^{x}}{x+1}$ for $-1 < x < \infty$. There is no transient term.

13. For $y' + \left(1 + \dfrac{2}{x}\right) y = \dfrac{e^x}{x^2}$ an integrating factor is $e^{\int [1+(2/x)]dx} = x^2 e^x$ so that $\dfrac{d}{dx}\left[x^2 e^x y\right] = e^{2x}$ and $y = \dfrac{1}{2}\dfrac{e^x}{x^2} + \dfrac{ce^{-x}}{x^2}$ for $0 < x < \infty$. The transient term is $\dfrac{ce^{-x}}{x^2}$.

14. For $y' + \left(1 + \dfrac{1}{x}\right) y = \dfrac{1}{x}e^{-x}\sin 2x$ an integrating factor is $e^{\int [1+(1/x)]dx} = xe^x$ so that $\dfrac{d}{dx}\left[xe^x y\right] = \sin 2x$ and $y = -\dfrac{1}{2x}e^{-x}\cos 2x + \dfrac{ce^{-x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

15. For $\dfrac{dx}{dy} - \dfrac{4}{y}x = 4y^5$ an integrating factor is $e^{-\int (4/y)dy} = e^{\ln y^{-4}} = y^{-4}$ so that $\dfrac{d}{dy}\left[y^{-4}x\right] = 4y$ and $x = 2y^6 + cy^4$ for $0 < y < \infty$. There is no transient term.

16. For $\dfrac{dx}{dy} + \dfrac{2}{y}x = e^y$ an integrating factor is $e^{\int (2/y)dy} = y^2$ so that $\dfrac{d}{dy}\left[y^2 x\right] = y^2 e^y$ and $x = e^y - \dfrac{2}{y}e^y + \dfrac{2}{y^2}e^y + \dfrac{c}{y^2}$ for $0 < y < \infty$. The transient term is $\dfrac{c}{y^2}$.

17. For $y' + (\tan x)y = \sec x$ an integrating factor is $e^{\int \tan x\, dx} = \sec x$ so that $\dfrac{d}{dx}\left[(\sec x)\, y\right] = \sec^2 x$ and $y = \sin x + c\cos x$ for $-\pi/2 < x < \pi/2$. There is no transient term.

18. For $y' + (\cot x)y = \sec^2 x \csc x$ an integrating factor is $e^{\int \cot x\, dx} = e^{\ln|\sin x|} = \sin x$ so that $\dfrac{d}{dx}\left[(\sin x)\, y\right] = \sec^2 x$ and $y = \sec x + c\csc x$ for $0 < x < \pi/2$. There is no transient term.

19. For $y' + \dfrac{x+2}{x+1}\, y = \dfrac{2xe^{-x}}{x+1}$ an integrating factor is $e^{\int [(x+2)/(x+1)]dx} = (x+1)e^x$, so $\dfrac{d}{dx}\left[(x+1)e^x y\right] = 2x$ and $y = \dfrac{x^2}{x+1}\, e^{-x} + \dfrac{c}{x+1}\, e^{-x}$ for $-1 < x < \infty$. The entire solution is transient.

20. For $y' + \dfrac{4}{x+2}\, y = \dfrac{5}{(x+2)^2}$ an integrating factor is $e^{\int [4/(x+2)]dx} = (x+2)^4$ so that $\dfrac{d}{dx}\left[(x+2)^4 y\right] = 5(x+2)^2$ and $y = \dfrac{5}{3}(x+2)^{-1} + c(x+2)^{-4}$ for $-2 < x < \infty$. The entire solution is transient.

21. For $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$ an integrating factor is $e^{\int \sec\theta\, d\theta} = e^{\ln|\sec x + \tan x|} = \sec\theta + \tan\theta$ so that $\dfrac{d}{d\theta}\left[(\sec\theta + \tan\theta)r\right] = 1 + \sin\theta$ and $(\sec\theta + \tan\theta)r = \theta - \cos\theta + c$ for $-\pi/2 < \theta < \pi/2$.

22. For $\dfrac{dP}{dt} + (2t-1)P = 4t - 2$ an integrating factor is $e^{\int (2t-1)\, dt} = e^{t^2-t}$ so that $\dfrac{d}{dt}\left[e^{t^2-t}P\right] = (4t-2)e^{t^2-t}$ and $P = 2 + ce^{t-t^2}$ for $-\infty < t < \infty$. The transient term is ce^{t-t^2}.

23. For $y' + \left(3 + \dfrac{1}{x}\right) y = \dfrac{e^{-3x}}{x}$ an integrating factor is $e^{\int [3+(1/x)]dx} = xe^{3x}$ so that $\dfrac{d}{dx}\left[xe^{3x}y\right] = 1$ and $y = e^{-3x} + \dfrac{ce^{-3x}}{x}$ for $0 < x < \infty$. The entire solution is transient.

24. For $y' + \dfrac{2}{x^2-1}\, y = \dfrac{x+1}{x-1}$ an integrating factor is $e^{\int [2/(x^2-1)]dx} = \dfrac{x-1}{x+1}$ so that $\dfrac{d}{dx}\left[\dfrac{x-1}{x+1}y\right] = 1$ and $(x-1)y = x(x+1) + c(x+1)$ for $-1 < x < 1$.

25. For $y' - 5y = x$ an integrating factor is $e^{\int -5\,dx} = e^{-5x}$ so that $\dfrac{d}{dx}[e^{-5x}y] = xe^{-5x}$ and

$$y = e^{5x}\int xe^{-5x}dx = e^{5x}\left(-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + c\right) = -\frac{1}{5}x - \frac{1}{25} + ce^{5x}.$$

If $y(0) = 3$ then $c = \frac{1}{25}$ and $y = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}$. The solution is defined on $I = (-\infty, \infty)$.

26. For $y' + 3y = 2x$ an integrating factor is $e^{\int 3\,dx} = e^{3x}$ so that $\dfrac{d}{dx}[e^{3x}y] = 2xe^{3x}$ and

$$y = e^{-3x}\int 2xe^{3x}dx = e^{-3x}\left(\frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + c\right) = \frac{2}{3}x - \frac{2}{9} + ce^{-3x}.$$

If $y(0) = \frac{1}{3}$ then $c = \frac{5}{9}$ and $y = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}$. The solution is defined on $I = (-\infty, \infty)$.

27. For $y' + \dfrac{1}{x}\, y = \dfrac{1}{x}\, e^{x}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\dfrac{d}{dx}\,[xy] = e^{x}$ and $y = \dfrac{1}{x}e^{x} + \dfrac{c}{x}$ for $0 < x < \infty$. If $y(1) = 2$ then $c = 2 - e$ and $y = \dfrac{1}{x}e^{x} + \dfrac{2-e}{x}$. The solution is defined on $I = (0, \infty)$.

28. For $\dfrac{dx}{dy} - \dfrac{1}{y}\, x = 2y$ an integrating factor is $e^{-\int (1/y)dy} = \dfrac{1}{y}$ so that $\dfrac{d}{dy}\left[\dfrac{1}{y}x\right] = 2$ and $x = 2y^2 + cy$ for $0 < y < \infty$. If $y(1) = 5$ then $c = -\dfrac{49}{5}$ and $x = 2y^2 - \dfrac{49}{5}y$. The solution is defined on $I = (0, \infty)$.

29. For $\dfrac{di}{dt} + \dfrac{R}{L}\, i = \dfrac{E}{L}$ an integrating factor is $e^{\int (R/L)\,dt} = e^{Rt/L}$ so that $\dfrac{d}{dt}\left[e^{Rt/L}\, i\right] = \dfrac{E}{L}e^{Rt/L}$ and $i = \dfrac{E}{R} + ce^{-Rt/L}$ for $-\infty < t < \infty$. If $i(0) = i_0$ then $c = i_0 - E/R$ and $i = \dfrac{E}{R} + \left(i_0 - \dfrac{E}{R}\right) e^{-Rt/L}$. The solution is defined on $I = (-\infty, \infty)$.

30. For $\dfrac{dT}{dt} - kT = -T_m k$ an integrating factor is $e^{\int (-k)dt} = e^{-kt}$ so that $\dfrac{d}{dt}\,[e^{-kt}T] = -T_m ke^{-kt}$ and $T = T_m + ce^{kt}$ for $-\infty < t < \infty$.If $T(0) = T_0$ then $c = T_0 - T_m$ and $T = T_m + (T_0 - T_m)e^{kt}$. The solution is defined on $I = (-\infty, \infty)$.

31. For $y' + \dfrac{1}{x}y = 4 + \dfrac{1}{x}$ an integrating factor is $e^{\int(1/x)\,dx} = x$ so that $\dfrac{d}{dx}[xy] = 4x + 1$ and

$$y = \frac{1}{x}\int(4x+1)dx = \frac{1}{x}\left(2x^2 + x + c\right) = 2x + 1 + \frac{c}{x}.$$

If $y(1) = 8$ then $c = 5$ and $y = 2x + 1 + \dfrac{5}{x}$. The solution is defined on $I = (0, \infty)$.

32. For $y' + 4xy = x^3e^{x^2}$ an integrating factor is $e^{4x\,dx} = e^{2x^2}$ so that $\dfrac{d}{dx}[e^{2x^2}y] = x^3e^{3x^2}$ and

$$y = e^{-2x^2}\int x^3e^{3x^2}dx = e^{-2x^2}\left(\frac{1}{6}x^2e^{3x^2} - \frac{1}{18}e^{3x^2} + c\right) = \frac{1}{6}x^2e^{x^2} - \frac{1}{18}e^{x^2} + ce^{-2x^2}.$$

If $y(0) = -1$ then $c = -\frac{17}{18}$ and $y = \frac{1}{6}x^2e^{x^2} - \frac{1}{18}e^{x^2} - \frac{17}{18}e^{-2x^2}$. The solution is defined on $I = (-\infty, \infty)$.

33. For $y' + \dfrac{1}{x+1}y = \dfrac{\ln x}{x+1}$ an integrating factor is $e^{\int[1/(x+1)]dx} = x+1$ so that $\dfrac{d}{dx}[(x+1)y] = \ln x$ and

$$y = \frac{x}{x+1}\ln x - \frac{x}{x+1} + \frac{c}{x+1}$$

for $0 < x < \infty$. If $y(1) = 10$ then $c = 21$ and $y = \dfrac{x}{x+1}\ln x - \dfrac{x}{x+1} + \dfrac{21}{x+1}$. The solution is defined on $I = (0, \infty)$.

34. For $y' + \dfrac{1}{x+1}y = \dfrac{1}{x(x+1)}$ an integrating factor is $e^{\int[1/(x+1)]dx} = x+1$ so that $\dfrac{d}{dx}[(x+1)y] = \dfrac{1}{x}$ and

$$y = \frac{1}{x+1}\int\frac{1}{x}dx = \frac{1}{x+1}(\ln x + c) = \frac{\ln x}{x+1} + \frac{c}{x+1}.$$

If $y(e) = 1$ then $c = e$ and $y = \dfrac{\ln x}{x+1} + \dfrac{e}{x+1}$. The solution is defined on $I = (0, \infty)$.

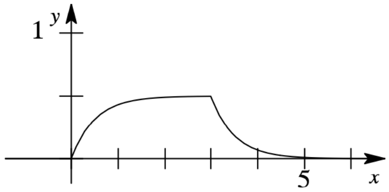
35. For $y' - (\sin x)y = 2\sin x$ an integrating factor is $e^{\int(-\sin x)\,dx} = e^{\cos x}$ so that $\dfrac{d}{dx}[e^{\cos x}y] = 2(\sin x)e^{\cos x}$ and

$$y = e^{-\cos x}\int 2(\sin x)e^{\cos x}dx = e^{-\cos x}\left(-2e^{\cos x} + c\right) = -2 + ce^{-\cos x}.$$

If $y(\pi/2) = 1$ then $c = 3$ and $y = -2 + 3e^{-\cos x}$. The solution is defined on $I = (-\infty, \infty)$.

36. For $y' + (\tan x)y = \cos^2 x$ an integrating factor is $e^{\int\tan x\,dx} = e^{\ln|\sec x|} = \sec x$ so that $\dfrac{d}{dx}[(\sec x)\,y] = \cos x$ and $y = \sin x\cos x + c\cos x$ for $-\pi/2 < x < \pi/2$. If $y(0) = -1$ then $c = -1$ and $y = \sin x\cos x - \cos x$. The solution is defined on $I = (-\pi/2, \pi/2)$.

37. For $y' + 2y = f(x)$ an integrating factor is e^{2x} so that

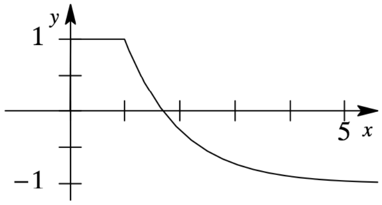


$$ye^{2x} = \begin{cases} \frac{1}{2}e^{2x} + c_1, & 0 \le x \le 3 \\ c_2, & x > 3. \end{cases}$$

If $y(0) = 0$ then $c_1 = -1/2$ and for continuity we must have $c_2 = \frac{1}{2}e^6 - \frac{1}{2}$ so that

$$y = \begin{cases} \frac{1}{2}(1 - e^{-2x}), & 0 \le x \le 3 \\ \frac{1}{2}(e^6 - 1)e^{-2x}, & x > 3. \end{cases}$$

38. For $y' + y = f(x)$ an integrating factor is e^x so that

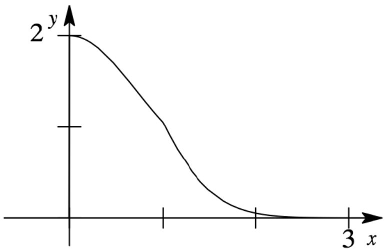


$$ye^{x} = \begin{cases} e^{x} + c_1, & 0 \le x \le 1 \\ -e^{x} + c_2, & x > 1. \end{cases}$$

If $y(0) = 1$ then $c_1 = 0$ and for continuity we must have $c_2 = 2e$ so that

$$y = \begin{cases} 1, & 0 \le x \le 1 \\ 2e^{1-x} - 1, & x > 1. \end{cases}$$

39. For $y' + 2xy = f(x)$ an integrating factor is e^{x^2} so that

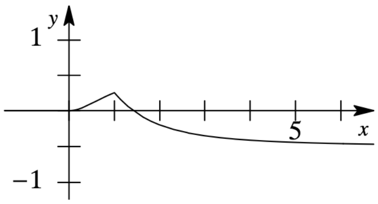


$$ye^{x^2} = \begin{cases} \frac{1}{2}e^{x^2} + c_1, & 0 \le x < 1 \\ c_2, & x \ge 1. \end{cases}$$

If $y(0) = 2$ then $c_1 = 3/2$ and for continuity we must have $c_2 = \frac{1}{2}e + \frac{3}{2}$ so that

$$y = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-x^2}, & 0 \le x < 1 \\ \left(\frac{1}{2}e + \frac{3}{2}\right)e^{-x^2}, & x \ge 1. \end{cases}$$

40. For



$$y' + \frac{2x}{1+x^2}\,y = \begin{cases} \dfrac{x}{1+x^2}, & 0 \le x \le 1 \\ \dfrac{-x}{1+x^2}, & x > 1, \end{cases}$$

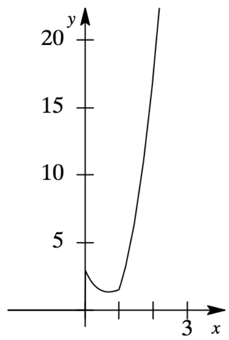
an integrating factor is $1 + x^2$ so that

$$\left(1 + x^2\right)y = \begin{cases} \dfrac{1}{2}x^2 + c_1, & 0 \le x \le 1 \\ -\dfrac{1}{2}x^2 + c_2, & x > 1. \end{cases}$$

If $y(0) = 0$ then $c_1 = 0$ and for continuity we must have $c_2 = 1$ so that

$$y = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2(1+x^2)}, & 0 \le x \le 1 \\ \dfrac{3}{2(1+x^2)} - \dfrac{1}{2}, & x > 1. \end{cases}$$

41. We first solve the initial-value problem $y' + 2y = 4x$, $y(0) = 3$ on the interval [0, 1]. The integrating factor is $e^{\int 2\,dx} = e^{2x}$, so



$$\begin{aligned} \frac{d}{dx}[e^{2x}y] &= 4xe^{2x} \\ e^{2x}y &= \int 4xe^{2x}dx = 2xe^{2x} - e^{2x} + c_1 \\ y &= 2x - 1 + c_1e^{-2x}. \end{aligned}$$

Using the initial condition, we find $y(0) = -1 + c_1 = 3$, so $c_1 = 4$ and $y = 2x - 1 + 4e^{-2x}$, $0 \le x \le 1$. Now, since $y(1) = 2 - 1 + 4e^{-2} = 1 + 4e^{-2}$, we solve the initial-value problem $y' - (2/x)y = 4x$, $y(1) = 1 + 4e^{-2}$ on the interval $(1, \infty)$. The integrating factor is $e^{\int(-2/x)dx} = e^{-2\ln x} = x^{-2}$, so

$$\begin{aligned} \frac{d}{dx}[x^{-2}y] &= 4xx^{-2} = \frac{4}{x} \\ x^{-2}y &= \int \frac{4}{x}\,dx = 4\ln x + c_2 \\ y &= 4x^2\ln x + c_2x^2. \end{aligned}$$

(We use $\ln x$ instead of $\ln|x|$ because $x > 1$.) Using the initial condition we find $y(1) = c_2 = 1 + 4e^{-2}$, so $y = 4x^2\ln x + (1 + 4e^{-2})x^2$, $x > 1$. Thus, the solution of the original initial-value problem is

$$y = \begin{cases} 2x - 1 + 4e^{-2x}, & 0 \le x \le 1 \\ 4x^2\ln x + (1 + 4e^{-2})x^2, & x > 1. \end{cases}$$

See Problem 48 in this section.

42. For $y' + e^x y = 1$ an integrating factor is e^{e^x}. Thus

$$\frac{d}{dx}\left[e^{e^x}y\right] = e^{e^x} \quad \text{and} \quad e^{e^x}y = \int_0^x e^{e^t}dt + c.$$

From $y(0) = 1$ we get $c = e$, so $y = e^{-e^x}\int_0^x e^{e^t}dt + e^{1-e^x}$.

When $y' + e^x y = 0$ we can separate variables and integrate:

$$\frac{dy}{y} = -e^x\,dx \quad \text{and} \quad \ln|y| = -e^x + c.$$

Thus $y = c_1e^{-e^x}$. From $y(0) = 1$ we get $c_1 = e$, so $y = e^{1-e^x}$.

When $y' + e^x y = e^x$ we can see by inspection that $y = 1$ is a solution.

43. An integrating factor for $y' - 2xy = 1$ is e^{-x^2}. Thus

$$\begin{aligned}\frac{d}{dx}[e^{-x^2}y] &= e^{-x^2}\\ e^{-x^2}y &= \int_0^x e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + c\\ y &= \frac{\sqrt{\pi}}{2}e^{x^2}\operatorname{erf}(x) + ce^{x^2}.\end{aligned}$$

From $y(1) = \dfrac{\sqrt{\pi}}{2}e\operatorname{erf}(1) + ce = 1$ we get $c = e^{-1} - \dfrac{\sqrt{\pi}}{2}\operatorname{erf}(1)$. The solution of the initial-value problem is

$$\begin{aligned}y &= \frac{\sqrt{\pi}}{2}e^{x^2}\operatorname{erf}(x) + \left(e^{-1} - \frac{\sqrt{\pi}}{2}\operatorname{erf}(1)\right)e^{x^2}\\ &= e^{x^2-1} + \frac{\sqrt{\pi}}{2}e^{x^2}\big(\operatorname{erf}(x) - \operatorname{erf}(1)\big).\end{aligned}$$

Discussion Problems

44. We want 4 to be a critical point, so we use $y' = 4 - y$.

45. **(a)** All solutions of the form $y = x^5e^x - x^4e^x + cx^4$ satisfy the initial condition. In this case, since $4/x$ is discontinuous at $x = 0$, the hypotheses of Theorem 1.2.1 are not satisfied and the initial-value problem does not have a unique solution.

(b) The differential equation has no solution satisfying $y(0) = y_0$, $y_0 > 0$.

(c) In this case, since $x_0 > 0$, Theorem 1.2.1 applies and the initial-value problem has a unique solution given by $y = x^5e^x - x^4e^x + cx^4$ where $c = y_0/x_0^4 - x_0e^{x_0} + e^{x_0}$.

46. On the interval $(-3, 3)$ the integrating factor is

$$e^{\int x\,dx/(x^2-9)} = e^{-\int x\,dx/(9-x^2)} = e^{\frac{1}{2}\ln(9-x^2)} = \sqrt{9-x^2},$$

and so

$$\frac{d}{dx}\left[\sqrt{9-x^2}\,y\right] = 0 \quad \text{and} \quad y = \frac{c}{\sqrt{9-x^2}}.$$

47. We want the general solution to be $y = 3x - 5 + ce^{-x}$. (Rather than e^{-x}, any function that approaches 0 as $x \to \infty$ could be used.) Differentiating we get

$$y' = 3 - ce^{-x} = 3 - (y - 3x + 5) = -y + 3x - 2,$$

so the differential equation $y' + y = 3x - 2$ has solutions asymptotic to the line $y = 3x - 5$.

48. The left-hand derivative of the function at $x = 1$ is $1/e$ and the right-hand derivative at $x = 1$ is $1 - 1/e$. Thus, y is not differentiable at $x = 1$.

49. **(a)** Differentiating $y_c = c/x^3$ we get

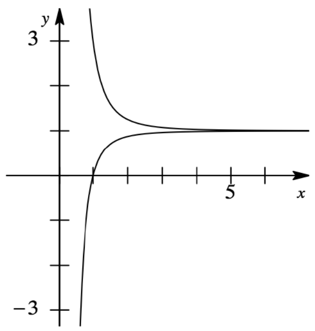
$$y_c' = -\frac{3c}{x^4} = -\frac{3}{x}\frac{c}{x^3} = -\frac{3}{x}\,y_c$$

so a differential equation with general solution $y_c = c/x^3$ is $xy' + 3y = 0$. Now

$$xy_p' + 3y_p = x(3x^2) + 3(x^3) = 6x^3,$$

so a differential equation with general solution $y = c/x^3 + x^3$ is $xy' + 3y = 6x^3$. This will be a general solution on $(0, \infty)$.

(b) Since $y(1) = 1^3 - 1/1^3 = 0$, an initial condition is $y(1) = 0$. Since $y(1) = 1^3 + 2/1^3 = 3$, an initial condition is $y(1) = 3$. In each case the interval of definition is $(0, \infty)$. The initial-value problem $xy' + 3y = 6x^3$, $y(0) = 0$ has solution $y = x^3$ for $-\infty < x < \infty$. In the figure the lower curve is the graph of $y(x) = x^3 - 1/x^3$, while the upper curve is the graph of $y = x^3 - 2/x^3$.



(c) The first two initial-value problems in part (b) are not unique. For example, setting $y(2) = 2^3 - 1/2^3 = 63/8$, we see that $y(2) = 63/8$ is also an initial condition leading to the solution $y = x^3 - 1/x^3$.

50. Since $e^{\int P(x)dx + c} = e^c e^{\int P(x)dx} = c_1 e^{\int P(x)dx}$, we would have

$$c_1 e^{\int P(x)dx} y = c_2 + \int c_1 e^{\int P(x)dx} f(x)\,dx \quad \text{and} \quad y = c_3 e^{-\int P(x)dx} + e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx,$$

which is the same as (4) in the text.

51. We see by inspection that $y = 0$ is a solution.

Mathematical Models

52. The solution of the first equation is $x = c_1 e^{-\lambda_1 t}$. From $x(0) = x_0$ we obtain $c_1 = x_0$ and so $x = x_0 e^{-\lambda_1 t}$. The second equation then becomes

$$\frac{dy}{dt} = x_0 \lambda_1 e^{-\lambda_1 t} - \lambda_2 y \qquad \text{or} \qquad \frac{dy}{dt} + \lambda_2 y = x_0 \lambda_1 e^{-\lambda_1 t},$$

which is linear. An integrating factor is $e^{\lambda_2 t}$. Thus

$$\frac{d}{dt}\left[e^{\lambda_2 t} y\right] = x_0 \lambda_1 e^{-\lambda_1 t} e^{\lambda_2 t} = x_0 \lambda_1 e^{(\lambda_2 - \lambda_1)t}$$

$$e^{\lambda_2 t} y = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{(\lambda_2 - \lambda_1)t} + c_2$$

$$y = \frac{x_0 \lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}.$$

From $y(0) = y_0$ we obtain $c_2 = (y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1)/(\lambda_2 - \lambda_1)$. The solution is

$$y = \frac{x_0\lambda_1}{\lambda_2 - \lambda_1}\, e^{-\lambda_1 t} + \frac{y_0\lambda_2 - y_0\lambda_1 - x_0\lambda_1}{\lambda_2 - \lambda_1}\, e^{-\lambda_2 t}.$$

53. Writing the differential equation as $\dfrac{dE}{dt} + \dfrac{1}{RC}\,E = 0$ we see that an integrating factor is $e^{t/RC}$. Then

$$\frac{d}{dt}[e^{t/RC}E] = 0$$
$$e^{t/RC}E = c$$
$$E = ce^{-t/RC}.$$

From $E(4) = ce^{-4/RC} = E_0$ we find $c = E_0 e^{4/RC}$. Thus, the solution of the initial-value problem is

$$E = E_0 e^{4/RC} e^{-t/RC} = E_0 e^{-(t-4)/RC}.$$

Computer Lab Assignments

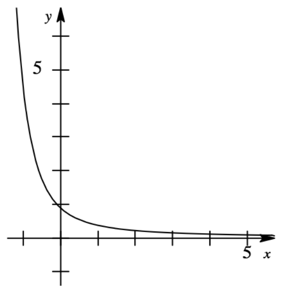
54. (a) An integrating factor for $y' - 2xy = -1$ is e^{-x^2}. Thus

$$\frac{d}{dx}[e^{-x^2}y] = -e^{-x^2}$$
$$e^{-x^2}y = -\int_0^x e^{-t^2}\,dt = -\frac{\sqrt{\pi}}{2}\,\text{erf}(x) + c.$$

From $y(0) = \sqrt{\pi}/2$, and noting that $\text{erf}(0) = 0$, we get $c = \sqrt{\pi}/2$. Thus

$$y = e^{x^2}\left(-\frac{\sqrt{\pi}}{2}\,\text{erf}(x) + \frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{2}\,e^{x^2}(1 - \text{erf}(x)) = \frac{\sqrt{\pi}}{2}\,e^{x^2}\text{erfc}(x).$$

(b) Using a CAS we find $y(2) \approx 0.226339$.



55. (a) An integrating factor for

$$y' + \frac{2}{x}\,y = \frac{10\sin x}{x^3}$$

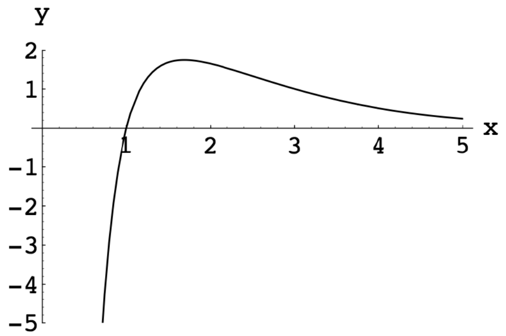
is x^2. Thus

$$\frac{d}{dx}\,[x^2y] = 10\frac{\sin x}{x}$$
$$x^2y = 10\int_0^x \frac{\sin t}{t}\,dt + c$$
$$y = 10x^{-2}\text{Si}(x) + cx^{-2}.$$

From $y(1) = 0$ we get $c = -10\,\mathrm{Si}(1)$. Thus

$$y = 10x^{-2}\,\mathrm{Si}(x) - 10x^{-2}\,\mathrm{Si}(1) = 10x^{-2}\big(\mathrm{Si}(x) - \mathrm{Si}(1)\big).$$

(b)



(c) From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for x we see that the absolute maximum is (1.688, 1.742).

56. (a) The integrating factor for $y' - (\sin x^2)y = 0$ is $e^{-\int_0^x \sin t^2\,dt}$. Then

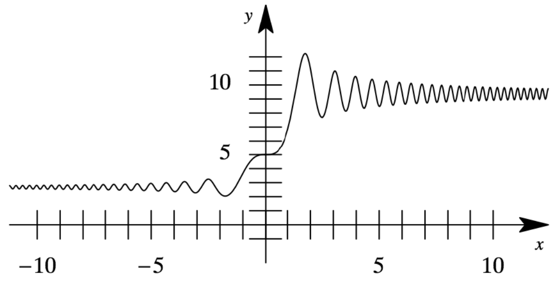
$$\frac{d}{dx}\left[e^{-\int_0^x \sin t^2 dt}y\right] = 0$$
$$e^{-\int_0^x \sin t^2\,dt}y = c_1$$
$$y = c_1 e^{\int_0^x \sin t^2 dt}.$$

Letting $t = \sqrt{\pi/2}\,u$ we have $dt = \sqrt{\pi/2}\,du$ and

$$\int_0^x \sin t^2\,dt = \sqrt{\frac{\pi}{2}}\int_0^{\sqrt{2/\pi}\,x} \sin\left(\frac{\pi}{2}u^2\right)du = \sqrt{\frac{\pi}{2}}\,S\left(\sqrt{\frac{2}{\pi}}\,x\right)$$

so $y = c_1 e^{\sqrt{\pi/2}\,S(\sqrt{2/\pi}\,x)}$. Using $S(0) = 0$ and $y(0) = c_1 = 5$ we have $y = 5e^{\sqrt{\pi/2}\,S(\sqrt{2/\pi}\,x)}$.

(b)



(c) From the graph we see that as $x \to \infty$, $y(x)$ oscillates with decreasing amplitudes approaching 9.35672. Since $\lim_{x\to\infty} 5S(x) = \frac{1}{2}$, $\lim_{x\to\infty} y(x) = 5e^{\sqrt{\pi/8}} \approx 9.357$, and since $\lim_{x\to-\infty} S(x) = -\frac{1}{2}$, $\lim_{x\to-\infty} y(x) = 5e^{-\sqrt{\pi/8}} \approx 2.672$.

(d) From the graph in part (b) we see that the absolute maximum occurs around $x = 1.7$ and the absolute minimum occurs around $x = -1.8$. Using the root-finding capability of a CAS and solving $y'(x) = 0$ for x, we see that the absolute maximum is (1.772, 12.235) and the absolute minimum is (−1.772, 2.044).

2.4 Exact Equations

1. Let $M = 2x - 1$ and $N = 3y + 7$ so that $M_y = 0 = N_x$. From $f_x = 2x - 1$ we obtain $f = x^2 - x + h(y)$, $h'(y) = 3y + 7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

2. Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

3. Let $M = 5x + 4y$ and $N = 4x - 8y^3$ so that $M_y = 4 = N_x$. From $f_x = 5x + 4y$ we obtain $f = \frac{5}{2}x^2 + 4xy + h(y)$, $h'(y) = -8y^3$, and $h(y) = -2y^4$. A solution is $\frac{5}{2}x^2 + 4xy - 2y^4 = c$.

4. Let $M = \sin y - y\sin x$ and $N = \cos x + x\cos y - y$ so that $M_y = \cos y - \sin x = N_x$. From $f_x = \sin y - y\sin x$ we obtain $f = x\sin y + y\cos x + h(y)$, $h'(y) = -y$, and $h(y) = -\frac{1}{2}y^2$. A solution is $x\sin y + y\cos x - \frac{1}{2}y^2 = c$.

5. Let $M = 2y^2x - 3$ and $N = 2yx^2 + 4$ so that $M_y = 4xy = N_x$. From $f_x = 2y^2x - 3$ we obtain $f = x^2y^2 - 3x + h(y)$, $h'(y) = 4$, and $h(y) = 4y$. A solution is $x^2y^2 - 3x + 4y = c$.

6. Let $M = 4x^3 - 3y\sin 3x - y/x^2$ and $N = 2y - 1/x + \cos 3x$ so that $M_y = -3\sin 3x - 1/x^2$ and $N_x = 1/x^2 - 3\sin 3x$. The equation is not exact.

7. Let $M = x^2 - y^2$ and $N = x^2 - 2xy$ so that $M_y = -2y$ and $N_x = 2x - 2y$. The equation is not exact.

8. Let $M = 1+\ln x+y/x$ and $N = -1+\ln x$ so that $M_y = 1/x = N_x$. From $f_y = -1+\ln x$ we obtain $f = -y+y\ln x+h(y)$, $h'(x) = 1+\ln x$, and $h(y) = x\ln x$. A solution is $-y+y\ln x+x\ln x = c$.

9. Let $M = y^3 - y^2\sin x - x$ and $N = 3xy^2 + 2y\cos x$ so that $M_y = 3y^2 - 2y\sin x = N_x$. From $f_x = y^3 - y^2\sin x - x$ we obtain $f = xy^3 + y^2\cos x - \frac{1}{2}x^2 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy^3 + y^2\cos x - \frac{1}{2}x^2 = c$.

10. Let $M = x^3 + y^3$ and $N = 3xy^2$ so that $M_y = 3y^2 = N_x$. From $f_x = x^3 + y^3$ we obtain $f = \frac{1}{4}x^4 + xy^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\frac{1}{4}x^4 + xy^3 = c$.

11. Let $M = y\ln y - e^{-xy}$ and $N = 1/y + x\ln y$ so that $M_y = 1 + \ln y + xe^{-xy}$ and $N_x = \ln y$. The equation is not exact.

12. Let $M = 3x^2y + e^y$ and $N = x^3 + xe^y - 2y$ so that $M_y = 3x^2 + e^y = N_x$. From $f_x = 3x^2y + e^y$ we obtain $f = x^3y + xe^y + h(y)$, $h'(y) = -2y$, and $h(y) = -y^2$. A solution is $x^3y + xe^y - y^2 = c$.

13. Let $M = y - 6x^2 - 2xe^x$ and $N = x$ so that $M_y = 1 = N_x$. From $f_x = y - 6x^2 - 2xe^x$ we obtain $f = xy - 2x^3 - 2xe^x + 2e^x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $xy - 2x^3 - 2xe^x + 2e^x = c$.

14. Let $M = 1 - 3/x + y$ and $N = 1 - 3/y + x$ so that $M_y = 1 = N_x$. From $f_x = 1 - 3/x + y$ we obtain $f = x - 3\ln|x| + xy + h(y)$, $h'(y) = 1 - \dfrac{3}{y}$, and $h(y) = y - 3\ln|y|$. A solution is $x + y + xy - 3\ln|xy| = c$.

15. Let $M = x^2y^3 - 1/\left(1 + 9x^2\right)$ and $N = x^3y^2$ so that $M_y = 3x^2y^2 = N_x$. From $f_x = x^2y^3 - 1/\left(1 + 9x^2\right)$ we obtain $f = \frac{1}{3}x^3y^3 - \frac{1}{3}\arctan(3x) + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $x^3y^3 - \arctan(3x) = c$.

16. Let $M = -2y$ and $N = 5y - 2x$ so that $M_y = -2 = N_x$. From $f_x = -2y$ we obtain $f = -2xy + h(y)$, $h'(y) = 5y$, and $h(y) = \frac{5}{2}y^2$. A solution is $-2xy + \frac{5}{2}y^2 = c$.

17. Let $M = \tan x - \sin x \sin y$ and $N = \cos x \cos y$ so that $M_y = -\sin x \cos y = N_x$. From $f_x = \tan x - \sin x \sin y$ we obtain $f = \ln|\sec x| + \cos x \sin y + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $\ln|\sec x| + \cos x \sin y = c$.

18. Let $M = 2y\sin x \cos x - y + 2y^2e^{xy^2}$ and $N = -x + \sin^2 x + 4xye^{xy^2}$ so that

$$M_y = 2\sin x \cos x - 1 + 4xy^3e^{xy^2} + 4ye^{xy^2} = N_x.$$

From $f_x = 2y\sin x\cos x - y + 2y^2e^{xy^2}$ we obtain $f = y\sin^2 x - xy + 2e^{xy^2} + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution is $y\sin^2 x - xy + 2e^{xy^2} = c$.

19. Let $M = 4t^3y - 15t^2 - y$ and $N = t^4 + 3y^2 - t$ so that $M_y = 4t^3 - 1 = N_t$. From $f_t = 4t^3y - 15t^2 - y$ we obtain $f = t^4y - 5t^3 - ty + h(y)$, $h'(y) = 3y^2$, and $h(y) = y^3$. A solution is $t^4y - 5t^3 - ty + y^3 = c$.

20. Let $M = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ and $N = ye^y + t/\left(t^2 + y^2\right)$ so that $M_y = \left(y^2 - t^2\right)/\left(t^2 + y^2\right)^2 = N_t$. From $f_t = 1/t + 1/t^2 - y/\left(t^2 + y^2\right)$ we obtain $f = \ln|t| - \dfrac{1}{t} - \arctan\left(\dfrac{t}{y}\right) + h(y)$, $h'(y) = ye^y$, and $h(y) = ye^y - e^y$. A solution is

$$\ln|t| - \frac{1}{t} - \arctan\left(\frac{t}{y}\right) + ye^y - e^y = c.$$

21. Let $M = x^2 + 2xy + y^2$ and $N = 2xy + x^2 - 1$ so that $M_y = 2(x + y) = N_x$. From $f_x = x^2 + 2xy + y^2$ we obtain $f = \frac{1}{3}x^3 + x^2y + xy^2 + h(y)$, $h'(y) = -1$, and $h(y) = -y$. The solution is $\frac{1}{3}x^3 + x^2y + xy^2 - y = c$. If $y(1) = 1$ then $c = 4/3$ and a solution of the initial-value problem is $\frac{1}{3}x^3 + x^2y + xy^2 - y = \frac{4}{3}$.

22. Let $M = e^x + y$ and $N = 2 + x + ye^y$ so that $M_y = 1 = N_x$. From $f_x = e^x + y$ we obtain $f = e^x + xy + h(y)$, $h'(y) = 2 + ye^y$, and $h(y) = 2y + ye^y - y$. The solution is $e^x + xy + 2y + ye^y - e^y = c$. If $y(0) = 1$ then $c = 3$ and a solution of the initial-value problem is $e^x + xy + 2y + ye^y - e^y = 3$.

23. Let $M = 4y + 2t - 5$ and $N = 6y + 4t - 1$ so that $M_y = 4 = N_t$. From $f_t = 4y + 2t - 5$ we obtain $f = 4ty + t^2 - 5t + h(y)$, $h'(y) = 6y - 1$, and $h(y) = 3y^2 - y$. The solution is $4ty + t^2 - 5t + 3y^2 - y = c$. If $y(-1) = 2$ then $c = 8$ and a solution of the initial-value problem is $4ty + t^2 - 5t + 3y^2 - y = 8$.

24. Let $M = t/2y^4$ and $N = \left(3y^2 - t^2\right)/y^5$ so that $M_y = -2t/y^5 = N_t$. From $f_t = t/2y^4$ we obtain $f = \dfrac{t^2}{4y^4} + h(y)$, $h'(y) = \dfrac{3}{y^3}$, and $h(y) = -\dfrac{3}{2y^2}$. The solution is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = c$. If $y(1) = 1$ then $c = -5/4$ and a solution of the initial-value problem is $\dfrac{t^2}{4y^4} - \dfrac{3}{2y^2} = -\dfrac{5}{4}$.

25. Let $M = y^2 \cos x - 3x^2 y - 2x$ and $N = 2y \sin x - x^3 + \ln y$ so that $M_y = 2y \cos x - 3x^2 = N_x$. From $f_x = y^2 \cos x - 3x^2 y - 2x$ we obtain $f = y^2 \sin x - x^3 y - x^2 + h(y)$, $h'(y) = \ln y$, and $h(y) = y \ln y - y$. The solution is $y^2 \sin x - x^3 y - x^2 + y \ln y - y = c$. If $y(0) = e$ then $c = 0$ and a solution of the initial-value problem is $y^2 \sin x - x^3 y - x^2 + y \ln y - y = 0$.

26. Let $M = y^2 + y \sin x$ and $N = 2xy - \cos x - 1/\left(1 + y^2\right)$ so that $M_y = 2y + \sin x = N_x$. From $f_x = y^2 + y \sin x$ we obtain $f = xy^2 - y \cos x + h(y)$, $h'(y) = -1/(1+y^2)$, and $h(y) = -\tan^{-1} y$. The solution is $xy^2 - y \cos x - \tan^{-1} y = c$. If $y(0) = 1$ then $c = -1 - \pi/4$ and a solution of the initial-value problem is $xy^2 - y \cos x - \tan^{-1} y = -1 - \pi/4$.

27. Equating $M_y = 3y^2 + 4kxy^3$ and $N_x = 3y^2 + 40xy^3$ we obtain $k = 10$.

28. Equating $M_y = 18xy^2 - \sin y$ and $N_x = 4kxy^2 - \sin y$ we obtain $k = 9/2$.

29. Let $M = -x^2 y^2 \sin x + 2xy^2 \cos x$ and $N = 2x^2 y \cos x$ so that $M_y = -2x^2 y \sin x + 4xy \cos x = N_x$. From $f_y = 2x^2 y \cos x$ we obtain $f = x^2 y^2 \cos x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $x^2 y^2 \cos x = c$.

30. Let $M = \left(x^2 + 2xy - y^2\right)/\left(x^2 + 2xy + y^2\right)$ and $N = \left(y^2 + 2xy - x^2\right)/\left(y^2 + 2xy + x^2\right)$ so that $M_y = -4xy/(x+y)^3 = N_x$. From $f_x = \left(x^2 + 2xy + y^2 - 2y^2\right)/(x+y)^2$ we obtain $f = x + \dfrac{2y^2}{x+y} + h(y)$, $h'(y) = -1$, and $h(y) = -y$. A solution of the differential equation is $x^2 + y^2 = c(x+y)$.

31. We note that $(M_y - N_x)/N = 1/x$, so an integrating factor is $e^{\int dx/x} = x$. Let $M = 2xy^2 + 3x^2$ and $N = 2x^2 y$ so that $M_y = 4xy = N_x$. From $f_x = 2xy^2 + 3x^2$ we obtain $f = x^2 y^2 + x^3 + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $x^2 y^2 + x^3 = c$.

32. We note that $(M_y - N_x)/N = 1$, so an integrating factor is $e^{\int dx} = e^x$. Let $M = xye^x + y^2 e^x + ye^x$ and $N = xe^x + 2ye^x$ so that $M_y = xe^x + 2ye^x + e^x = N_x$. From $f_y = xe^x + 2ye^x$ we obtain $f = xye^x + y^2 e^x + h(x)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $xye^x + y^2 e^x = c$.

33. We note that $(N_x - M_y)/M = 2/y$, so an integrating factor is $e^{\int 2dy/y} = y^2$. Let $M = 6xy^3$ and $N = 4y^3 + 9x^2 y^2$ so that $M_y = 18xy^2 = N_x$. From $f_x = 6xy^3$ we obtain $f = 3x^2 y^3 + h(y)$, $h'(y) = 4y^3$, and $h(y) = y^4$. A solution of the differential equation is $3x^2 y^3 + y^4 = c$.

34. We note that $(M_y - N_x)/N = -\cot x$, so an integrating factor is $e^{-\int \cot x\, dx} = \csc x$. Let $M = \cos x \csc x = \cot x$ and $N = (1 + 2/y)\sin x \csc x = 1 + 2/y$, so that $M_y = 0 = N_x$. From $f_x = \cot x$ we obtain $f = \ln(\sin x) + h(y)$, $h'(y) = 1 + 2/y$, and $h(y) = y + \ln y^2$. A solution of the differential equation is $\ln(\sin x) + y + \ln y^2 = c$.

35. We note that $(M_y - N_x)/N = 3$, so an integrating factor is $e^{\int 3\, dx} = e^{3x}$. Let

$$M = (10 - 6y + e^{-3x})e^{3x} = 10e^{3x} - 6ye^{3x} + 1 \qquad \text{and} \qquad N = -2e^{3x},$$

so that $M_y = -6e^{3x} = N_x$. From $f_x = 10e^{3x} - 6ye^{3x} + 1$ we obtain $f = \frac{10}{3}e^{3x} - 2ye^{3x} + x + h(y)$, $h'(y) = 0$, and $h(y) = 0$. A solution of the differential equation is $\frac{10}{3}e^{3x} - 2ye^{3x} + x = c$.

36. We note that $(N_x - M_y)/M = -3/y$, so an integrating factor is $e^{-3\int dy/y} = 1/y^3$. Let

$$M = (y^2 + xy^3)/y^3 = 1/y + x \quad \text{and} \quad N = (5y^2 - xy + y^3 \sin y)/y^3 = 5/y - x/y^2 + \sin y,$$

so that $M_y = -1/y^2 = N_x$. From $f_x = 1/y + x$ we obtain $f = x/y + \frac{1}{2}x^2 + h(y)$, $h'(y) = 5/y + \sin y$, and $h(y) = 5\ln|y| - \cos y$. A solution of the differential equation is $x/y + \frac{1}{2}x^2 + 5\ln|y| - \cos y = c$.

37. We note that $(M_y - N_x)/N = 2x/(4 + x^2)$, so an integrating factor is $e^{-2\int x\, dx/(4+x^2)} = 1/(4 + x^2)$. Let $M = x/(4 + x^2)$ and $N = (x^2y + 4y)/(4 + x^2) = y$, so that $M_y = 0 = N_x$. From $f_x = x(4 + x^2)$ we obtain $f = \frac{1}{2}\ln(4 + x^2) + h(y)$, $h'(y) = y$, and $h(y) = \frac{1}{2}y^2$. A solution of the differential equation is $\frac{1}{2}\ln(4 + x^2) + \frac{1}{2}y^2 = c$. Multiplying both sides by 2 and then exponentiating we find $e^{y^2}(4 + x^2) = c_1$. Using the initial condition $y(4) = 0$ we see that $c_1 = 20$ and the solution of the initial-value problem is $e^{y^2}(4 + x^2) = 20$.

38. We note that $(M_y - N_x)/N = -3/(1 + x)$, so an integrating factor is $e^{-3\int dx/(1+x)} = 1/(1 + x)^3$. Let $M = (x^2 + y^2 - 5)/(1 + x)^3$ and $N = -(y + xy)/(1 + x)^3 = -y/(1 + x)^2$, so that $M_y = 2y/(1 + x)^3 = N_x$. From $f_y = -y/(1 + x)^2$ we obtain $f = -\frac{1}{2}y^2/(1 + x)^2 + h(x)$, $h'(x) = (x^2 - 5)/(1 + x)^3$, and $h(x) = 2/(1 + x)^2 + 2/(1 + x) + \ln|1 + x|$. A solution of the differential equation is

$$-\frac{y^2}{2(1+x)^2} + \frac{2}{(1+x)^2} + \frac{2}{(1+x)} + \ln|1+x| = c.$$

Using the initial condition $y(0) = 1$ we see that $c = \frac{7}{2}$ and the solution of the initial-value problem is

$$-\frac{y^2}{2(1+x)^2} + \frac{2}{(1+x)^2} + \frac{2}{(1+x)} + \ln|1+x| = \frac{7}{2}.$$

39. (a) Implicitly differentiating $x^3 + 2x^2y + y^2 = c$ and solving for dy/dx we obtain

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy + 2y\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} = -\frac{3x^2 + 4xy}{2x^2 + 2y}.$$

By writing the last equation in differential form we get $(4xy + 3x^2)dx + (2y + 2x^2)dy = 0$.

(b) Setting $x = 0$ and $y = -2$ in $x^3 + 2x^2y + y^2 = c$ we find $c = 4$, and setting $x = y = 1$ we also find $c = 4$. Thus, both initial conditions determine the same implicit solution.

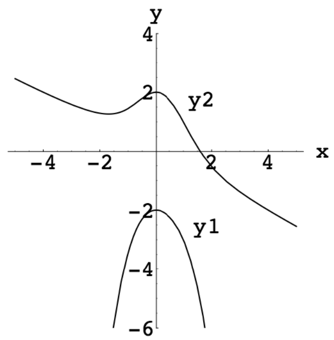
(c) Solving $x^3 + 2x^2y + y^2 = 4$ for y we get

$$y_1(x) = -x^2 - \sqrt{4 - x^3 + x^4}$$

and

$$y_2(x) = -x^2 + \sqrt{4 - x^3 + x^4}\,.$$

Observe in the figure that $y_1(0) = -2$ and $y_2(1) = 1$.



Discussion Problems

40. To see that the equations are not equivalent consider $dx = -(x/y)dy$. An integrating factor is $\mu(x, y) = y$ resulting in $y\,dx + x\,dy = 0$. A solution of the latter equation is $y = 0$, but this is not a solution of the original equation.

41. The explicit solution is $y = \sqrt{(3 + \cos^2 x)/(1 - x^2)}$. Since $3 + \cos^2 x > 0$ for all x we must have $1 - x^2 > 0$ or $-1 < x < 1$. Thus, the interval of definition is $(-1, 1)$.

42. **(a)** Since $f_y = N(x, y) = xe^{xy} + 2xy + \dfrac{1}{x}$ we obtain $f = e^{xy} + xy^2 + \dfrac{y}{x} + h(x)$ so that $f_x = ye^{xy} + y^2 - \dfrac{y}{x^2} + h'(x)$. Let $M(x, y) = ye^{xy} + y^2 - \dfrac{y}{x^2}$.

(b) Since $f_x = M(x, y) = y^{1/2}x^{-1/2} + x\left(x^2 + y\right)^{-1}$ we obtain $f = 2y^{1/2}x^{1/2} + \frac{1}{2}\ln\left|x^2 + y\right| + g(y)$ so that $f_y = y^{-1/2}x^{1/2} + \frac{1}{2}\left(x^2 + y\right)^{-1} + g'(x)$. Let $N(x, y) = y^{-1/2}x^{1/2} + \frac{1}{2}\left(x^2 + y\right)^{-1}$.

43. First note that

$$d\left(\sqrt{x^2 + y^2}\right) = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy.$$

Then $x\,dx + y\,dy = \sqrt{x^2 + y^2}\,dx$ becomes

$$\frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy = d\left(\sqrt{x^2 + y^2}\right) = dx.$$

The left side is the total differential of $\sqrt{x^2 + y^2}$ and the right side is the total differential of $x + c$. Thus $\sqrt{x^2 + y^2} = x + c$ is a solution of the differential equation.

44. To see that the statement is true, write the separable equation as $-g(x)\,dx + dy/h(y) = 0$. Identifying $M = -g(x)$ and $N = 1/h(y)$, we see that $M_y = 0 = N_x$, so the differential equation is exact.

Mathematical Model

45. (a) In differential form we have $(v^2-32x)dx+xv\,dv=0$. This is not an exact form, but $\mu(x)=x$ is an integrating factor. Multiplying by x we get $(xv^2-32x^2)dx+x^2v\,dv=0$. This form is the total differential of $u=\frac{1}{2}x^2v^2-\frac{32}{3}x^3$, so an implicit solution is $\frac{1}{2}x^2v^2-\frac{32}{3}x^3=c$. Letting $x=3$ and $v=0$ we find $c=-288$. Solving for v we get

$$v=8\sqrt{\frac{x}{3}-\frac{9}{x^2}}\,.$$

(b) The chain leaves the platform when $x=8$, so the velocity at this time is

$$v(8)=8\sqrt{\frac{8}{3}-\frac{9}{64}}\approx 12.7\text{ ft/s}.$$

Computer Lab Assignments

46. (a) Letting

$$M(x,y)=\frac{2xy}{(x^2+y^2)^2}\qquad\text{and}\qquad N(x,y)=1+\frac{y^2-x^2}{(x^2+y^2)^2}$$

we compute

$$M_y=\frac{2x^3-8xy^2}{(x^2+y^2)^3}=N_x,$$

so the differential equation is exact. Then we have

$$\frac{\partial f}{\partial x}=M(x,y)=\frac{2xy}{(x^2+y^2)^2}=2xy(x^2+y^2)^{-2}$$
$$f(x,y)=-y(x^2+y^2)^{-1}+g(y)=-\frac{y}{x^2+y^2}+g(y)$$
$$\frac{\partial f}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}+g'(y)=N(x,y)=1+\frac{y^2-x^2}{(x^2+y^2)^2}\,.$$

Thus, $g'(y)=1$ and $g(y)=y$. The solution is $y-\dfrac{y}{x^2+y^2}=c$. When $c=0$ the solution is $x^2+y^2=1$.

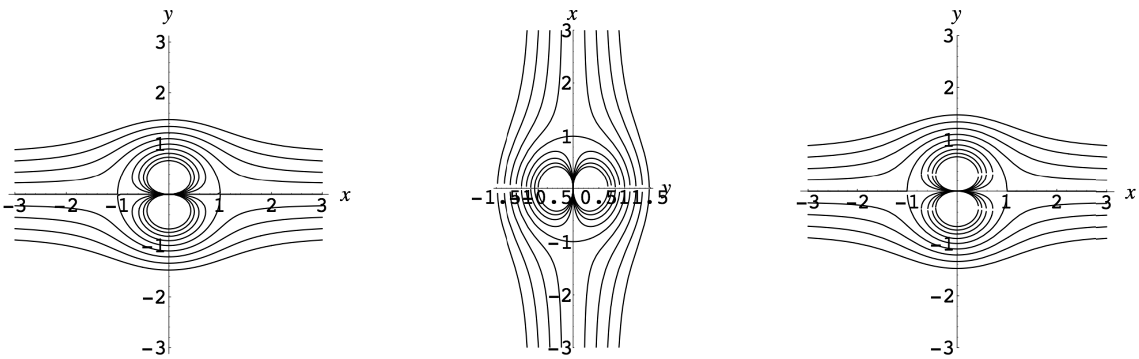
(b) The first graph below is obtained in *Mathematica* using $f(x,y)=y-y/(x^2+y^2)$ and

```
ContourPlot[f[x, y], { x, -3, 3}, { y, -3, 3},
  Axes->True, AxesOrigin->{0, 0}, AxesLabel->{ x, y},
  Frame->False, PlotPoints->100, ContourShading->False,
  Contours->{ 0, -0.2, 0.2, -0.4, 0.4, -0.6, 0.6, -0.8, 0.8} ]
```

The second graph uses

$$x=-\sqrt{\frac{y^3-cy^2-y}{c-y}}\qquad\text{and}\qquad x=\sqrt{\frac{y^3-cy^2-y}{c-y}}\,.$$

In this case the x-axis is vertical and the y-axis is horizontal. To obtain the third graph, we solve $y-y/(x^2+y^2)=c$ for y in a CAS. This appears to give one real and two complex solutions. When graphed in *Mathematica* however, all three solutions contribute to the graph. This is because the solutions involve the square root of expressions containing c. For some values of c the expression is negative, causing an apparent complex solution to actually be real.



2.5 Solutions by Substitutions

1. Letting $y=ux$ we have

$$\begin{aligned} (x-ux)\,dx + x(u\,dx + x\,du) &= 0 \\ dx + x\,du &= 0 \\ \frac{dx}{x} + du &= 0 \\ \ln|x| + u &= c \\ x\ln|x| + y &= cx. \end{aligned}$$

2. Letting $y=ux$ we have

$$\begin{aligned} (x+ux)\,dx + x(u\,dx + x\,du) &= 0 \\ (1+2u)\,dx + x\,du &= 0 \\ \frac{dx}{x} + \frac{du}{1+2u} &= 0 \\ \ln|x| + \frac{1}{2}\ln|1+2u| &= c \\ x^2\left(1+2\frac{y}{x}\right) &= c_1 \\ x^2+2xy &= c_1. \end{aligned}$$

3. Letting $x = vy$ we have

$$\begin{aligned}
vy(v\,dy + y\,dv) + (y - 2vy)\,dy &= 0 \\
vy^2\,dv + y\left(v^2 - 2v + 1\right)dy &= 0 \\
\frac{v\,dv}{(v-1)^2} + \frac{dy}{y} &= 0 \\
\ln|v-1| - \frac{1}{v-1} + \ln|y| &= c \\
\ln\left|\frac{x}{y} - 1\right| - \frac{1}{x/y - 1} + \ln y &= c \\
(x-y)\ln|x-y| - y &= c(x-y).
\end{aligned}$$

4. Letting $x = vy$ we have

$$\begin{aligned}
y(v\,dy + y\,dv) - 2(vy + y)\,dy &= 0 \\
y\,dv - (v+2)\,dy &= 0 \\
\frac{dv}{v+2} - \frac{dy}{y} &= 0 \\
\ln|v+2| - \ln|y| &= c \\
\ln\left|\frac{x}{y} + 2\right| - \ln|y| &= c \\
x + 2y &= c_1 y^2.
\end{aligned}$$

5. Letting $y = ux$ we have

$$\begin{aligned}
\left(u^2x^2 + ux^2\right)dx - x^2(u\,dx + x\,du) &= 0 \\
u^2\,dx - x\,du &= 0 \\
\frac{dx}{x} - \frac{du}{u^2} &= 0 \\
\ln|x| + \frac{1}{u} &= c \\
\ln|x| + \frac{x}{y} &= c \\
y\ln|x| + x &= cy.
\end{aligned}$$

6. Letting $y = ux$ and using partial fractions, we have

$$\begin{aligned}
\left(u^2x^2 + ux^2\right) dx + x^2(u\,dx + x\,du) &= 0 \\
x^2\left(u^2 + 2u\right) dx + x^3\,du &= 0 \\
\frac{dx}{x} + \frac{du}{u(u+2)} &= 0 \\
\ln|x| + \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| &= c \\
\frac{x^2u}{u+2} &= c_1 \\
x^2\frac{y}{x} &= c_1\left(\frac{y}{x} + 2\right) \\
x^2y &= c_1(y + 2x).
\end{aligned}$$

7. Letting $y = ux$ we have

$$\begin{aligned}
(ux - x)\,dx - (ux + x)(u\,dx + x\,du) &= 0 \\
\left(u^2 + 1\right) dx + x(u+1)\,du &= 0 \\
\frac{dx}{x} + \frac{u+1}{u^2+1}\,du &= 0 \\
\ln|x| + \frac{1}{2}\ln\left(u^2+1\right) + \tan^{-1}u &= c \\
\ln x^2\left(\frac{y^2}{x^2} + 1\right) + 2\tan^{-1}\frac{y}{x} &= c_1 \\
\ln\left(x^2 + y^2\right) + 2\tan^{-1}\frac{y}{x} &= c_1.
\end{aligned}$$

8. Letting $y = ux$ we have

$$\begin{aligned}
(x + 3ux)\,dx - (3x + ux)(u\,dx + x\,du) &= 0 \\
\left(u^2 - 1\right) dx + x(u+3)\,du &= 0 \\
\frac{dx}{x} + \frac{u+3}{(u-1)(u+1)}\,du &= 0 \\
\ln|x| + 2\ln|u-1| - \ln|u+1| &= c \\
\frac{x(u-1)^2}{u+1} &= c_1 \\
x\left(\frac{y}{x} - 1\right)^2 &= c_1\left(\frac{y}{x} + 1\right) \\
(y - x)^2 &= c_1(y + x).
\end{aligned}$$

9. Letting $y = ux$ we have

$$\begin{aligned}
-ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0 \\
(x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0 \\
\left(u^{-3/2} + \frac{1}{u}\right) du + \frac{dx}{x} &= 0 \\
-2u^{-1/2} + \ln|u| + \ln|x| &= c \\
\ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c \\
y(\ln|y| - c)^2 &= 4x.
\end{aligned}$$

10. Letting $y = ux$ we have

$$\begin{aligned}
\left(ux + \sqrt{x^2 - (ux)^2}\right) dx - x(udx + xdu)\,du &= 0 \\
\sqrt{x^2 - u^2x^2}\,dx - x^2\,du &= 0 \\
x\sqrt{1 - u^2}\,dx - x^2\,du &= 0, \quad (x > 0) \\
\frac{dx}{x} - \frac{du}{\sqrt{1 - u^2}} &= 0 \\
\ln x - \sin^{-1} u &= c \\
\sin^{-1} u &= \ln x + c_1 \\
\sin^{-1} \frac{y}{x} &= \ln x + c_2 \\
\frac{y}{x} &= \sin(\ln x + c_2) \\
y &= x\sin(\ln x + c_2).
\end{aligned}$$

See Problem 33 in this section for an analysis of the solution.

11. Letting $y = ux$ we have

$$\begin{aligned}
\left(x^3 - u^3x^3\right) dx + u^2x^3(u\,dx + x\,du) &= 0 \\
dx + u^2x\,du &= 0 \\
\frac{dx}{x} + u^2\,du &= 0 \\
\ln|x| + \frac{1}{3}u^3 &= c \\
3x^3\ln|x| + y^3 &= c_1x^3.
\end{aligned}$$

Using $y(1) = 2$ we find $c_1 = 8$. The solution of the initial-value problem is $3x^3\ln|x| + y^3 = 8x^3$.

12. Letting $y = ux$ we have

$$\begin{aligned}(x^2 + 2u^2x^2)dx - ux^2(u\,dx + x\,du) &= 0\\ x^2(1+u^2)dx - ux^3\,du &= 0\\ \frac{dx}{x} - \frac{u\,du}{1+u^2} &= 0\\ \ln|x| - \frac{1}{2}\ln(1+u^2) &= c\\ \frac{x^2}{1+u^2} &= c_1\\ x^4 &= c_1(x^2+y^2).\end{aligned}$$

Using $y(-1) = 1$ we find $c_1 = 1/2$. The solution of the initial-value problem is $2x^4 = y^2 + x^2$.

13. Letting $y = ux$ we have

$$\begin{aligned}(x + uxe^u)\,dx - xe^u(u\,dx + x\,du) &= 0\\ dx - xe^u\,du &= 0\\ \frac{dx}{x} - e^u\,du &= 0\\ \ln|x| - e^u &= c\\ \ln|x| - e^{y/x} &= c.\end{aligned}$$

Using $y(1) = 0$ we find $c = -1$. The solution of the initial-value problem is $\ln|x| = e^{y/x} - 1$.

14. Letting $x = vy$ we have

$$\begin{aligned}y(v\,dy + y\,dv) + vy(\ln vy - \ln y - 1)\,dy &= 0\\ y\,dv + v\ln v\,dy &= 0\\ \frac{dv}{v\ln v} + \frac{dy}{y} &= 0\\ \ln|\ln|v|| + \ln|y| &= c\\ y\ln\left|\frac{x}{y}\right| &= c_1.\end{aligned}$$

Using $y(1) = e$ we find $c_1 = -e$. The solution of the initial-value problem is $y\ln\left|\dfrac{x}{y}\right| = -e$.

15. From $y' + \dfrac{1}{x}y = \dfrac{1}{x}y^{-2}$ and $w = y^3$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{x}w = \dfrac{3}{x}$. An integrating factor is x^3 so that $x^3w = x^3 + c$ or $y^3 = 1 + cx^{-3}$.

16. From $y' - y = e^xy^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + w = -e^x$. An integrating factor is e^x so that $e^xw = -\frac{1}{2}e^{2x} + c$ or $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$.

17. From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} - 3w = -3x$. An integrating factor is e^{-3x} so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

18. From $y' - \left(1 + \dfrac{1}{x}\right) y = y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + \left(1 + \dfrac{1}{x}\right) w = -1$. An integrating factor is xe^x so that $xe^x w = -xe^x + e^x + c$ or $y^{-1} = -1 + \dfrac{1}{x} + \dfrac{c}{x}e^{-x}$.

19. From $y' - \dfrac{1}{t}y = -\dfrac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dt} + \dfrac{1}{t}w = \dfrac{1}{t^2}$. An integrating factor is t so that $tw = \ln t + c$ or $y^{-1} = \dfrac{1}{t}\ln t + \dfrac{c}{t}$. Writing this in the form $\dfrac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1 t$.

20. From $y' + \dfrac{2}{3(1+t^2)}y = \dfrac{2t}{3(1+t^2)}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dt} - \dfrac{2t}{1+t^2}w = \dfrac{-2t}{1+t^2}$. An integrating factor is $\dfrac{1}{1+t^2}$ so that $\dfrac{w}{1+t^2} = \dfrac{1}{1+t^2} + c$ or $y^{-3} = 1 + c\left(1+t^2\right)$.

21. From $y' - \dfrac{2}{x}y = \dfrac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} + \dfrac{6}{x}w = -\dfrac{9}{x^2}$. An integrating factor is x^6 so that $x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.

22. From $y' + y = y^{-1/2}$ and $w = y^{3/2}$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{2}w = \dfrac{3}{2}$. An integrating factor is $e^{3x/2}$ so that $e^{3x/2}w = e^{3x/2} + c$ or $y^{3/2} = 1 + ce^{-3x/2}$. If $y(0) = 4$ then $c = 7$ and $y^{3/2} = 1 + 7e^{-3x/2}$.

23. Let $u = x + y + 1$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = u^2$ or $\dfrac{1}{1+u^2}\,du = dx$. Thus $\tan^{-1} u = x + c$ or $u = \tan(x + c)$, and $x + y + 1 = \tan(x + c)$ or $y = \tan(x + c) - x - 1$.

24. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \dfrac{1-u}{u}$ or $u\,du = dx$. Thus $\frac{1}{2}u^2 = x + c$ or $u^2 = 2x + c_1$, and $(x + y)^2 = 2x + c_1$.

25. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \tan^2 u$ or $\cos^2 u\,du = dx$. Thus $\frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$ or $2u + \sin 2u = 4x + c_1$, and $2(x + y) + \sin 2(x + y) = 4x + c_1$ or $2y + \sin 2(x + y) = 2x + c_1$.

26. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \sin u$ or $\dfrac{1}{1 + \sin u}\,du = dx$. Multiplying by $(1 - \sin u)/(1 - \sin u)$ we have $\dfrac{1 - \sin u}{\cos^2 u}\,du = dx$ or $(\sec^2 u - \sec u \tan u)du = dx$. Thus $\tan u - \sec u = x + c$ or $\tan(x + y) - \sec(x + y) = x + c$.

27. Let $u = y - 2x + 3$ so that $du/dx = dy/dx - 2$. Then $\dfrac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\dfrac{1}{\sqrt{u}}\,du = dx$. Thus $2\sqrt{u} = x + c$ and $2\sqrt{y - 2x + 3} = x + c$.

28. Let $u = y - x + 5$ so that $du/dx = dy/dx - 1$. Then $\dfrac{du}{dx} + 1 = 1 + e^u$ or $e^{-u}du = dx$. Thus $-e^{-u} = x + c$ and $-e^{y-x+5} = x + c$.

29. Let $u = x + y$ so that $du/dx = 1 + dy/dx$. Then $\dfrac{du}{dx} - 1 = \cos u$ and $\dfrac{1}{1+\cos u}\,du = dx$. Now

$$\frac{1}{1+\cos u} = \frac{1-\cos u}{1-\cos^2 u} = \frac{1-\cos u}{\sin^2 u} = \csc^2 u - \csc u \cot u,$$

so we have $\int(\csc^2 u - \csc u \cot u)du = \int dx$ and $-\cot u + \csc u = x + c$. Thus $-\cot(x+y) + \csc(x+y) = x + c$. Setting $x = 0$ and $y = \pi/4$ we obtain $c = \sqrt{2} - 1$. The solution is

$$\csc(x+y) - \cot(x+y) = x + \sqrt{2} - 1.$$

30. Let $u = 3x + 2y$ so that $du/dx = 3 + 2\,dy/dx$. Then $\dfrac{du}{dx} = 3 + \dfrac{2u}{u+2} = \dfrac{5u+6}{u+2}$ and $\dfrac{u+2}{5u+6}\,du = dx$. Now by long division

$$\frac{u+2}{5u+6} = \frac{1}{5} + \frac{4}{25u+30}$$

so we have

$$\int\left(\frac{1}{5} + \frac{4}{25u+30}\right)du = dx$$

and $\frac{1}{5}u + \frac{4}{25}\ln|25u+30| = x + c$. Thus

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x+50y+30| = x + c.$$

Setting $x = -1$ and $y = -1$ we obtain $c = \frac{4}{25}\ln 95$. The solution is

$$\frac{1}{5}(3x+2y) + \frac{4}{25}\ln|75x+50y+30| = x + \frac{4}{25}\ln 95$$

or $\quad 5y - 5x + 2\ln|75x+50y+30| = 2\ln 95.$

Discussion Problems

31. We write the differential equation $M(x, y)dx + N(x, y)dy = 0$ as $dy/dx = f(x, y)$ where

$$f(x, y) = -\frac{M(x, y)}{N(x, y)}\,.$$

The function $f(x, y)$ must necessarily be homogeneous of degree 0 when M and N are homogeneous of degree α. Since M is homogeneous of degree α, $M(tx, ty) = t^\alpha M(x, y)$, and letting $t = 1/x$ we have

$$M(1, y/x) = \frac{1}{x^\alpha}M(x, y) \qquad \text{or} \qquad M(x, y) = x^\alpha M(1, y/x).$$

Thus

$$\frac{dy}{dx} = f(x, y) = -\frac{x^\alpha M(1, y/x)}{x^\alpha N(1, y/x)} = -\frac{M(1, y/x)}{N(1, y/x)} = F\left(\frac{y}{x}\right).$$

32. Rewrite $(5x^2 - 2y^2)dx - xy\,dy = 0$ as

$$xy\,\frac{dy}{dx} = 5x^2 - 2y^2$$

and divide by xy, so that

$$\frac{dy}{dx} = 5\,\frac{x}{y} - 2\,\frac{y}{x}\,.$$

We then identify

$$F\left(\frac{y}{x}\right) = 5\left(\frac{y}{x}\right)^{-1} - 2\left(\frac{y}{x}\right).$$

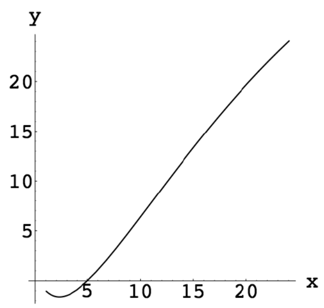
33. **(a)** By inspection $y = x$ and $y = -x$ are solutions of the differential equation and not members of the family $y = x\sin(\ln x + c_2)$.

(b) Letting $x = 5$ and $y = 0$ in $\sin^{-1}(y/x) = \ln x + c_2$ we get $\sin^{-1} 0 = \ln 5 + c$ or $c = -\ln 5$. Then $\sin^{-1}(y/x) = \ln x - \ln 5 = \ln(x/5)$. Because the range of the arcsine function is $[-\pi/2, \pi/2]$ we must have

$$-\frac{\pi}{2} \le \ln\frac{x}{5} \le \frac{\pi}{2}$$

$$e^{-\pi/2} \le \frac{x}{5} \le e^{\pi/2}$$

$$5e^{-\pi/2} \le x \le 5e^{\pi/2}.$$



The interval of definition of the solution is approximately $[1.04, 24.05]$.

34. As $x \to -\infty$, $e^{6x} \to 0$ and $y \to 2x+3$. Now write $(1+ce^{6x})/(1-ce^{6x})$ as $(e^{-6x}+c)/(e^{-6x}-c)$. Then, as $x \to \infty$, $e^{-6x} \to 0$ and $y \to 2x-3$.

35. **(a)** The substitutions

$$y = y_1 + u \qquad \text{and} \qquad \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

lead to

$$\begin{aligned}\frac{dy_1}{dx} + \frac{du}{dx} &= P + Q(y_1+u) + R(y_1+u)^2 \\ &= P + Qy_1 + Ry_1^2 + Qu + 2y_1Ru + Ru^2\end{aligned}$$

or

$$\frac{du}{dx} - (Q + 2y_1R)u = Ru^2.$$

This is a Bernoulli equation with $n=2$ which can be reduced to the linear equation

$$\frac{dw}{dx} + (Q + 2y_1R)w = -R$$

by the substitution $w = u^{-1}$.

(b) Identify $P(x) = -4/x^2$, $Q(x) = -1/x$, and $R(x) = 1$. Then $\dfrac{dw}{dx} + \left(-\dfrac{1}{x} + \dfrac{4}{x}\right) w = -1$. An integrating factor is x^3 so that $x^3w = -\frac{1}{4}x^4 + c$ or $u = \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}$. Thus, $y = \dfrac{2}{x} + u$.

36. Write the differential equation in the form $x(y'/y) = \ln x + \ln y$ and let $u = \ln y$. Then $du/dx = y'/y$ and the differential equation becomes $x(du/dx) = \ln x + u$ or $du/dx - u/x = (\ln x)/x$, which is first-order and linear. An integrating factor is $e^{-\int dx/x} = 1/x$, so that (using integration by parts)

$$\frac{d}{dx}\left[\frac{1}{x}\,u\right] = \frac{\ln x}{x^2} \quad \text{and} \quad \frac{u}{x} = -\frac{1}{x} - \frac{\ln x}{x} + c.$$

The solution is

$$\ln y = -1 - \ln x + cx \quad \text{or} \quad y = \frac{e^{cx-1}}{x}.$$

Mathematical Models

37. Write the differential equation as

$$\frac{dv}{dx} + \frac{1}{x}\,v = 32v^{-1},$$

and let $u = v^2$ or $v = u^{1/2}$. Then

$$\frac{dv}{dx} = \frac{1}{2}u^{-1/2}\,\frac{du}{dx},$$

and substituting into the differential equation, we have

$$\frac{1}{2}u^{-1/2}\,\frac{du}{dx} + \frac{1}{x}\,u^{1/2} = 32u^{-1/2} \qquad \text{or} \qquad \frac{du}{dx} + \frac{2}{x}\,u = 64.$$

The latter differential equation is linear with integrating factor $e^{\int (2/x)dx} = x^2$, so

$$\frac{d}{dx}\left[x^2 u\right] = 64x^2$$

and

$$x^2 u = \frac{64}{3}x^3 + c \qquad \text{or} \qquad v^2 = \frac{64}{3}x + \frac{c}{x^2}.$$

38. Write the differential equation as $dP/dt - aP = -bP^2$ and let $u = P^{-1}$ or $P = u^{-1}$. Then

$$\frac{dp}{dt} = -u^{-2}\frac{du}{dt},$$

and substituting into the differential equation, we have

$$-u^{-2}\frac{du}{dt} - au^{-1} = -bu^{-2} \qquad \text{or} \qquad \frac{du}{dt} + au = b.$$

The latter differential equation is linear with integrating factor $e^{\int a\,dt} = e^{at}$, so

$$\frac{d}{dt}\left[e^{at}u\right] = be^{at}$$

and

$$\begin{aligned} e^{at}u &= \frac{b}{a}e^{at} + c \\ e^{at}P^{-1} &= \frac{b}{a}e^{at} + c \\ P^{-1} &= \frac{b}{a} + ce^{-at} \\ P &= \frac{1}{b/a + ce^{-at}} = \frac{a}{b + c_1 e^{-at}}. \end{aligned}$$

2.6 A Numerical Method

1. We identify $f(x, y) = 2x - 3y + 1$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(2x_n - 3y_n + 1) = 0.2x_n + 0.7y_n + 0.1,$$

and

$$\begin{aligned} y(1.1) &\approx y_1 = 0.2(1) + 0.7(5) + 0.1 = 3.8 \\ y(1.2) &\approx y_2 = 0.2(1.1) + 0.7(3.8) + 0.1 = 2.98. \end{aligned}$$

For $h = 0.05$,

$$y_{n+1} = y_n + 0.05(2x_n - 3y_n + 1) = 0.1x_n + 0.85y_n + 0.05,$$

and

$$\begin{aligned} y(1.05) &\approx y_1 = 0.1(1) + 0.85(5) + 0.05 = 4.4 \\ y(1.1) &\approx y_2 = 0.1(1.05) + 0.85(4.4) + 0.05 = 3.895 \\ y(1.15) &\approx y_3 = 0.1(1.1) + 0.85(3.895) + 0.05 = 3.47075 \\ y(1.2) &\approx y_4 = 0.1(1.15) + 0.85(3.47075) + 0.05 = 3.11514. \end{aligned}$$

2. We identify $f(x, y) = x + y^2$. Then, for $h = 0.1$,

$$y_{n+1} = y_n + 0.1(x_n + y_n^2) = 0.1x_n + y_n + 0.1y_n^2,$$

and

$$\begin{aligned} y(0.1) &\approx y_1 = 0.1(0) + 0 + 0.1(0)^2 = 0 \\ y(0.2) &\approx y_2 = 0.1(0.1) + 0 + 0.1(0)^2 = 0.01. \end{aligned}$$

For $h = 0.05$,

$$y_{n+1} = y_n + 0.05(x_n + y_n^2) = 0.05x_n + y_n + 0.05y_n^2,$$

and

$$\begin{aligned} y(0.05) &\approx y_1 = 0.05(0) + 0 + 0.05(0)^2 = 0 \\ y(0.1) &\approx y_2 = 0.05(0.05) + 0 + 0.05(0)^2 = 0.0025 \\ y(0.15) &\approx y_3 = 0.05(0.1) + 0.0025 + 0.05(0.0025)^2 = 0.0075 \\ y(0.2) &\approx y_4 = 0.05(0.15) + 0.0075 + 0.05(0.0075)^2 = 0.0150. \end{aligned}$$

3. Separating variables and integrating, we have

$$\frac{dy}{y} = dx \qquad \text{and} \qquad \ln|y| = x + c.$$

Thus $y = c_1 e^x$ and, using $y(0) = 1$, we find $c = 1$, so $y = e^x$ is the solution of the initial-value problem.

h=0.1

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.10	1.1000	1.1052	0.0052	0.47
0.20	1.2100	1.2214	0.0114	0.93
0.30	1.3310	1.3499	0.0189	1.40
0.40	1.4641	1.4918	0.0277	1.86
0.50	1.6105	1.6487	0.0382	2.32
0.60	1.7716	1.8221	0.0506	2.77
0.70	1.9487	2.0138	0.0650	3.23
0.80	2.1436	2.2255	0.0820	3.68
0.90	2.3579	2.4596	0.1017	4.13
1.00	2.5937	2.7183	0.1245	4.58

h=0.05

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
0.00	1.0000	1.0000	0.0000	0.00
0.05	1.0500	1.0513	0.0013	0.12
0.10	1.1025	1.1052	0.0027	0.24
0.15	1.1576	1.1618	0.0042	0.36
0.20	1.2155	1.2214	0.0059	0.48
0.25	1.2763	1.2840	0.0077	0.60
0.30	1.3401	1.3499	0.0098	0.72
0.35	1.4071	1.4191	0.0120	0.84
0.40	1.4775	1.4918	0.0144	0.96
0.45	1.5513	1.5683	0.0170	1.08
0.50	1.6289	1.6487	0.0198	1.20
0.55	1.7103	1.7333	0.0229	1.32
0.60	1.7959	1.8221	0.0263	1.44
0.65	1.8856	1.9155	0.0299	1.56
0.70	1.9799	2.0138	0.0338	1.68
0.75	2.0789	2.1170	0.0381	1.80
0.80	2.1829	2.2255	0.0427	1.92
0.85	2.2920	2.3396	0.0476	2.04
0.90	2.4066	2.4596	0.0530	2.15
0.95	2.5270	2.5857	0.0588	2.27
1.00	2.6533	2.7183	0.0650	2.39

4. Separating variables and integrating, we have

$$\frac{dy}{y} = 2x\,dx \qquad \text{and} \qquad \ln|y| = x^2 + c.$$

Thus $y = c_1 e^{x^2}$ and, using $y(1) = 1$, we find $c = e^{-1}$, so $y = e^{x^2-1}$ is the solution of the initial-value problem.

h=0.1

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2000	1.2337	0.0337	2.73
1.20	1.4640	1.5527	0.0887	5.71
1.30	1.8154	1.9937	0.1784	8.95
1.40	2.2874	2.6117	0.3243	12.42
1.50	2.9278	3.4903	0.5625	16.12

h=0.05

x_n	y_n	Actual Value	Abs. Error	% Rel. Error
1.00	1.0000	1.0000	0.0000	0.00
1.05	1.1000	1.1079	0.0079	0.72
1.10	1.2155	1.2337	0.0182	1.47
1.15	1.3492	1.3806	0.0314	2.27
1.20	1.5044	1.5527	0.0483	3.11
1.25	1.6849	1.7551	0.0702	4.00
1.30	1.8955	1.9937	0.0982	4.93
1.35	2.1419	2.2762	0.1343	5.90
1.40	2.4311	2.6117	0.1806	6.92
1.45	2.7714	3.0117	0.2403	7.98
1.50	3.1733	3.4903	0.3171	9.08

5. h=0.1

x_n	y_n
0.00	0.0000
0.10	0.1000
0.20	0.1905
0.30	0.2731
0.40	0.3492
0.50	0.4198

h=0.05

x_n	y_n
0.00	0.0000
0.05	0.0500
0.10	0.0976
0.15	0.1429
0.20	0.1863
0.25	0.2278
0.30	0.2676
0.35	0.3058
0.40	0.3427
0.45	0.3782
0.50	0.4124

6. h=0.1

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2220
0.30	1.3753
0.40	1.5735
0.50	1.8371

h=0.05

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1053
0.15	1.1668
0.20	1.2360
0.25	1.3144
0.30	1.4039
0.35	1.5070
0.40	1.6267
0.45	1.7670
0.50	1.9332

7. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5431
0.30	0.5548
0.40	0.5613
0.50	0.5639

h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5232
0.15	0.5322
0.20	0.5395
0.25	0.5452
0.30	0.5496
0.35	0.5527
0.40	0.5547
0.45	0.5559
0.50	0.5565

8. h=0.1

x_n	y_n
0.00	1.0000
0.10	1.1000
0.20	1.2159
0.30	1.3505
0.40	1.5072
0.50	1.6902

h=0.05

x_n	y_n
0.00	1.0000
0.05	1.0500
0.10	1.1039
0.15	1.1619
0.20	1.2245
0.25	1.2921
0.30	1.3651
0.35	1.4440
0.40	1.5293
0.45	1.6217
0.50	1.7219

9. h=0.1

x_n	y_n
1.00	1.0000
1.10	1.0000
1.20	1.0191
1.30	1.0588
1.40	1.1231
1.50	1.2194

h=0.05

x_n	y_n
1.00	1.0000
1.05	1.0000
1.10	1.0049
1.15	1.0147
1.20	1.0298
1.25	1.0506
1.30	1.0775
1.35	1.1115
1.40	1.1538
1.45	1.2057
1.50	1.2696

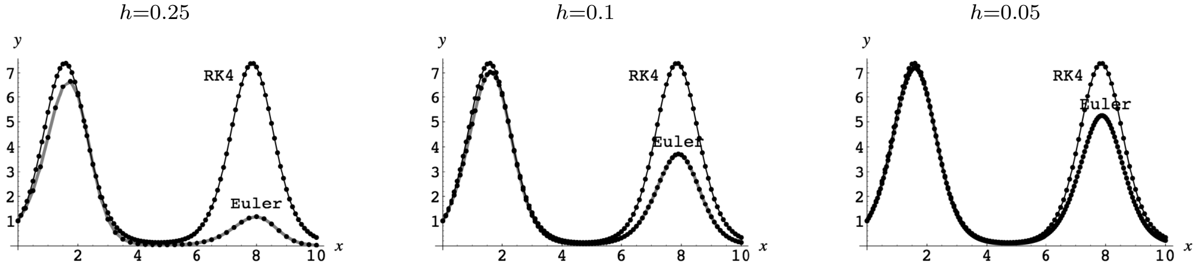
10. h=0.1

x_n	y_n
0.00	0.5000
0.10	0.5250
0.20	0.5499
0.30	0.5747
0.40	0.5991
0.50	0.6231

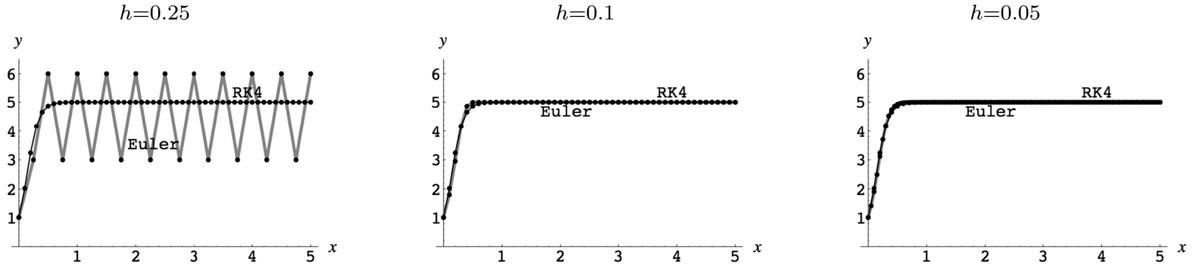
h=0.05

x_n	y_n
0.00	0.5000
0.05	0.5125
0.10	0.5250
0.15	0.5375
0.20	0.5499
0.25	0.5623
0.30	0.5746
0.35	0.5868
0.40	0.5989
0.45	0.6109
0.50	0.6228

11. Tables of values were computed using the Euler and RK4 methods. The resulting points were plotted and joined using **ListPlot** in *Mathematica*.



12. Tables of values were computed using the Euler and RK4 methods. The resulting points were plotted and joined using **ListPlot** in *Mathematica*.



Discussion Problems

13. Tables of values, shown below, were first computed using Euler's method with $h = 0.1$ and $h = 0.05$, and then using the RK4 method with the same values of h. Using separation of variables we find that the solution of the differential equation is $y = 1/(1 - x^2)$, which is undefined at $x = 1$, where the graph has a vertical asymptote. Because the actual solution of the differential equation becomes unbounded at x approaches 1, very small changes in the inputs x will result in large changes in the corresponding outputs y. This can be expected to have a serious effect on numerical procedures.

h=0.1 (Euler)

x_n	y_n
0.00	1.0000
0.10	1.0000
0.20	1.0200
0.30	1.0616
0.40	1.1292
0.50	1.2313
0.60	1.3829
0.70	1.6123
0.80	1.9763
0.90	2.6012
1.00	3.8191

h=0.05 (Euler)

x_n	y_n
0.00	1.0000
0.05	1.0000
0.10	1.0050
0.15	1.0151
0.20	1.0306
0.25	1.0518
0.30	1.0795
0.35	1.1144
0.40	1.1579
0.45	1.2115
0.50	1.2776
0.55	1.3592
0.60	1.4608
0.65	1.5888
0.70	1.7529
0.75	1.9679
0.80	2.2584
0.85	2.6664
0.90	3.2708
0.95	4.2336
1.00	5.9363

h=0.1 (RK4)

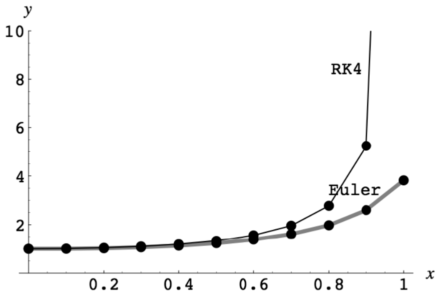
x_n	y_n
0.00	1.0000
0.10	1.0101
0.20	1.0417
0.30	1.0989
0.40	1.1905
0.50	1.3333
0.60	1.5625
0.70	1.9607
0.80	2.7771
0.90	5.2388
1.00	42.9931

h=0.05 (RK4)

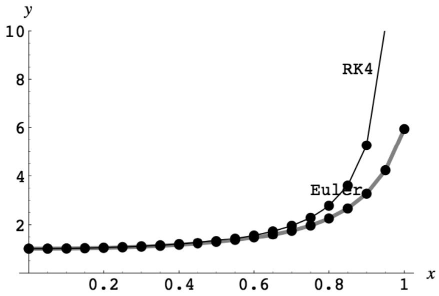
x_n	y_n
0.00	1.0000
0.05	1.0025
0.10	1.0101
0.15	1.0230
0.20	1.0417
0.25	1.0667
0.30	1.0989
0.35	1.1396
0.40	1.1905
0.45	1.2539
0.50	1.3333
0.55	1.4337
0.60	1.5625
0.65	1.7316
0.70	1.9608
0.75	2.2857
0.80	2.7777
0.85	3.6034
0.90	5.2609
0.95	10.1973
1.00	84.0132

The points in the tables above were plotted and joined using **ListPlot** in *Mathematica*.

h=0.1

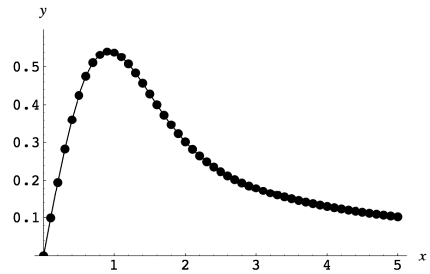


h=0.05



Computer Lab Assignments

14. (a) The graph to the right was obtained using RK4 and **ListPlot** in *Mathematica* with $h = 0.1$.



(b) Writing the differential equation in the form $y' + 2xy = 1$ we see that an integrating factor is $e^{\int 2x dx} = e^{x^2}$, so

$$\frac{d}{dx}[e^{x^2} y] = e^{x^2}$$

and

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + ce^{-x^2}.$$

This solution can also be expressed in terms of the inverse error function as

$$y = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x) + ce^{-x^2}.$$

Letting $x = 0$ and $y(0) = 0$ we find $c = 0$, so the solution of the initial-value problem is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt = \frac{\sqrt{\pi}}{2}\,e^{-x^2}\operatorname{erfi}(x).$$

(c) Using **FindRoot** in *Mathematica* we see that $y'(x) = 0$ when $x = 0.924139$. Since $y(0.924139) = 0.541044$, we see from the graph in part (a) that $(0.924139, 0.541044)$ is a relative maximum. Now, using the substitution $u = -t$ in the integral below, we have

$$y(-x) = e^{-(-x)^2}\int_0^{-x} e^{t^2}\,dt = e^{-x^2}\int_0^x e^{(-u)^2}(-du) = -e^{-x^2}\int_0^x e^{u^2}\,du = -y(x).$$

Thus, $y(x)$ is an odd function and $(-0.924139, -0.541044)$ is a relative minimum.

2.R Chapter 2 in Review

1. Writing the differential equation in the form $y' = k(y + A/k)$ we see that the critical point $-A/k$ is a repeller for $k > 0$ and an attractor for $k < 0$.

2. Separating variables and integrating we have

$$\frac{dy}{y} = \frac{4}{x}\,dx$$
$$\ln y = 4\ln x + c = \ln x^4 + c$$
$$y = c_1x^4.$$

We see that when $x = 0$, $y = 0$, so the initial-value problem has an infinite number of solutions for $k = 0$ and no solutions for $k \neq 0$.

3. True; $y = k_2/k_1$ is always a solution for $k_1 \neq 0$.

4. True; writing the differential equation as $a_1(x)\,dy + a_2(x)y\,dx = 0$ and separating variables yields

$$\frac{dy}{y} = -\frac{a_2(x)}{a_1(x)}\,dx.$$

5. An example of a nonlinear third-order differential equation in normal form is $\dfrac{d^3y}{dx^3} = xe^y$. There are many possible answers.

6. False, because $r\theta + r + \theta + 1 = (r+1)(\theta+1)$ and the differential equation can be written as

$$\frac{dr}{r+1} = (\theta+1)d\theta.$$

7. True, because the differential equation can be written as $\dfrac{dy}{f(y)} = dx$.

8. Since the differential equation is autonomous, $2 - |y| = 0$ implies that $y = 2$ and $y = -2$ are critical points and hence solutions of the differential equation.

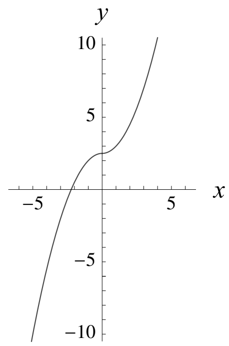
9. The differential equation is separable so

$$\frac{dy}{y} = e^x dx \qquad \text{implies} \qquad \ln|y| = e^{e^x} + c,$$

and thus $y = c_1 e^{e^x}$ is the general solution of the differential equation.

10. We have

$$y' = |x| = \begin{cases} -x, & x < 0 \\ x, & x \ge 0 \end{cases} \quad \text{implies} \quad y = \begin{cases} -\frac{1}{2}x^2 + c_1, & x < 0 \\ \frac{1}{2}x^2 + c_2, & x \ge 0. \end{cases}$$



The initial condition $y(-1) = 2$ implies that $-\frac{1}{2}(-1)^2 + c_1 = 2$ so $c_1 = \frac{5}{2}$. Since $y(x)$ is supposed to be continuous at $x = 0$, the two parts of the function must agree. That is, c_2 must also be $\frac{5}{2}$, and

$$y(x) = \begin{cases} -\frac{1}{2}x^2 + \frac{5}{2}, & x < 0 \\ \frac{1}{2}x^2 + \frac{5}{2}, & x \ge 0 \end{cases} = \begin{cases} -\frac{1}{2}(5 - x^2), & x < 0 \\ \frac{1}{2}(5 + x^2), & x \ge 0. \end{cases}$$

11. Differentiating we find

$$\frac{dy}{dx} = e^{\cos x} x e^{-\cos x} + (-\sin x)\cos x \int_0^x te^{-\cos t} dt = x - (\sin x)y.$$

Thus the linear differential equation is $\dfrac{dy}{dx} + (\sin x)y = x$.

12. An example of an autonomous linear first-order differential equation with a single critical point at -3 is $\dfrac{dy}{dx} = y + 3$, whereas an autonomous nonlinear first-order differential equation with a single critical point -3 is $\dfrac{dy}{dx} = (y+3)^2$.

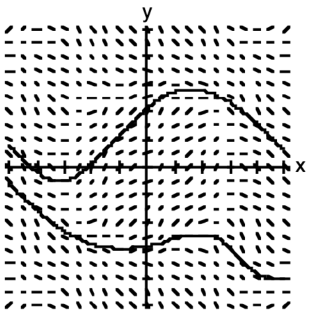
13. $\dfrac{dy}{dx} = (y-1)^2(y-3)^2$

14. $\dfrac{dy}{dx} = y(y-2)^2(y-4)$

15. When n is odd, $x^n < 0$ for $x < 0$ and $x^n > 0$ for $x > 0$. In this case 0 is unstable. When n is even, $x^n > 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable. When n is odd, $-x^n > 0$ for $x < 0$ and $-x^n < 0$ for $x > 0$. In this case 0 is asymptotically stable. When n is even, $-x^n < 0$ for $x < 0$ and for $x > 0$. In this case 0 is semi-stable. Technically, 0^0 is an indeterminant form; however for all values of x except 0, $x^0 = 1$. Thus, we define 0^0 to be 1 in this case.

16. Using a CAS we find that the zero of f occurs at approximately $P = 1.3214$. From the graph we observe that $dP/dt > 0$ for $P < 1.3214$ and $dP/dt < 0$ for $P > 1.3214$, so $P = 1.3214$ is an asymptotically stable critical point. Thus, $\lim_{t\to\infty} P(t) = 1.3214$.

17.



18. **(a)** linear in y, homogeneous, exact

(b) linear in x

(c) separable, exact, linear in x and y

(d) Bernoulli in x

(e) separable

(f) separable, linear in x, Bernoulli

(g) linear in x

(h) homogeneous

(i) Bernoulli

(j) homogeneous, exact, Bernoulli

(k) linear in x and y, exact, separable, homogeneous

(l) exact, linear in y

(m) homogeneous

(n) separable

19. Separating variables and using the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\cos^2 x\,dx = \frac{y}{y^2+1}\,dy,$$

$$\frac{1}{2}x + \frac{1}{4}\sin 2x = \frac{1}{2}\ln\left(y^2+1\right) + c,$$

and

$$2x + \sin 2x = 2\ln\left(y^2+1\right) + c.$$

20. Write the differential equation in the form

$$y \ln \frac{x}{y}\, dx = \left(x \ln \frac{x}{y} - y \right) dy.$$

This is a homogeneous equation, so let $x = uy$. Then $dx = u\,dy + y\,du$ and the differential equation becomes

$$y \ln u(u\,dy + y\,du) = (uy \ln u - y)\,dy \quad \text{or} \quad y \ln u\,du = -dy.$$

Separating variables, we obtain

$$\ln u\,du = -\frac{dy}{y}$$

$$u \ln |u| - u = -\ln |y| + c$$

$$\frac{x}{y} \ln \left| \frac{x}{y} \right| - \frac{x}{y} = -\ln |y| + c$$

$$x(\ln x - \ln y) - x = -y \ln |y| + cy.$$

21. The differential equation

$$\frac{dy}{dx} + \frac{2}{6x+1} y = -\frac{3x^2}{6x+1} y^{-2}$$

is Bernoulli. Using $w = y^3$, we obtain the linear equation

$$\frac{dw}{dx} + \frac{6}{6x+1} w = -\frac{9x^2}{6x+1}.$$

An integrating factor is $6x + 1$, so

$$\frac{d}{dx}\left[(6x+1)w\right] = -9x^2,$$

$$w = -\frac{3x^3}{6x+1} + \frac{c}{6x+1},$$

and

$$(6x+1)y^3 = -3x^3 + c.$$

Note: The differential equation is also exact.

22. Write the differential equation in the form $(3y^2 + 2x)dx + (4y^2 + 6xy)dy = 0$. Letting $M = 3y^2 + 2x$ and $N = 4y^2 + 6xy$ we see that $M_y = 6y = N_x$, so the differential equation is exact. From $f_x = 3y^2 + 2x$ we obtain $f = 3xy^2 + x^2 + h(y)$. Then $f_y = 6xy + h'(y) = 4y^2 + 6xy$ and $h'(y) = 4y^2$ so $h(y) = \frac{4}{3}y^3$. A one-parameter family of solutions is

$$3xy^2 + x^2 + \frac{4}{3}y^3 = c.$$

23. Write the equation in the form

$$\frac{dQ}{dt} + \frac{1}{t}Q = t^3 \ln t.$$

An integrating factor is $e^{\ln t} = t$, so

$$\frac{d}{dt}[tQ] = t^4 \ln t$$

$$tQ = -\frac{1}{25}t^5 + \frac{1}{5}t^5 \ln t + c$$

and

$$Q = -\frac{1}{25}t^4 + \frac{1}{5}t^4 \ln t + \frac{c}{t}.$$

24. Letting $u = 2x + y + 1$ we have

$$\frac{du}{dx} = 2 + \frac{dy}{dx},$$

and so the given differential equation is transformed into

$$u\left(\frac{du}{dx} - 2\right) = 1 \quad \text{or} \quad \frac{du}{dx} = \frac{2u+1}{u}.$$

Separating variables and integrating we get

$$\frac{u}{2u+1}\, du = dx$$

$$\left(\frac{1}{2} - \frac{1}{2}\frac{1}{2u+1}\right) du = dx$$

$$\frac{1}{2}u - \frac{1}{4}\ln|2u+1| = x + c$$

$$2u - \ln|2u+1| = 2x + c_1.$$

Resubstituting for u gives the solution

$$4x + 2y + 2 - \ln|4x + 2y + 3| = 2x + c_1$$

or

$$2x + 2y + 2 - \ln|4x + 2y + 3| = c_1.$$

25. Write the equation in the form

$$\frac{dy}{dx} + \frac{8x}{x^2+4}y = \frac{2x}{x^2+4}.$$

An integrating factor is $\left(x^2+4\right)^4$, so

$$\frac{d}{dx}\left[\left(x^2+4\right)^4 y\right] = 2x\left(x^2+4\right)^3$$

$$\left(x^2+4\right)^4 y = \frac{1}{4}\left(x^2+4\right)^4 + c$$

and

$$y = \frac{1}{4} + c\left(x^2+4\right)^{-4}.$$

26. Letting $M = 2r^2\cos\theta\sin\theta + r\cos\theta$ and $N = 4r + \sin\theta - 2r\cos^2\theta$ we see that $M_r = 4r\cos\theta\sin\theta + \cos\theta = N_\theta$, so the differential equation is exact. From $f_\theta = 2r^2\cos\theta\sin\theta + r\cos\theta$ we obtain $f = -r^2\cos^2\theta + r\sin\theta + h(r)$. Then $f_r = -2r\cos^2\theta + \sin\theta + h'(r) = 4r + \sin\theta - 2r\cos^2\theta$ and $h'(r) = 4r$ so $h(r) = 2r^2$. The solution is

$$-r^2\cos^2\theta + r\sin\theta + 2r^2 = c.$$

27. The differential equation has the form $(d/dx)\,[(\sin x)y] = 0$. Integrating, we have $(\sin x)y = c$ or $y = c/\sin x$. The initial condition implies $c = -2\sin(7\pi/6) = 1$. Thus, $y = 1/\sin x = \csc x$, where the interval $\pi < x < 2\pi$ is chosen to include $x = 7\pi/6$.

28. Separating variables and integrating we have

$$\frac{dy}{y^2} = -2(t+1)\,dt$$

$$-\frac{1}{y} = -(t+1)^2 + c$$

$$y = \frac{1}{(t+1)^2 + c_1} \qquad \leftarrow \text{letting } -c = c_1 \qquad .$$

The initial condition $y(0) = -\frac{1}{8}$ implies $c_1 = -9$, so a solution of the initial-value problem is

$$y = \frac{1}{(t+1)^2 - 9} \qquad \text{or} \qquad y = \frac{1}{t^2 + 2t - 8},$$

where $-4 < t < 2$.

29. **(a)** For $y < 0$, $\sqrt{y}$ is not a real number.

(b) Separating variables and integrating we have

$$\frac{dy}{\sqrt{y}} = dx \qquad \text{and} \qquad 2\sqrt{y} = x + c.$$

Letting $y(x_0) = y_0$ we get $c = 2\sqrt{y_0} - x_0$, so that

$$2\sqrt{y} = x + 2\sqrt{y_0} - x_0 \quad \text{and} \quad y = \frac{1}{4}(x + 2\sqrt{y_0} - x_0)^2.$$

Since $\sqrt{y} > 0$ for $y \neq 0$, we see that $dy/dx = \frac{1}{2}(x + 2\sqrt{y_0} - x_0)$ must be positive. Thus, the interval on which the solution is defined is $(x_0 - 2\sqrt{y_0}, \infty)$.

30. **(a)** The differential equation is homogeneous and we let $y = ux$. Then

$$(x^2 - y^2)\,dx + xy\,dy = 0$$

$$(x^2 - u^2x^2)\,dx + ux^2(u\,dx + x\,du) = 0$$

$$dx + ux\,du = 0$$

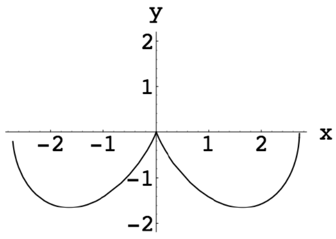
$$u\,du = -\frac{dx}{x}$$

$$\frac{1}{2}u^2 = -\ln|x| + c$$

$$\frac{y^2}{x^2} = -2\ln|x| + c_1.$$

The initial condition gives $c_1 = 2$, so an implicit solution is $y^2 = x^2(2 - 2\ln|x|)$.

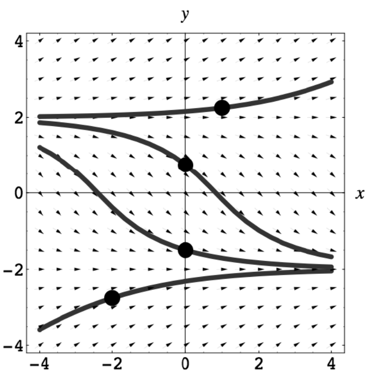
(b) Solving for y in part (a) and being sure that the initial condition is still satisfied, we have $y = -\sqrt{2}\,|x|(1 - \ln|x|)^{1/2}$, where $-e \le x \le e$ so that $1 - \ln|x| \ge 0$. The graph of this function indicates that the derivative is not defined at $x = 0$ and $x = e$. Thus, the solution of the initial-value problem is $y = -\sqrt{2}\,x(1 - \ln x)^{1/2}$, for $0 < x < e$.



31. The graph of $y_1(x)$ is the portion of the closed blue curve lying in the fourth quadrant. Its interval of definition is approximately (0.7, 4.3). The graph of $y_2(x)$ is the portion of the left-hand blue curve lying in the third quadrant. Its interval of definition is $(-\infty, 0)$.

32. The first step of Euler's method gives $y(1.1) \approx 9 + 0.1(1 + 3) = 9.4$. Applying Euler's method one more time gives $y(1.2) \approx 9.4 + 0.1(1 + 1.1\sqrt{9.4}\,) \approx 9.8373$.

33. Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has critical points at -2 (an attractor) and at 2 (a repeller). Thus, -2 is an asymptotically stable critical point and 2 is an unstable critical point.



34. Since the differential equation is autonomous, all lineal elements on a given horizontal line have the same slope. The direction field is then as shown in the figure at the right. It appears from the figure that the differential equation has no critical points.